Mathematical Derivation of Drawdown and Stream Depletion Produced by Pumping in the Vicinity of a Finite-Width Stream of Shallow Penetration

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I. Introduction

In this report, the mathematical derivation of the solution of Butler et al. (2001) for drawdown and stream depletion produced by pumping in the vicinity of a finite-width stream of shallow penetration is presented and the definition of the Hunt (1999) leakance parameter is examined. For the sake of generality, the solution is obtained in a dimensionless form. See Butler et al. (2001) for notation definitions that are not given in this report.

II. Drawdown Solution

The drawdown solution was obtained using a straightforward extension of the approach described in Butler and Liu (1991). Equations (1)-(10) of Butler et al. (2001) describe the flow conditions of interest here. Dimensionless forms of these equations are as follows:

$$\frac{\partial^2 \Phi_1}{\partial \xi^2} + \frac{\partial^2 \Phi_1}{\partial \eta^2} = P_1 \frac{\partial \Phi_1}{\partial \tau}, -X_{LB} \le \xi \le -1, \quad -\infty < \eta < \infty, \tau > 0 \tag{1}$$

$$\frac{\partial^2 \Phi_2}{\partial \xi^2} + \frac{\partial^2 \Phi_2}{\partial \eta^2} - \mathrm{B}\Phi_2 = P_2 \frac{\partial \Phi_2}{\partial \tau}, \ -1 \le \xi \le 0, \quad -\infty < \eta < \infty, \ \tau > 0 \tag{2}$$

$$\frac{\partial^2 \Phi_3}{\partial \xi^2} + \frac{\partial^2 \Phi_3}{\partial \eta^2} + \delta(\xi - \alpha) = \frac{\partial \Phi_3}{\partial \tau}, 0 \le \xi \le X_{RB}, \quad -\infty < \eta < \infty, \tau > 0 \tag{3}$$

$$\Phi_i(\xi, \eta, 0) = 0, -X_{LB} \le \xi \le X_{RB}, -\infty < \eta < \infty, \mathrm{i} = 1,3 \tag{4}$$

$$\frac{\partial \Phi_1}{\partial \xi}(-X_{LB}, \eta, \tau) = \frac{\partial \Phi_3}{\partial \xi}(X_{RB}, \eta, \tau) = 0, -\infty < \eta < \infty, \quad \tau > 0 \tag{5}$$

$$\Phi_i(\xi, \pm\infty, \tau) = 0, -X_{LB} \le \xi \le X_{RB}, \ \tau > 0 \tag{6}$$

$$\Phi_1(-1, \eta, \tau) = \Phi_2(-1, \eta, \tau), -\infty < \eta < \infty, \ \tau > 0 \tag{7}$$

$$\frac{\partial \Phi_1}{\partial \xi}(-1,\eta,\tau) = \gamma_1 \frac{\partial \Phi_2}{\partial \xi}(-1,\eta,\tau), -\infty < \eta < \infty, \tau > 0 \tag{8}$$

$$\Phi_2(0,\eta,\tau) = \Phi_3(0,\eta,\tau), -\infty < \eta < \infty,\ \tau > 0 \tag{9}$$

$$\frac{\partial \Phi_2}{\partial \xi}(0,\eta,\tau) = \gamma_2 \frac{\partial \Phi_3}{\partial \xi}(0,\eta,\tau), -\infty < \eta < \infty,\ \tau > 0 \tag{10}$$

where

Φ_i (dimensionless drawdown) $=s_i T_3/Q$, i=1,3;
τ (dimensionless time)$=(T_3 t)/(w^2 S_3)$;
$\xi=x/w$; $\eta=y/w$; $\alpha = a/w$;
B (stream leakance) = $(k'w^2)/(b'T_2)$;
$X_{RB}=x_{rb}/w$; $X_{LB}=x_{lb}/w$; $\gamma_i=T_{i+1}/T_i$, i=1,2;
$P_i= \mu_i/\mu_3$, i=1,2; $\mu_i=S_i/T_i$, i=1,3.

Note that these dimensionless parameters are obtained by simply grouping terms in the dimensional equations (eqns. (1)-(10)) of Butler et al. (2001).

A solution can be obtained for equations (1)-(10) through use of integral transforms (Robinson, 1968; Churchill, 1972). A Laplace transform in time followed by a Fourier exponential transform in the η direction produces Fourier-Laplace space analogues to (1)-(3) of the following form:

$$\frac{\partial^2 \overline{\overline{\Phi_1}}}{\partial \xi^2} - \lambda_1^2 \overline{\overline{\Phi_1}} = 0, \quad -X_{LB} \le \xi \le -1 \tag{11}$$

$$\frac{\partial^2 \overline{\overline{\Phi_2}}}{\partial \xi^2} - \lambda_2^2 \overline{\overline{\Phi_2}} = 0, \quad -1 \le \xi \le 0 \tag{12}$$

$$\frac{\partial^2 \overline{\overline{\Phi_3}}}{\partial \xi^2} - \lambda_3^2 \overline{\overline{\Phi_3}} = \frac{-\delta(\xi-\alpha)}{p\sqrt{2\pi}}, \; 0 \le \xi \le X_{RB} \tag{13}$$

where

$\overline{\overline{\Phi_i}}$ =the Fourier-Laplace transform of Φ_i, i=1,3;
p=Laplace transform variable;
ω=Fourier transform variable;
$\lambda_1 = (\omega^2+P_1 p)^{0.5}$;
$\lambda_2 = (\omega^2+B+P_2 p)^{0.5}$;
$\lambda_3 = (\omega^2+p)^{0.5}$.

The Fourier-Laplace space solutions to (11) and (12) are quite straightforward:

$$\overline{\overline{\Phi}}_1 = C_1 e^{\lambda_1 \xi} + C_2 e^{-\lambda_1 \xi} \qquad (14)$$

$$\overline{\overline{\Phi}}_2 = C_3 e^{\lambda_2 \xi} + C_4 e^{-\lambda_2 \xi} \qquad (15)$$

The Fourier-Laplace space solution to (13) cannot be found as easily owing to the non-homogeneous delta function term in that expression. The approach used for obtaining a solution to (13) was to divide zone 3 into two subregions, subregion 31 ($0 \leq \xi \leq \alpha$) and subregion 32 ($\alpha < \xi \leq X_{RB}$). The solution for subregion 31 consists of a homogeneous part and a particular solution ($\overline{\overline{\Phi}}_{3p}$):

$$\overline{\overline{\Phi}}_{31} = C_5 e^{\lambda_3 \xi} + C_6 e^{-\lambda_3 \xi} + \overline{\overline{\Phi}}_{3p} \qquad (16)$$

Using Theorem 3.13 of Boyce and DiPrima (1986), the particular solution can be written as:

$$\overline{\overline{\Phi}}_{3p} = \frac{-1}{2 \lambda_3 p \sqrt{2\pi}} [-e^{\lambda_3 (\alpha - \xi)} + e^{\lambda_3 (\xi - \alpha)}], \; \xi = \alpha \qquad (17)$$

$$= 0, \quad \text{elsewhere}$$

The solution for subregion 32 consists solely of a homogeneous part:

$$\overline{\overline{\Phi}}_{32} = C_7 e^{\lambda_3 \xi} + C_8 e^{-\lambda_3 \xi} \qquad (18)$$

The division of region 3 into two subregions requires two additional boundary conditions to ensure continuity across the division:

$$\overline{\overline{\Phi}}_{31}(\alpha, \omega, p) = \overline{\overline{\Phi}}_{32}(\alpha, \omega, p) \qquad (19)$$

$$\frac{\partial \overline{\overline{\Phi}}_{31}}{\partial \xi}(\alpha, \omega, p) = \frac{\partial \overline{\overline{\Phi}}_{32}}{\partial \xi}(\alpha, \omega, p) \qquad (20)$$

The constants in equations (14)-(16) and (18) can be evaluated by substituting these expressions into (19)-(20) and the following Fourier-Laplace space analogues of (5) and (7)-(10):

$$\frac{\partial \overline{\overline{\Phi}}_1}{\partial \xi}(-X_{LB},\omega,p) = \frac{\partial \overline{\overline{\Phi}}_3}{\partial \xi}(X_{RB},\omega,p) = 0 \qquad (21)$$

$$\overline{\overline{\Phi}}_1(-1,\omega,p) = \overline{\overline{\Phi}}_2(-1,\omega,p) \qquad (22)$$

$$\frac{\partial \overline{\overline{\Phi}}_1}{\partial \xi}(-1,\omega,p) = \gamma_1 \frac{\partial \overline{\overline{\Phi}}_2}{\partial \xi}(-1,\omega,p) \qquad (23)$$

$$\overline{\overline{\Phi}}_2(0,\omega,p) = \overline{\overline{\Phi}}_3(0,\omega,p) \qquad (24)$$

$$\frac{\partial \overline{\overline{\Phi}}_2}{\partial \xi}(0,\omega,p) = \gamma_2 \frac{\partial \overline{\overline{\Phi}}_3}{\partial \xi}(0,\omega,p) \qquad (25)$$

Evaluation of the constants is a straightforward but tedious algebraic exercise. Once the constants are found, they are substituted back into equations (14)-(16) and (18) to obtain the following expressions:

$$\overline{\overline{\Phi}}_1(\xi,\omega,p) = (T_1)[e^{2\lambda_1 X_{LB}+\lambda_1\xi} + e^{-\lambda_1\xi}] \qquad (26)$$

$$\overline{\overline{\Phi}}_2(\xi,\omega,p) = (T_1)[(A_1)e^{\lambda_2\xi} + (B_1)e^{-\lambda_2\xi}] \qquad (27)$$

$$\overline{\overline{\Phi}}_3(\xi,\omega,p) = (T_1)[(D_1)e^{\lambda_3\xi} + (E_1)e^{-\lambda_3\xi}], 0 \le \xi \le \alpha \qquad (28)$$

$$\overline{\overline{\Phi}}_3(\xi,\omega,p) = (\frac{(T_1)(G_1)}{(H_1)})[e^{\lambda_3\xi} + e^{2\lambda_3 X_{RB}-\lambda_3\xi}], \alpha < \xi \le X_{RB} \qquad (29)$$

where

$$A_1=\frac{1}{2}(e^{2\lambda_1 X_{LB}-\lambda_1+\lambda_2} + e^{\lambda_1+\lambda_2}) + \frac{\lambda_1}{2\gamma_1\lambda_2}(e^{2\lambda_1 X_{LB}-\lambda_1+\lambda_2} - e^{\lambda_1+\lambda_2})$$

$$B_1=\frac{1}{2}(e^{2\lambda_1 X_{LB}-\lambda_1-\lambda_2} + e^{\lambda_1-\lambda_2}) - \frac{\lambda_1}{2\gamma_1\lambda_2}(e^{2\lambda_1 X_{LB}-\lambda_1-\lambda_2} - e^{-\lambda_1-\lambda_2})$$

$$D_1=\frac{1}{2}[(A_1)+(B_1)] + \frac{\lambda_2}{2\gamma_2\lambda_3}[(A_1)-(B_1)]$$

$E_1 = \frac{1}{2}[(A_1) + (B_1)] - \frac{\lambda_2}{2\gamma_2\lambda_3}[(A_1) - (B_1)]$

$F_1 = -1/(\lambda_3 p\sqrt{2\pi})$

$G_1 = (D_1)e^{\lambda_3\alpha} + (E_1)e^{-\lambda_3\alpha}$

$H_1 = e^{\lambda 3\alpha} + e^{2\lambda 3 X_{RB} - \lambda 3\alpha}$

$J_1 = (D_1)e^{\lambda_3\alpha} - (E_1)e^{-\lambda_3\alpha}$

$K_1 = e^{\lambda 3\alpha} - e^{2\lambda 3 X_{RB} - \lambda 3\alpha}$

$T_1 = (F_1)(H_1)/[(G_1)(K_1) - (J_1)(H_1)]$

Equations (26)-(29) form the Fourier-Laplace space solutions to (1)-(10). Substitution of (26)-(29) into the transform-space analogues of (1)-(10) will demonstrate the viability of the proposed solutions.

The Fourier-Laplace space solution must be transformed back to real space for practical applications. Butler et al. (2001) discuss the numerical inversion schemes used in this work and compare the numerically inverted solution, which is computed using Butler and Tsou (1999), with existing analytical and numerical models.

III. Stream Depletion Solution

The solution for stream depletion was obtained following the approach outlined by Hunt (1999). Butler et al. (2001) define stream depletion in equation (11), the dimensionless form of which is:

$$\Delta Q(\tau) = \frac{\mathrm{B}}{\gamma_2} \int_{-\infty}^{\infty} \int_{-1}^{0} \Phi_2 \, d\xi d\eta, \ \ \tau > 0 \tag{30}$$

Application of the Laplace transform to equation (30) and switching the ξ and η integrals results in:

$$\Delta \overline{Q}(p) = \frac{\mathrm{B}}{\gamma_2} \int_{-1}^{0} \left(\int_{-\infty}^{\infty} \overline{\Phi}_2 d\eta \right) d\xi \tag{31}$$

where

$\Delta \overline{Q}(p)$=Laplace transform of ΔQ.

The term in parentheses is simply the Fourier-Laplace transform of Φ_2 for ω=0:

$$\Delta \overline{Q}(p) = \frac{\mathrm{B}}{\gamma_2} \int_{-1}^{0} \overline{\overline{\Phi}}_2(\xi, 0, p) d\xi \tag{32}$$

Substitution of equation (27) into (32) and performing the integration results in:

$$\Delta \overline{Q}(p) = \frac{B}{\gamma_2 \lambda_2^*}(T_1)[(A_1)(1 - e^{-\lambda_2^*}) - (B_1)(1 - e^{\lambda_2^*})] \qquad (33)$$

where

$\lambda_2^* = (B + P_2 p)^{0.5}$.

Equation (33) is the Laplace-space solution for stream depletion. Butler et al. (2001) describe the numerical scheme used to invert equation (33) to real space, and compare the resulting solution to existing analytical and numerical models. As with the drawdown solution, the numerical inversion scheme is implemented in Butler and Tsou (1999).

Zlotnik et al. (1999) and Hunt (1999) obtain closed-form analytical solutions to various simplifications of equation (33) for laterally infinite aquifers. For pumping wells relatively close to the stream ($\alpha < 5$), equation (11) of Zlotnik et al. (1999) is the preferred approach because stream width cannot be considered negligible relative to the distance from the pumping well to the stream. For pumping wells at larger normalized distances from the stream, the two approaches produce approximately the same results when the Hunt leakance parameter is defined as discussed in the following section.

IV. Hunt Leakance Parameter

In this section, the Hunt leakance parameter is defined in terms of streambed and aquifer characteristics, and its correspondence with the retardation coefficient of Hantush is discussed.

The model proposed by Hunt (1999) is based on the assumption that stream width is small relative to the distance from the stream to the pumping well. Under that condition and the assumptions of aquifer homogeneity and a laterally unbounded aquifer, equations (1)-(3) of Butler et al. (2001) can be rewritten as a single expression:

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} - \frac{k'w}{b'T} s\,\delta(x) + \frac{Q}{T}\delta(x-a)\delta(y) = \frac{S}{T}\frac{\partial s}{\partial t}, -\infty < x < \infty, \qquad (34)$$
$$-\infty < y < \infty, t > 0$$

Comparison of equation (34) with equation (7) of Hunt (1999) shows that the Hunt leakance parameter (λ) can be defined as:

$$\lambda = (k'w)/b' \qquad (35)$$

The dimensionless form (distance from the stream to the pumping well (a) used as the normalizing length) of equation (35) is:

$$\lambda_D = (k'wa)/(b'T) = \mathrm{B}\,\alpha \qquad (36)$$

which is given as equation (12) in Butler et al. (2001).

Hunt (1999) derives the condition for which the Hunt and Hantush models are equivalent. This equivalence occurs when the retardation coefficient of Hantush (L) is defined as:

$$L = (2T)/\lambda \qquad (37)$$

Substitution of equation (35) and the definition of the retardation coefficient ($L=(Kb')/k'$, where K is the hydraulic conductivity of the aquifer) into equation (37) results in:

$$b = w/2 \qquad (38)$$

where b is aquifer thickness. Thus, if the aquifer thickness is set to half the stream width, the Hantush solution will be equivalent to that of Hunt. This condition was used to generate the plots of Figure 7 in Butler et al. (2001). These plots, in addition to Figure 3A in Butler et al. (2001), demonstrate the viability of the Hunt and Hantush solutions for estimation of stream depletion in a laterally infinite aquifer when the distance from the stream to the pumping well is at least five times the stream width ($\alpha \geq 5$) and the model of Figure 2 in Butler et al. (2001) is a reasonable representation of stream-aquifer interactions. Note that these solutions are also appropriate for stream-channel penetrations that cannot be assumed negligible relative to aquifer thickness. In that case, the k' parameter should be redefined as ($k'l/w$), where l is the length along the perimeter of the streambed.

V. References

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