Mathematical Derivations of Expressions for the Total Pumping-Induced Leakage Entering an Aquifer

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I. Introduction

In this report, derivations of the Laplace-space expressions for the volume of pumping-induced leakage entering an aquifer (total leakage) are presented. These expressions, which are used by Butler and Tsou (2003) to demonstrate the scale invariance of total leakage, are obtained from the solutions of Hantush and Jacob (1955), Hantush (1960), and Butler et al. (2001). For the sake of generality, the expressions are presented in a dimensionless form. In all three cases, the Laplace-space expressions can be numerically inverted into real space using the Stehfest (1970) algorithm, as is done in Butler et al. (2001) and Butler and Tsou (2003).

II. Hantush and Jacob (1955) Model

Hantush and Jacob (1955) present a solution for the drawdown produced by pumping in an infinite semiconfined aquifer (Figure 1). The specific storage of the confining unit is assumed negligible and the head in the overlying aquifer is assumed to be unaffected by pumping. The dimensionless forms of equations (1) – (2c) of Hantush and Jacob (1955) can be written as follows:

$$\frac{\partial^2 s_d}{\partial r_d^2} + \frac{1}{r_d}\frac{\partial s_d}{\partial r_d} - \frac{s_d}{B_d^2} = \frac{\partial s_d}{\partial t_d} \qquad (1)$$

$$s_d(r_d, 0) = 0,\ r_d \geq 0 \qquad (2a)$$

$$s_d(\infty, t_d) = 0,\ t_d \geq 0 \qquad (2b)$$

$$r_d \frac{\partial s_d}{\partial r_d} = -1,\ r_d \to 0 \qquad (2c)$$

where

B_d (dimensionless leakage parameter) = $(Kb'/K'b)^{0.5}$;
s_d (dimensionless drawdown) = $2\pi Ts/Q$;
r_d (dimensionless radial distance) = r/b;
t_d (dimensionless time)=$(Tt)/(b^2S)$;
b = aquifer thickness;
b' = thickness of confining layer;
K = hydraulic conductivity of aquifer;
K' = hydraulic conductivity of confining layer;
Q = pumping rate;
r = radial direction;
S = storativity of aquifer;
s = drawdown;
T = transmissivity of aquifer;
t = time.

The Laplace-space analogues of equations (1)-(2c) are:

$$\frac{\partial^2 \overline{s_d}}{\partial \mathrm{r}_\mathrm{d}^2} + \frac{1}{r_d}\frac{\partial \overline{s_d}}{\partial r_d} - \frac{\overline{s_d}}{B_d^2} = p\overline{s_d} \tag{3}$$

$$\overline{s_d}(\infty, p) = 0 \tag{4a}$$

$$r_d \frac{\partial \overline{s_d}}{\partial r_d} = -\frac{1}{p}, \; r_d \to 0 \tag{4b}$$

where

p = Laplace transform variable;

$\overline{s_d}$ = Laplace transform of s_d, overbar ($\bar{}$) will be used for the remainder of this report to designate Laplace-space variables.

Equation (3) can be rewritten as:

$$\frac{\partial^2 \overline{s_d}}{\partial r_d^2} + \frac{1}{r_d}\frac{\partial \overline{s_d}}{\partial r_d} - \overline{s_d}\left(\frac{1}{B_d^2} + p\right) = \frac{\partial^2 \overline{s_d}}{\partial r_d^2} + \frac{1}{r_d}\frac{\partial \overline{s_d}}{\partial r_d} - \overline{s_d}\lambda^2 = 0 \tag{5}$$

where

$$\lambda = \left(\frac{1}{B_d^2} + p\right)^{1/2}.$$

Equation (5) is the modified Bessel equation, so a solution can be written as:

$$\overline{s_d} = C_1 K_0(\lambda r_d) + C_2 I_0(\lambda r_d) \tag{6}$$

where

C_i = constant to be evaluated from boundary conditions;

I_0 = modified Bessel function of the first kind of order zero;

K_0 = modified Bessel function of the second kind of order zero.

From equation (4a) and the properties of modified Bessel functions, C_2 must equal zero and thus equation (6) can be rewritten as

$$\overline{s_d} = C_1 K_0(\lambda r_d) \quad (7).$$

Substitution of equation (7) into equation (4b) and utilizing the following relationships

$$K_0^{'}(\lambda r_d) = -\lambda K_1(\lambda r_d)$$

$$K_1(\lambda r_d) = \frac{1}{\lambda r_d},\ r_d \rightarrow 0$$

where

K_1 = modified Bessel function of the second kind of order one,

results in

$$C_1 = \frac{1}{p} \quad (8).$$

The Laplace-space expression for drawdown in a semiconfined aquifer can be obtained by substituting equation (8) into equation (7):

$$\overline{s_d} = \frac{K_0(\lambda r_d)}{p} \quad (9).$$

This is the expression given as equation (1) in Butler and Tsou (2003), and is the Laplace transform of equation (6) of Hantush and Jacob (1955).

Since the head in the overlying aquifer does not change, an expression for the leakage (q_z) into the pumped aquifer through an infinitely thin ring centered on the pumping well can be written in dimensional form as

$$q_z = (2\pi r dr)\frac{K'}{b'}s \quad (10a)$$

which can be written in dimensionless form as

$$q_{zd} = r_d dr_d \frac{s_d}{B_d^2} \quad (10b)$$

where

q_{zd} = dimensionless leakage = q_z/Q.

Taking the Laplace transform of equation (10b) results in

$$\overline{q_{zd}} = r_d dr_d \frac{\overline{s_d}}{B_d^2} \quad (11)$$

which is the same expression as equation (2) of Butler and Tsou (2003).

Substitution of equation (9) into equation (11) produces the following expression for pumping-induced leakage:

$$\overline{q_{zd}} = r_d dr_d \frac{K_0(\lambda r_d)}{pB_d^2} \quad (12).$$

An expression for the pumping-induced leakage over the entire aquifer, henceforth total leakage ($Q_{zd_{\inf_{HJ}}}$), can be obtained by integrating equation (12) over r_d from 0 to ∞:

$$\overline{Q_{zd_{\inf_{HJ}}}} = \frac{1}{pB_d^2} \int_0^\infty r_d K_0(\lambda r_d) dr_d \quad (13)$$

which is the same expression as equation (3) of Butler and Tsou (2003). Note that the hydraulic conductivity of the aquifer (K) will be much larger than that of the confining layer (K') in semiconfined aquifer systems, so values for B_d should usually fall between 1 and 100.

III. Hantush (1960) Model

Hantush (1960) presents a solution for the drawdown produced by pumping in an infinite semiconfined aquifer (Figure 1) that incorporates the specific storage of the confining unit (the head in the overlying aquifer is again assumed to be unaffected by pumping). The dimensionless forms of the governing equations and auxiliary conditions of Hantush (1960) can be written as follows:

$$\frac{\partial^2 s_d}{\partial r_d^2} + \frac{1}{r_d}\frac{\partial s_d}{\partial r_d} - \frac{1}{B_d^2}\frac{\partial s_{1_d}(r_d,1,t_d)}{\partial z_d} = \frac{\partial s_d}{\partial t_d} \quad (14)$$

$$\frac{\partial^2 s_{1_d}}{\partial z_d^2} = D_d \frac{\partial s_{1_d}}{\partial t_d} \quad (15)$$

$$s_d(r_d,0) = s_{1_d}(r_d, z_d, 0) = 0,\ r_d \geq 0,\ 0 \leq z_d \leq 1 \quad (16a)$$

$$s_d(\infty, t_d) = 0,\ t_d \geq 0 \quad (16b)$$

$$r_d \frac{\partial s_d}{\partial r_d} = -1, \; r_d \to 0 \qquad (16c)$$

$$s_d(r_d, t_d) = s_{1_d}(r_d, 1, t_d), \; t_d \geq 0 \qquad (16d)$$

$$s_{1_d}(r_d, 0, t_d) = 0, \; t_d \geq 0 \qquad (16e)$$

where

D_d=dimensionless diffusivity for confining layer = $\frac{S^{'}}{S} B_d^2$;

s_{1_d} = dimensionless drawdown in confining layer = $(2\pi T s_1)/Q$

z_d = dimensionless vertical distance in confining layer = z/b';

S' = storativity of the confining layer;

s_1 = drawdown in confining layer;

z = vertical distance in confining layer (= 0 at top of the confining layer).

The Laplace-space analogues of equations (14)-(16e) are:

$$\frac{\partial^2 \overline{s_d}}{\partial r_d^2} + \frac{1}{r_d} \frac{\partial \overline{s_d}}{\partial r_d} - \frac{1}{B_d^2} \frac{\partial \overline{s_{1_d}}(r_d, 1, p)}{\partial z_d} = p\overline{s_d} \qquad (17)$$

$$\frac{\partial^2 \overline{s_{1_d}}}{\partial z_d^2} = pD_d \overline{s_{1_d}} \qquad (18)$$

$$\overline{s_d}(\infty, p) = 0 \qquad (19a)$$

$$r_d \frac{\partial \overline{s_d}}{\partial r_d} = -\frac{1}{p}, \; r_d \to 0 \qquad (19b)$$

$$\overline{s_d}(r_d, p) = \overline{s_{1_d}}(r_d, 1, p) \qquad (19c)$$

$$\overline{s_{1_d}}(r_d, 0, p) = 0 \qquad (19d).$$

A solution to equation (18) can be written as:

$$\overline{s_{1_d}} = C_1 \cosh(\sqrt{pD_d}\, z_d) + C_2 \sinh(\sqrt{pD_d}\, z_d) \qquad (20)$$

where

cosh = hyperbolic cosine function = $\frac{1}{2}(e^x + e^{-x})$;

sinh = hyperbolic sine function = $\frac{1}{2}(e^x - e^{-x})$.

Substitution of equation (20) into equation (19d) and recognizing that cosh(0)≠0 yields

$$\overline{s_{1_d}} = C_2 \sinh(\sqrt{pD_d}\, z_d) \qquad (21).$$

Substitution of equation (21) into equation (19c) and solving for C_2 result in

$$C_2 = \frac{\overline{s_d}}{\sinh(\sqrt{pD_d})} \qquad (22).$$

Thus, equation (21) can be rewritten as

$$\overline{s_{1_d}} = \overline{s_d} \frac{\sinh(\sqrt{pD_d}\, z_d)}{\sinh(\sqrt{pD_d})} \qquad (23).$$

The derivative of $\overline{s_{1_d}}$ with respect to z_d evaluated at the boundary between the confining bed and the aquifer (z_d=1) is

$$\frac{\partial \overline{s_{1_d}}}{\partial z_d} = \overline{s_d} \sqrt{pD_d} \frac{\cosh(\sqrt{pD_d})}{\sinh(\sqrt{pD_d})} = \overline{s_d} \sqrt{pD_d} \coth(\sqrt{pD_d}),\ z_d = 1 \qquad (24)$$

where

coth = hyperbolic cotangent function.

Substitution of equation (24) into equation (17) yields

$$\frac{\partial^2 \overline{s_d}}{\partial r_d^2} + \frac{1}{r_d} \frac{\partial \overline{s_d}}{\partial r_d} - \frac{1}{B_d^2} \sqrt{pD_d} \coth(\sqrt{pD_d}) \overline{s_d} = p\overline{s_d} \qquad (25a)$$

which can be rearranged to

$$\frac{\partial^2 \overline{s_d}}{\partial r_d^2} + \frac{1}{r_d}\frac{\partial \overline{s_d}}{\partial r_d} - (p + \frac{1}{B_d^2}\sqrt{pD_d}\coth(\sqrt{pD_d}))\overline{s_d} = 0 \qquad (25b)$$

and finally

$$\frac{\partial^2 \overline{s_d}}{\partial r_d^2} + \frac{1}{r_d}\frac{\partial \overline{s_d}}{\partial r_d} - \gamma^2 \overline{s_d} = 0 \qquad (25c)$$

where

$$\gamma = (p + \frac{1}{B_d^2}\sqrt{pD_d}\coth(\sqrt{pD_d}))^{1/2}.$$

Equation (25c) is in a similar form to equation (5), so a solution can be found using the same approach. The solution can therefore be written as:

$$\overline{s_d} = \frac{K_0(\gamma r_d)}{p} \qquad (26)$$

which is the same as equation (40) of Hantush (1960) except for the definition of γ (Hantush provided an expression for the case of overlying and underlying confining layers).

An expression for the leakage into the aquifer through an infinitely thin ring centered on the pumping well can be written in dimensional form as

$$q_z = (2\pi r dr)K'\frac{\partial s_1}{\partial z}(z = b') \qquad (27a)$$

which can be written in dimensionless form as

$$q_{zd} = \frac{r_d dr_d}{B_d^2}\frac{\partial s_{1_d}}{\partial z_d}(z_d = 1) \qquad (27b).$$

Taking the Laplace transform of equation (27b) results in

$$\overline{q_{zd}} = \frac{r_d dr_d}{B_d^2}\frac{\partial \overline{s_{1_d}}}{\partial z_d}(z_d = 1) \qquad (28).$$

Substitution of equations (24) and (26) into equation (28) produces the following expression for pumping-induced leakage:

$$\overline{q_{zd}} = r_d dr_d \left(\frac{\sqrt{pD_d}\coth(\sqrt{pD_d})}{B_d^2} \right) \frac{K_0(\gamma r_d)}{p} = \frac{r_d dr_d}{\Gamma^2} \frac{K_0(\gamma r_d)}{p} \quad (29)$$

where

$$\Gamma = \left(\frac{B_d^2}{\sqrt{pD_d}\coth(\sqrt{pD_d})} \right)^{1/2}.$$

An expression for total leakage can be obtained by integrating equation (29) over r_d from 0 to ∞:

$$\overline{Q_{zd_{\inf_H}}} = \frac{1}{p\Gamma^2} \int_0^\infty r_d K_0(\gamma r_d) dr_d \quad (30)$$

which is the same expression as equation (4) of Butler and Tsou (2003).

This derivation was for the case of a constant-head aquifer overlying the confining layer. If the upper boundary of the confining layer is a no-flow boundary, equation (30) does not change but Γ is now defined as

$$\Gamma = \left(\frac{B_d^2}{\sqrt{pD_d}\tanh(\sqrt{pD_d})} \right)^{1/2}$$

where

tanh = hyperbolic tangent function.

Although more involved expressions are obtained for Γ if the configuration consists of both an overlying and underlying confining layer, equation (30) will still be the appropriate expression for total leakage. Note that Butler and Tsou (2003) use S_d and B_d, instead of combining the terms in the D_d notation. Since a porous confining layer should be more compressible than an aquifer, one would expect that S_d would usually be greater than one. In fractured settings, however, S_d may be considerably less than one.

IV. Butler et al. (2001) Model

Butler et al. (2001) present a solution for drawdown and stream depletion (total leakage) produced by pumping in an interconnected stream-aquifer system of infinite length in which the stream and the aquifer are separated by a thin streambed of low permeability (Figure 2). The streambed is of finite width, storage in the streambed is negligible, and the water level in the stream is unaffected by pumping. Details of the derivation of that solution are presented in Butler and Tsou (2000). The notation of

Butler and Tsou (2000) is modified here to be consistent with that used in the previous sections.

An expression for the leakage into the aquifer through an infinitely thin strip of the streambed oriented perpendicular to the stream in dimensional form is

$$q_z = \frac{K'}{b'}\left(\int_{-w}^{0} s\,dx\right)dy \qquad (31a)$$

which can be written in dimensionless form as

$$q_{zd} = \frac{1}{B_{d_{sa}}^2}\left(\int_{-1}^{0} s_d\,dx_d\right)dy_d \qquad (31b)$$

where

$B_{d_{sa}}$ = dimensionless leakage parameter for stream-aquifer systems = $(b'T/K'w^2)^{1/2}$;
s_d = dimensionless drawdown = sT/Q;
x_d = dimensionless distance perpendicular to stream = x/w;
y_d = dimensionless distance parallel to stream = y/w;
w = stream width;
x = distance perpendicular to stream;
y = distance parallel to stream.

Taking the Laplace transform of equation (31b) results in

$$\overline{q_{zd}} = \frac{1}{B_{d_{sa}}^2}\left(\int_{-1}^{0} \overline{s_d}\,dx_d\right)dy_d \qquad (32)$$

which is the same expression as equation (5) of Butler and Tsou (2003).

An expression for the total pumping-induced leakage through the streambed (stream depletion) can be obtained by integrating equation (32) in the y direction from -∞ to +∞:

$$\overline{Q_{zd_{\inf_{BZT}}}} = \frac{1}{B_{d_{sa}}^2}\int_{-\infty}^{\infty}\int_{-1}^{0} \overline{s_d}\,dx_d\,dy_d \qquad (33)$$

which is the same expression as equation (6) of Butler and Tsou (2003). Note that the hydraulic conductivity of the aquifer (K) will normally be larger than that of the streambed (K') and the stream will normally be much wider than the square root of the product of b and b', so values for B_d should usually fall between 0.01 and 100.

V. References

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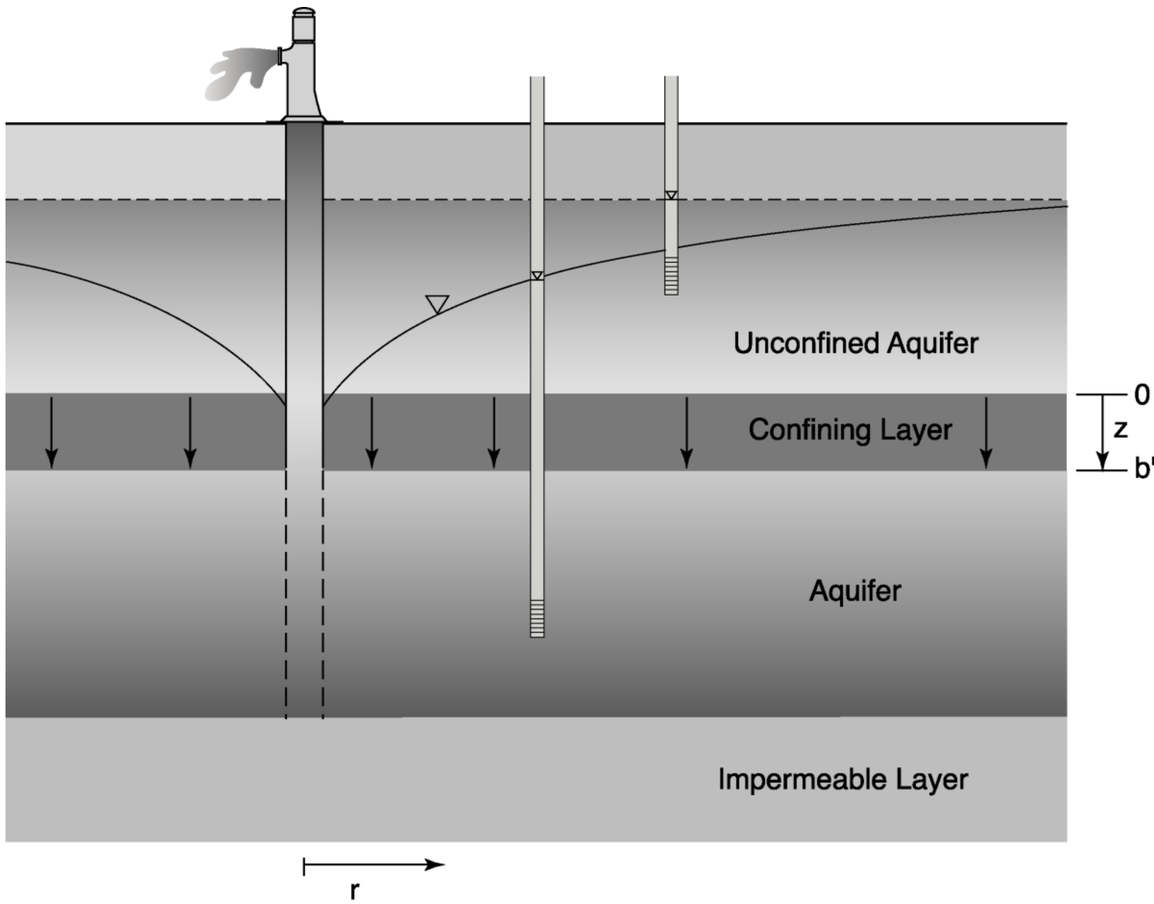

Figure 1 – Cross-sectional view of a hypothetical semiconfined aquifer and adjacent units. Pumping in the semiconfined aquifer induces vertical flow across the confining layer; head in the unconfined aquifer is unaffected by pumping.

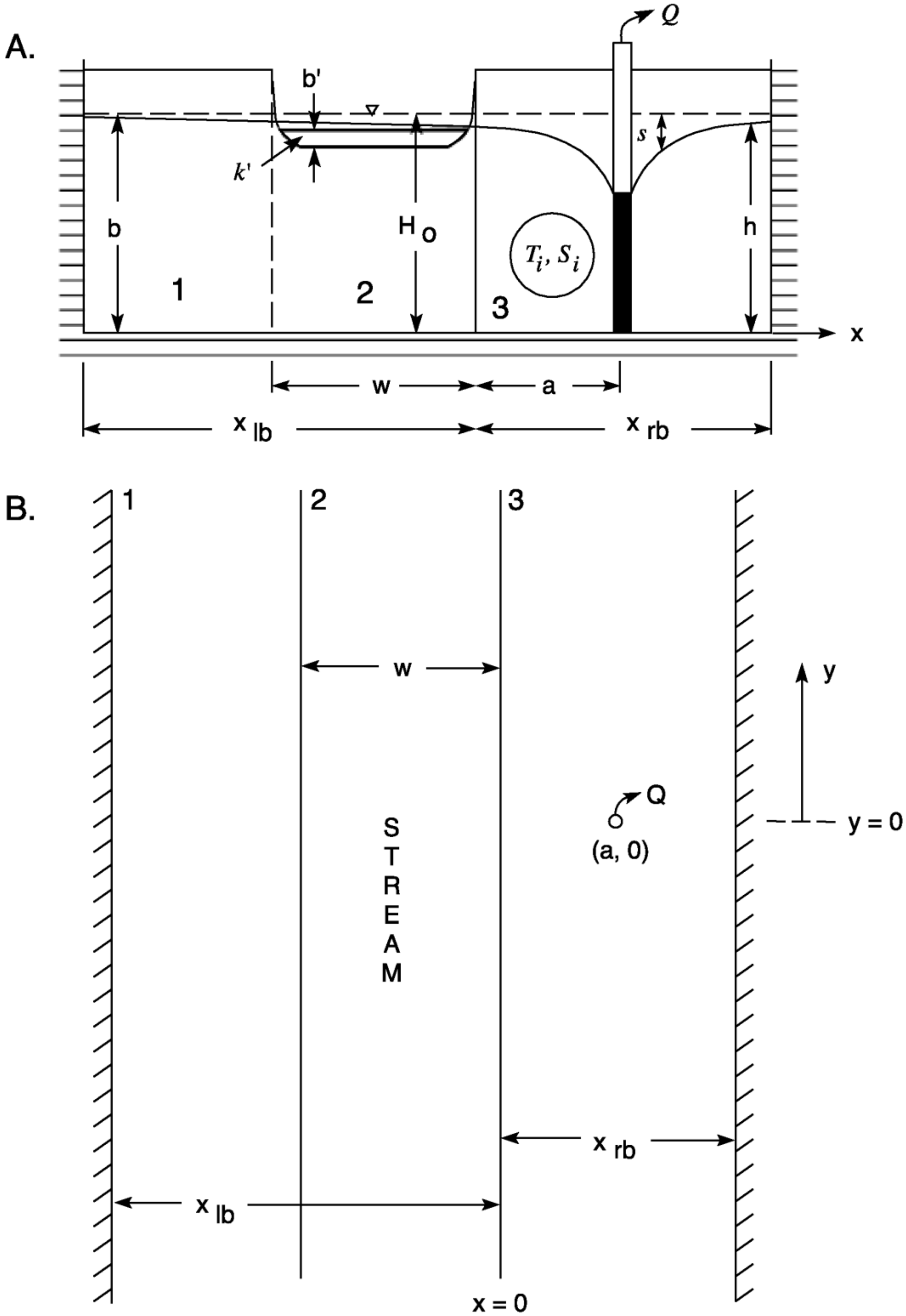

Figure 2 – Cross-sectional (a) and areal (b) views of a hypothetical stream-aquifer system (stream depletion in this configuration consists of vertical leakage across the low-permeability streambed; after Butler et al. (2001)).