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Adaptive Moving-mesh Modeling of Two-dimensional Groundwater Flow and Transport

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Abstract

A two-dimensional adaptive moving-mesh approach to ground-water flow and transport modeling is developed based on the moving-mesh partial differential equation method [1]. The approach is a generalization of the one-dimensional version studied previously by the authors [2, 3]. Advantages of adopting adaptive moving meshes in two dimensions are demonstrated in terms of numerical accuracy, efficiency, and reliability. Multi-level mesh movement and the error-controlled monitor function are employed to improve the computational performance. The approach is applied to the modeling of various two-dimensional ground-water flow and transport problems. Its effectiveness for handling sharp interfaces is demonstrated.

1 Introduction

Flow and transport in porous media often involves moving and sharp front or interface. Using uniform or fixed meshes to capture sharp front is expensive in terms of human and computer resources. A reasonably accurate solution can be obtained only using a dynamically adaptive-mesh method. The following example shows the difference in solution using fixed and moving meshes for a simple advection-dispersion equation (see equation 4) with $V = 1$ and $D = 10^{-5}$.

In Figure 1, the result is obtained with 5121 uniform and fixed nodes. The computation takes 3407 seconds of CPU time on a SGI Origin 2000 processor and has an error $||e|| = 2.74 \times 10^{-3}$. The second-order convergence is observed only for large numbers of nodes.

Figure 2 shows the result obtained using the moving-mesh method with 81 nodes. The computation takes 79.3 seconds of CPU time and the error is $||e|| = 4.72 \times 10^{-4}$. The second-order convergence is observed even with a small number of nodes.

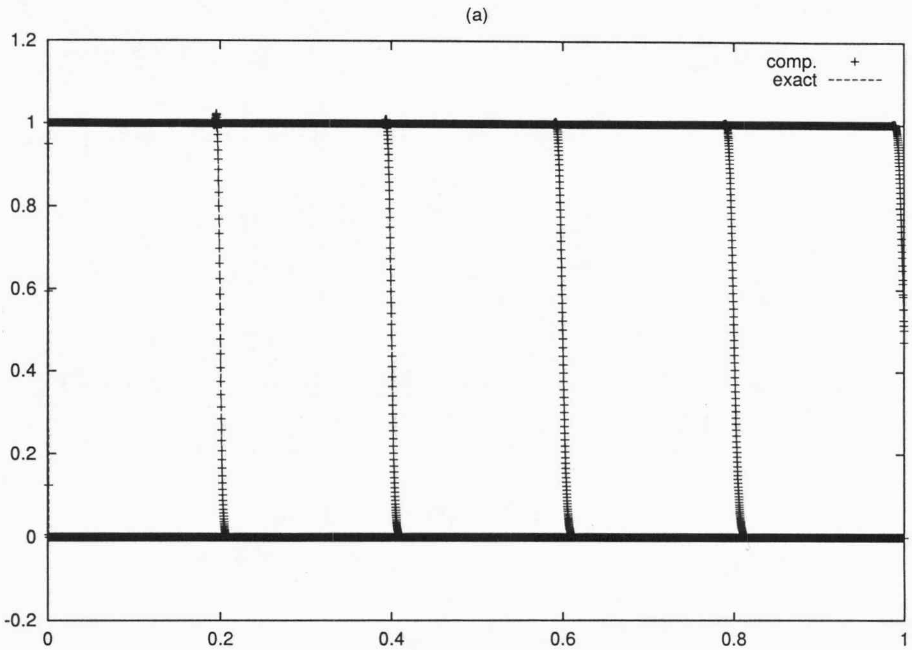

Figure 1: fixed mesh solution

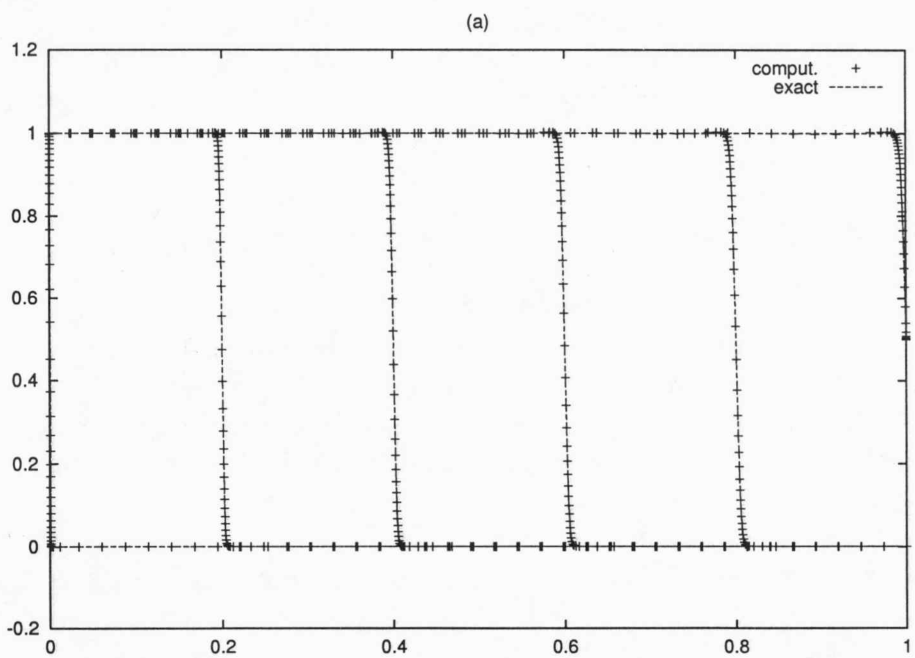

Figure 2: moving mesh solution

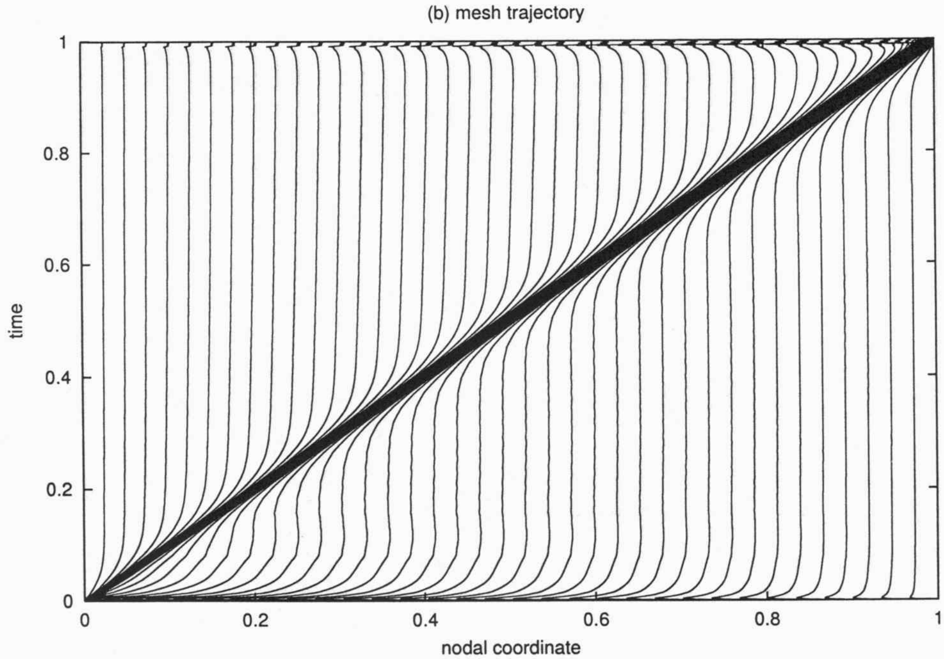

Figure 3: moving mesh trajectories

The monitor function, which connects the physical solution and mesh, is based on the interpolation error indicator. The mesh-concentration parameter for allocating the node distribution is taken as $\beta = 0.5$ or 50%. The solution at various time instants and the mesh trajectories (figure 3) clearly show the ability of the moving-mesh method to catch the moving front while maintaining sufficient mesh points in the front area.

To get this level of accuracy, more than 10,241 uniform nodes have to be used and more than a factor of 65 CPU time is needed. Scaling up from one dimension to two dimensions will cost much more!

2 A 2-D Adaptive Moving Mesh PDE Strategy

Adaptive moving meshes are generated as images of a reference mesh through an evolutionary coordinate transformation, $x = x(\xi, \eta, t), y = y(\xi, \eta, t)$, between the computational and physical domains. The coordinate transformation is defined as the solution of the moving-mesh partial differential equation or MMPDE,

$$\frac{\partial x}{\partial t} = \frac{\tau}{p}\left[a_{11}\frac{\partial^2 x}{\partial \xi^2} + 2a_{12}\frac{\partial^2 x}{\partial \xi \partial \eta} + a_{22}\frac{\partial^2 x}{\partial \eta^2} + b_1\frac{\partial x}{\partial \xi} + b_2\frac{\partial x}{\partial \eta}\right]$$
$$\frac{\partial y}{\partial t} = \frac{\tau}{p}\left[a_{11}\frac{\partial^2 y}{\partial \xi^2} + 2a_{12}\frac{\partial^2 y}{\partial \xi \partial \eta} + a_{22}\frac{\partial^2 y}{\partial \eta^2} + b_1\frac{\partial y}{\partial \xi} + b_2\frac{\partial y}{\partial \eta}\right]$$

where $p = \sqrt{a_{11}^2 + a_{22}^2 + b_1^2 + b_2^2}$ and $\tau > 0$ is the user-defined parameter used for adjusting the time scale of mesh movement. The coefficients are defined by

$$a_{11} = \boldsymbol{a} \cdot G^{-1} \cdot \boldsymbol{a}, \quad a_{12} = \boldsymbol{a} \cdot G^{-1} \cdot \boldsymbol{b}, \quad a_{22} = \boldsymbol{b} \cdot G^{-1} \cdot \boldsymbol{b},$$
$$b_1 = -\boldsymbol{a} \cdot \left(\frac{\partial G^{-1}}{\partial \xi} \boldsymbol{a} + \frac{\partial G^{-1}}{\partial \eta} \boldsymbol{b} \right), \quad b_2 = -\boldsymbol{b} \cdot \left(\frac{\partial G^{-1}}{\partial \xi} \boldsymbol{a} + \frac{\partial G^{-1}}{\partial \eta} \boldsymbol{b} \right).$$
$$J = x_\xi y_\eta - x_\eta y_\xi, \quad \boldsymbol{a} = \frac{1}{J} \begin{bmatrix} y_\eta \\ -x_\eta \end{bmatrix}, \quad \boldsymbol{b} = \frac{1}{J} \begin{bmatrix} -y_\xi \\ x_\xi \end{bmatrix}$$

and G, the so-called monitor function, is a two-by-two symmetric positive definite matrix.

The key to the success of the above strategy is to define a proper monitior function G, which connects the coordinate transformation to the physical solution. For example, we can define it based on solution gradients, i.e.

$$\mathbf{G} = \mathbf{I} + \sum_i w_i \nabla u_i \cdot \nabla u_i{}^T, \tag{1}$$

where w_i's are weights with $\sum_i w_i = 1$, $\mathbf{I}$ the identity matrix, and u_i's are the physical solutions. The monitor function can also be defined based on error indicators.

The coordinate transformation and the physical solution can be obtained by either simultaneously or alternately solving the MMPDE and the physical PDE [1].

Alternating Solution:

$$(x^n, u^n) \to (MMPDESolver) \to (x^{n+1}, u^n) \to (PDESolver) \to (x^{n+1}, u^{n+1}) \tag{2}$$

Simultaneous Solution:

$$(x^n, u^n) \to (MMPDESolver + PDESolver) \to (x^{n+1}, u^{n+1}) \tag{3}$$

where

u^n – the nth step physical solution;

x^n – the nth step mesh location.

3 Applications

3.1 Transport from single source in a heterogeneous aquifer

The first example shows solute transport from a single source in the west or left boundary in a heterogeneous aquifer. The governing equation is advection-dispersion equation

$$\frac{\partial C}{\partial t} = \nabla \cdot (\mathbf{D} \nabla C) - \mathbf{v} \nabla C \tag{4}$$

where

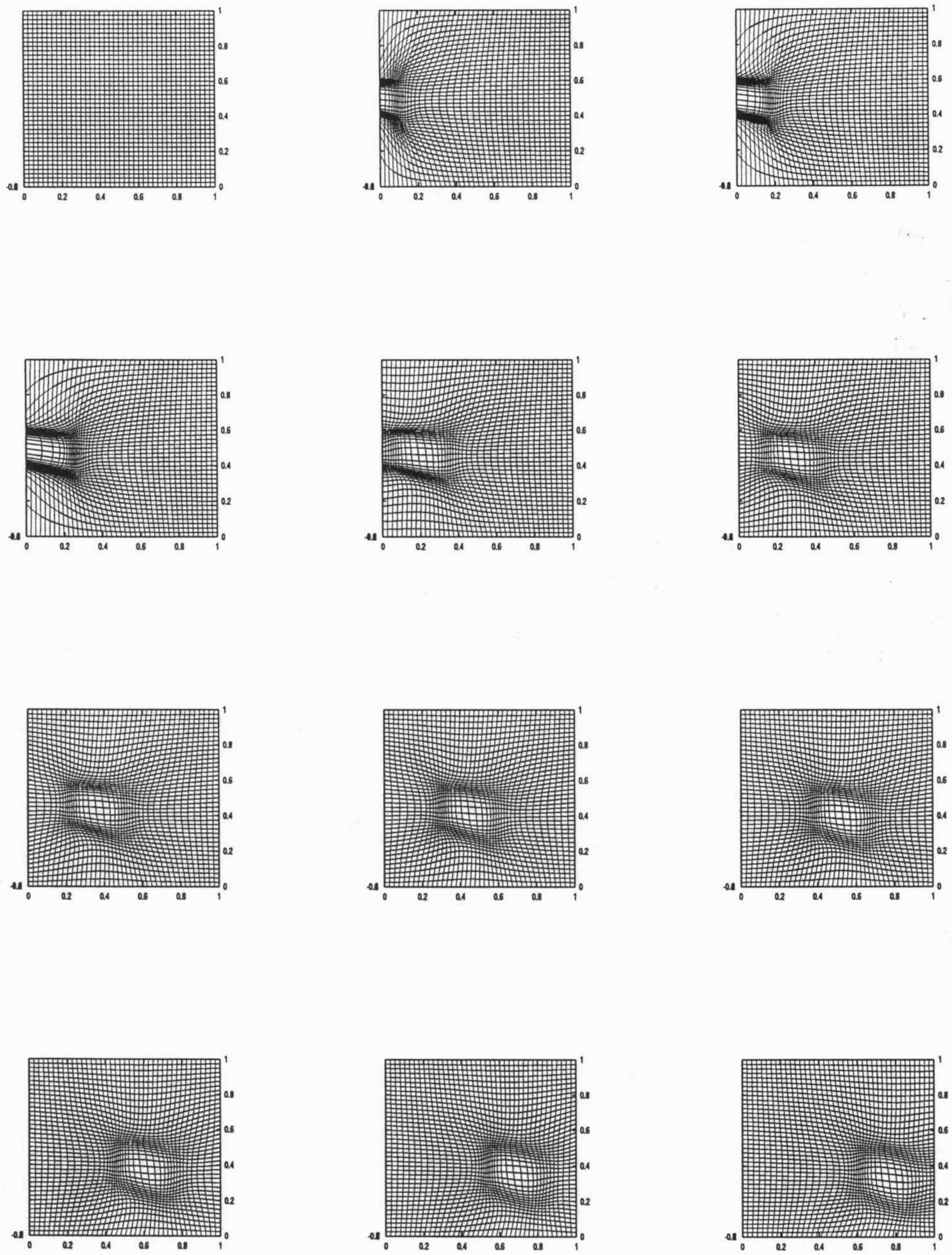

Figure 4: moving mesh for example 1

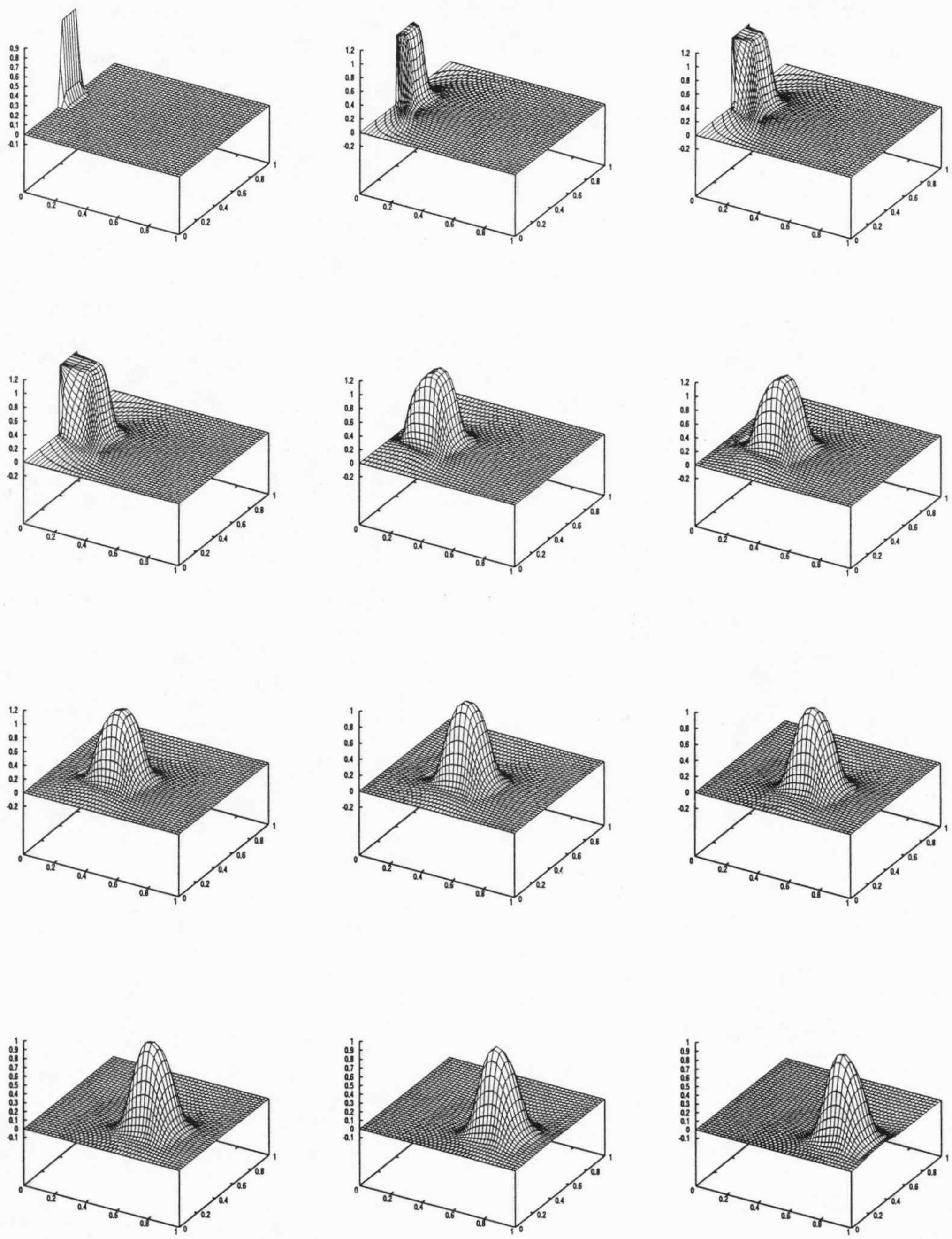

Figure 5: solution for example 1

C – Concentration;

D – Dispersion Coefficient, a spatial function due to heterogeneity;

v – Velocity, a spatial function due to heterogeneity.

The boundary conditions for south, north, and east are flux free and the initial condition is zero everywhere. The numerical solution is presented in Figure 4 and 5 for both mesh evolution in two-dimensional view and concentration variation in three-dimensional view. Both mesh and concentration are moving southeast due to the non-uniform property of the aquifer. Mesh movment follows the gradient.

3.2 Transport from multiple sources in a heterogeneous aquifer

The second example demonstrates solute transport from multiple sources, one in the west boundary and two within the domain, which appear at different time intervals in a heterogeneous aquifer. The governing equation is similar to the equation in the previous subsection, but some source and sink terms are added into the equation. Boundary and initial conditions are similar to those in the above subsection.

The numerical solutions are presented in Figure 6 and 7 for both mesh evolution in two-dimensional view and concentration variation in three-dimensional view. Both mesh and concentration around three peaks are moving southeast due to the non-uniform property of the aquifer. The mesh adapts well to the concentration variation. Emerging of new peaks is captured automatically.

3.3 Sharp front movement along the diagonal direction

The last example illustrates a sharp front moving along the diagonal direction. The governing equation is Burger's equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u) - u\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial y} \tag{5}$$

where

u – Velocity;

D – Dispersion Coefficient, $D = 0.005$;

The exact solution for this problem is given by

$$u = \frac{1}{1 + exp(\frac{x+y-t}{2D})} \tag{6}$$

which indicates u is constant along $x = -y$ or moving at an angle of 45°. The initial and boundary conditions for numerical modeling are specified by the exact solution.

The numerical solutions are presented in Figure 8 and 9 for both mesh evolution in two-dimensional view and velocity in three-dimensional view. Sharp front movement is well captured with the moving-mesh method.

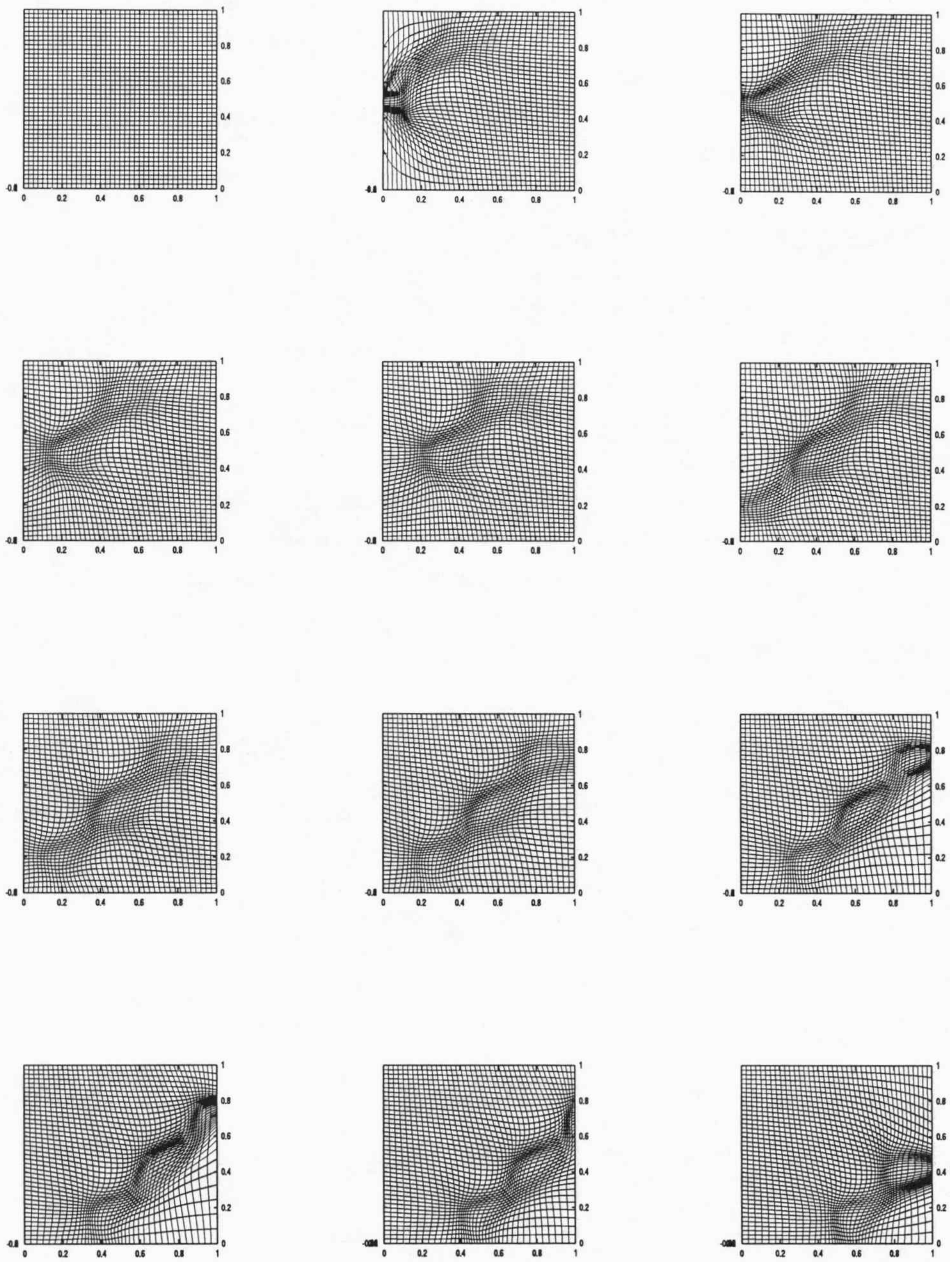

Figure 6: moving mesh for example 2

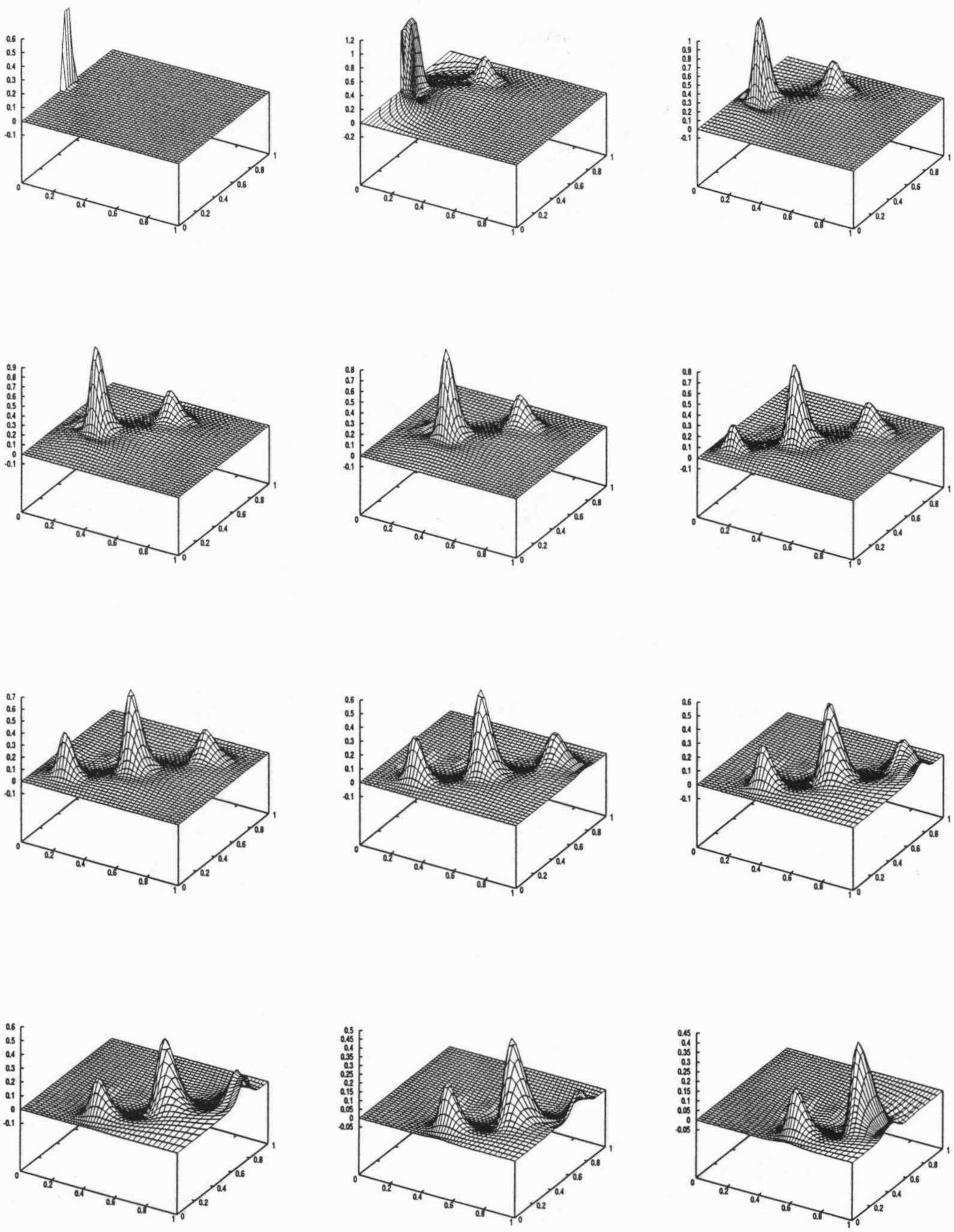

Figure 7: solution for example 2

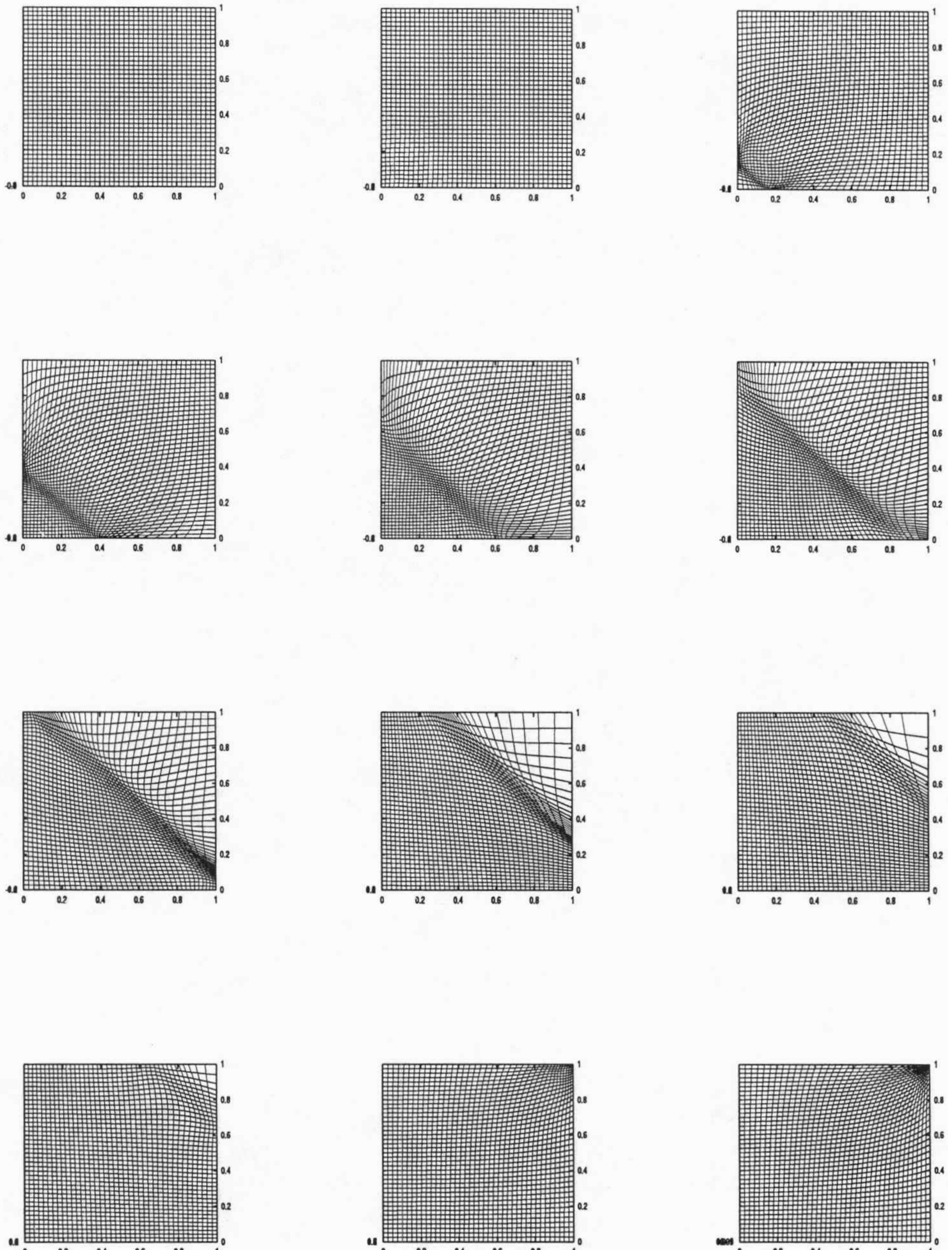

Figure 8: moving mesh for example 3

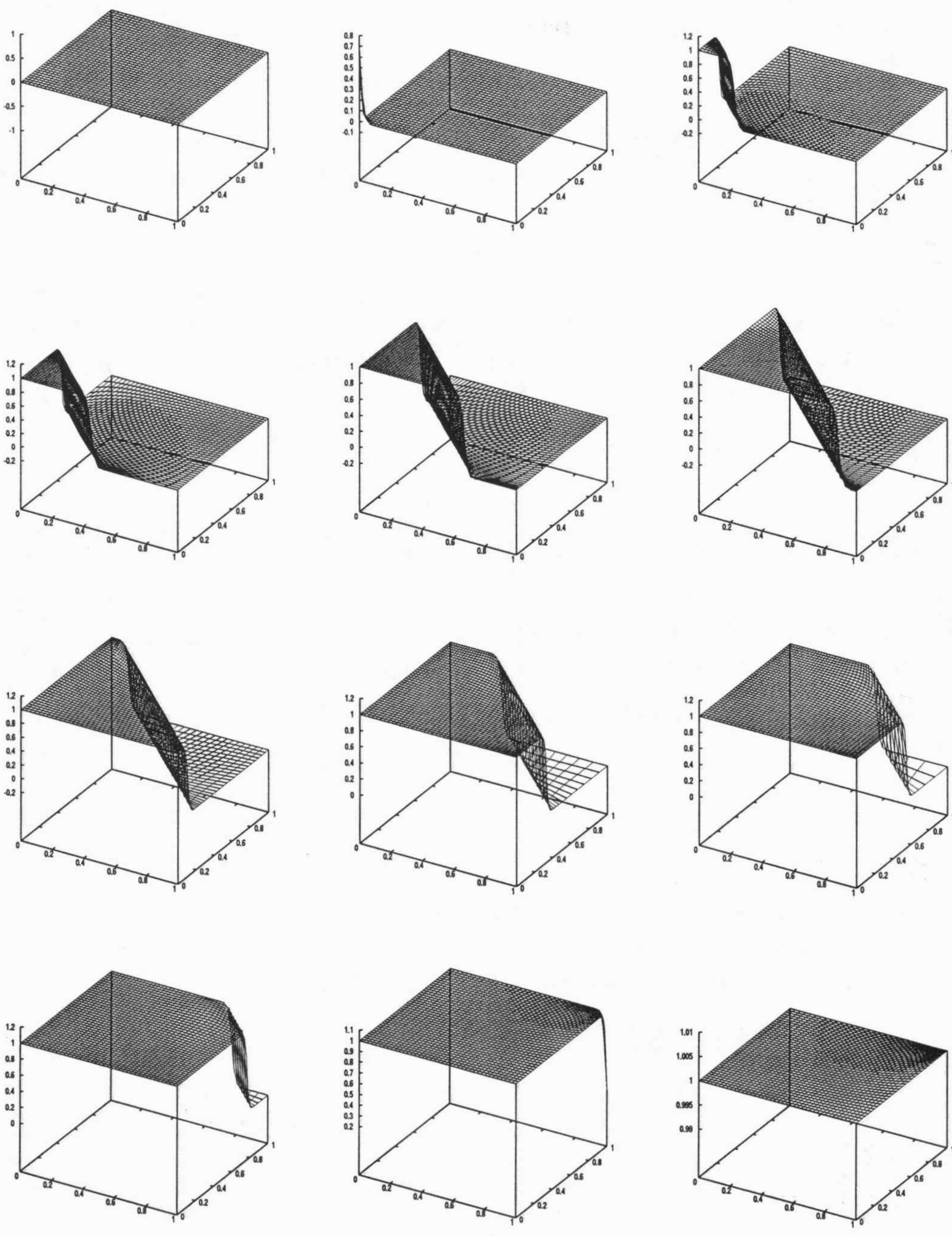

Figure 9: solution for example 3

4 Final Remarks

The application of the moving-mesh method based on moving-mesh PDES (MMPDES) has been extended to two dimensions. The implementation strategies, including the construction of the monitor function based on the interpolation error indicator, controlling of mesh concentration, and multi-level mesh movement, have been undertaken. The advection-dispersion equation and the Burger's equation are used to demonstrate the effectiveness of these strategies. Of particular interest, the strategy of controlling the mesh concentration and the two-layer mesh-movement strategy improve significantly the efficiency of the moving-mesh method.

The MMPDE approach provides an efficient means for capturing the evolution of sharp interfaces or fronts in ground-water systems. The advantages of MMPDE include efficiency smoothness, orthogonality, and non-singularity for meshing . Further research will focus on the application to three-dimensional ground-water systems.

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References

[1] W. Huang, Practical aspects of formulation and solution of moving mesh partial differential equations, Mathematics Research Report 99-11-02, Department of Mathematics, The University of Kansas, Lawrence, KS 66049.

[2] W. Huang, L. Zheng, & X. Zhan, Adaptive Moving Mesh Methods for Simulating Groundwater Problems with Sharp Moving Fronts, submitted to Int. J. Numerical Methods in Engineering, 2000.

[3] L. Zheng, X. Zhan, & W. Huang, Moving Mesh PDE Approach to Groundwater Flow and Transport Modeling (abstract & poster), AGU Spring 2000.