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Slug Tests With Observation Wells:
Extension of Hyder et al. (1994) Solution to Case
of Head in an Observation Well

by

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I. Introduction

In this report, the mathematical derivations of the extensions of the Hyder et al. (1994) solution to the case of head in a well other than the test well are presented. For the sake of generality, the solutions are obtained in a dimensionless form. Software implementing these solutions is given in Liu and Butler (1995). See Hyder et al. (1994) for notation definitions that are not given in this report.

II. Confined Aquifer Solution

Equations (1)-(9) of Hyder et al. (1994) describe the flow conditions of interest here. To work with the most general form of the solution, this derivation is performed using dimensionless forms of these equations. The dimensionless analogues of (1)-(9) of Hyder et al. (1994) are as follows:

$$\frac{\partial^2 \phi_i}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \phi_i}{\partial \xi} + \psi_i^2 \frac{\partial^2 \phi_i}{\partial \eta^2} = R_i \frac{\partial \phi_i}{\partial \tau} \qquad (1)$$

$$\phi_i(\xi, \eta, 0) = 0, \ \xi \geq 1, \ 0 \leq \eta \leq \beta \qquad (2)$$

$$\Phi(0) = 1 \tag{3}$$

$$\phi_2(\infty,\eta,\tau) = 0, \ \tau>0, \ 0<\eta<\beta \tag{4}$$

$$\frac{\partial \phi_i(\xi,0,\tau)}{\partial \eta} = \frac{\partial \phi_i(\xi,\beta,\tau)}{\partial \eta} = 0, \ \xi \geq 1, \ \tau>0 \tag{5}$$

$$\int_{\zeta}^{\zeta+1} \phi_1(1,\eta,\tau)\,d\eta = \Phi(\tau), \ \tau>0 \tag{6}$$

$$\frac{\partial \phi_1(1,\eta,\tau)}{\partial \xi} = \frac{\gamma}{2}\frac{d\Phi(\tau)}{d\tau}\square(\eta), \ \tau>0 \tag{7}$$

$$\phi_1(\xi_{sk},\eta,\tau) = \phi_2(\xi_{sk},\eta,\tau), \ 0<\eta<\beta, \ \tau>0 \tag{8}$$

$$\frac{\partial \phi_1(\xi_{sk},\eta,\tau)}{\partial \xi} = \gamma\frac{\partial \phi_2(\xi_{sk},\eta,\tau)}{\partial \xi}, \ 0<\eta<\beta, \ \tau>0 \tag{9}$$

where

$\phi_i = h_i/H_0$;

$\xi = r/r_w$;

$\eta = z/b$;

$\tau = (tbK_{r2})/(r_c^2)$;

$\psi_i = (A_i/a^2)^{.5}$;

$A_i = K_{zi}/K_{ri}$;

$a = b/r_w$;

$R_i = \gamma\alpha/2\lambda$, $i = 1$,

$= \alpha/2$, $i = 2$;

$\lambda = S_{s2}/S_{s1}$;

$\beta = B/b$;

Φ = head in the stressed well = H/H_0;

$\gamma = K_{r2}/K_{r1}$;

$\alpha = (2r_w^2 bS_{s2})/r_c^2$;

$\square(\eta)$ = boxcar function = 0, $\eta < \varsigma$, $\eta > \varsigma + 1$,

= 1, elsewhere;

$\varsigma = d/b$;

$\xi_{sk} = r_{sk}/r_w$.

A solution can be obtained for (1)-(9) through the use of integral transforms (Churchill, 1972). A Laplace transform in time followed by a finite Fourier cosine transform in the η direction produce a Fourier-Laplace space analogue to (1) of the following form:

$$\frac{\partial^2\overline{\Phi_i}}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\overline{\Phi_i}}{\partial\xi} - (\psi_i^2\omega^2 + R_i p)\overline{\Phi_i} = 0 \qquad (10)$$

where

$\overline{\Phi_i}$ = the Fourier-Laplace transform of ϕ_i, $f(\xi,\omega,p)$;

ω = the Fourier-transform variable = $(n\pi)/\beta$, n=0,1,2,..;

p = the Laplace-transform variable.

The Fourier-Laplace space solution to (10) is quite straightforward, as (10) is simply a form of the modified Bessel equation (Haberman, 1987). A solution can therefore be proposed in the form:

$$\overline{\Phi_i} = C_i K_0(\nu_i \xi) + D_i I_0(\nu_i \xi) \tag{11}$$

where

$\nu_i = (\psi_i^2 \omega^2 + R_i p)^{.5}$;

C_i, D_i = constants;

K_i = modified Bessel function of the second kind of order i;

I_i = modified Bessel function of the first kind of order i.

Using the transform-space analogues of auxiliary conditions (4) and (6)-(9), the constants in (11) can be evaluated. The transform-space expression for head at an arbitrary radial distance can be written as:

$$\overline{\Phi_i}(\xi,\omega,p) = \frac{\gamma}{2}[1-p\Phi(p)]F_c(\omega)f_{i+1} \tag{12}$$

where

$\Phi(p)$ = Laplace transform of $\Phi(t)$, the nondimensional form of $H(t)$,

$$= \frac{\frac{\gamma}{2}\Omega}{[1 + \frac{\gamma}{2}p\Omega]}$$

$$\Omega = \int_{\varsigma}^{\varsigma+1} (F_c^{-1}(F_c(\omega) f_1))\, d\eta;$$

F_c^{-1} = inverse finite Fourier cosine transform;

$F_c(\omega)$ = finite Fourier cosine transform of □(z)

$$= \frac{2}{\omega}\sin(\frac{\omega}{2})\cos(\frac{\omega(1+2\varsigma)}{2}),\ \omega=n\pi/\beta,\ n=1,2,3...,$$

$$= 1,\ \omega=0;$$

$$f_1 = \frac{[\Delta_2 K_0(\nu_1) - \Delta_1 I_0(\nu_1)]}{\nu_1[\Delta_2 K_1(\nu_1) + \Delta_1 I_1(\nu_1)]};$$

$$f_2 = \frac{[\Delta_2 K_0(\nu_1\xi) - \Delta_1 I_0(\nu_1\xi)]}{\nu_1[\Delta_2 K_1(\nu_1) + \Delta_1 I_1(\nu_1)]};$$

$$f_3 = \frac{[\Delta_2 K_0(\nu_1\xi_{sk}) - \Delta_1 I_0(\nu_1\xi_{sk})]K_0(\nu_2\xi)}{K_0(\nu_2\xi_{sk})\nu_1[\Delta_2 K_1(\nu_1) + \Delta_1 I_1(\nu_1)]};$$

$$\Delta_1 = K_0(\nu_1\xi_{sk})K_1(\nu_2\xi_{sk}) - [\frac{N}{\gamma}]K_0(\nu_2\xi_{sk})K_1(\nu_1\xi_{sk});$$

$$\Delta_2 = I_0(\nu_1\xi_{sk})K_1(\nu_2\xi_{sk}) + [\frac{N}{\gamma}]K_0(\nu_2\xi_{sk})I_1(\nu_1\xi_{sk});$$

$$N = \nu_1/\nu_2.$$

The application of an inverse finite Fourier cosine transform to (12) produces the following Laplace-space expression:

$$\overline{\Phi_i}(\xi,\eta,p) = \frac{\gamma/2}{(1+(\gamma/2)p\Omega)}\Omega_i \qquad (13)$$

where

$$\Omega_i = F_c^{-1}[F_c(\omega)f_i]$$

$$= \frac{f_i(n=0)}{\beta} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{f_i(n)}{n}\sin(\frac{n\pi}{2\beta})\cos(\frac{n\pi(1+2\varsigma)}{2\beta})\cos(\frac{n\pi\eta}{\beta})$$

Equation (13) was developed for the case of an observation well that can be represented as a point (ξ,η). Although this approach might be reasonable for a well with an extremely short screen, it is clearly not appropriate in the general case. A more general approach would be to define the observation-well head as a vertical average:

$$\phi_{OW_i} = \frac{1}{\Delta\zeta}\int_{\zeta_1}^{\zeta_2}\phi_i(\xi,\eta,\tau)\,d\eta \qquad (14)$$

where

$$\Delta\zeta = \zeta_2 - \zeta_1;$$

$\zeta_i = z_i/b;$

z_i = distance from top of formation to top (i=1) or bottom (i=2) of observation-well screen.

Employing the transform-space analogue to (14), the application of an inverse finite Fourier cosine transform to (12) produces a Laplace-space expression equivalent to (13), except that the Ω_i expression is defined as

$$\Omega_i = \frac{1}{\Delta\zeta}\int_{\zeta_1}^{\zeta_2}\left(\frac{f_i(n=0)}{\beta} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{f_i(n)}{n}\sin\left(\frac{n\pi}{2\beta}\right)\cos\left(\frac{n\pi(1+2\varsigma)}{2\beta}\right)\cos\left(\frac{n\pi\eta}{\beta}\right)\right)d\eta$$

$$= \frac{f_i(n=0)}{\beta} +$$

$$\frac{4}{\pi\Delta\zeta}\sum_{n=1}^{\infty}\frac{f_i(n)}{n}\sin(\frac{n\pi}{2\beta})\cos(\frac{n\pi(1+2\varsigma)}{2\beta})\int_{\zeta_1}^{\zeta_2}\cos(\frac{n\pi\eta}{\beta})d\eta \qquad (15)$$

The integer in (15) can be rewritten as

$$\begin{aligned}\int_{\zeta_1}^{\zeta_2}\cos(\frac{n\pi\eta}{\beta})d\eta &= \frac{\beta}{n\pi}[\sin(\frac{n\pi\zeta_2}{\beta}) - \sin(\frac{n\pi\zeta_1}{\beta})] \\ &= \frac{2\beta}{n\pi}(\cos(\frac{n\pi(\zeta_1+\zeta_2)}{2\beta})\sin(\frac{n\pi\Delta\zeta}{2\beta}))\end{aligned} \qquad (16)$$

Substituting (16) into (15) produces the following expression for Ω_i:

$$= \frac{f_i(n=0)}{\beta} +$$

$$\frac{8\beta}{\Delta\zeta\pi^2}\sum_{n=1}^{\infty}\frac{f_i(n)}{n^2}\sin(\frac{n\pi}{2\beta})\sin(\frac{n\pi\Delta\zeta}{2\beta}) \qquad (17)$$

$$*\cos(\frac{n\pi(1+2\varsigma)}{2\beta})\cos(\frac{n\pi(\zeta_1+\zeta_2)}{2\beta})$$

Equation (13) with the Ω_i definition given in (17) is employed to calculate observation-well heads in Liu and Butler (1995). Note that equation (13) is based on the assumption that the observation well has been packed off (i.e. well-bore storage effects can be neglected). Well-bore storage at the observation well could be included by utilizing an approach similar to that of Tongpenyai and Raghavan (1981) and Novakowski (1989).

In the homogeneous (no skin) case,

$$\Omega_2 = \Omega_3 = \frac{f_i^*(n{=}0)}{\beta} +$$
$$\frac{8\beta}{\Delta\zeta\pi^2}\sum_{n=1}^{\infty}\frac{f_i^*(n)}{n^2}\sin\left(\frac{n\pi}{2\beta}\right)\sin\left(\frac{n\pi\Delta\zeta}{2\beta}\right) \qquad (18)$$
$$*\cos\left(\frac{n\pi(1+2\varsigma)}{2\beta}\right)\cos\left(\frac{n\pi(\zeta_1+\zeta_2)}{2\beta}\right)$$

where

$f_i* = K_0(\nu\xi)/(\nu K_1(\nu))$

Note also that f_1 in the Ω term of (12) becomes $K_0(\nu)/(\nu K_1(\nu))$.

III. Unconfined Aquifer Solution

For the unconfined case, (5) is replaced by the dimensionless analogues of (11) and (12) of Hyder et al. (1994):

$$\phi_i(\xi,0,\tau) = 0, \quad \xi\geq 1, \quad \tau>0 \qquad (19)$$

$$\frac{\partial\phi_i(\xi,\beta,\tau)}{\partial\eta} = 0, \quad \xi\geq 1, \quad \tau>0 \qquad (20)$$

A solution for (1)-(4), (6)-(9), and (19)-(20) is obtained using the same approach as in the confined case. The Fourier-Laplace expression for head at an arbitrary radial distance in the unconfined case can be written as:

$$\overline{\phi_{i_{uc}}}(\xi,\omega^*,p) = \frac{\gamma}{2}[1-p\Phi_{uc}(p)]F_s(\omega^*)f_{i+1} \tag{21}$$

where

$\overline{\phi_{i_{uc}}}$ = the Fourier-Laplace transform of $\phi_{i_{uc}}$, the nondimensional form of h_i for the unconfined case;

$\Phi_{uc}(p)$ = the Laplace transform of the nondimensional form of H(t) for the unconfined case,

$$= \frac{\frac{\gamma}{2}\Omega^*}{[1 + \frac{\gamma}{2}p\Omega^*]}$$

$$\Omega^* = \int_{\varsigma}^{\varsigma+1} (F_s^{-1}(F_s(\omega^*)f_1))d\eta;$$

F_s^{-1} = inverse modified finite Fourier sine transform;

$F_s(\omega^*)$ = modified finite Fourier sine transform of □(z)

$$= \frac{2}{\omega^*}\sin(\frac{\omega^*(2\varsigma+1)}{2})\sin(\frac{\omega^*}{2});$$

ω^* =Fourier transform variable for the modified sine transform

$= (n\pi)/2\beta$, n=1,3,5,...

The application of an inverse modified finite Fourier sine transform to (21) for the case of an observation well that can be represented as a point (ξ,η) produces the following expression:

$$\overline{\Phi_{i_{uc}}}(\xi,\eta,p) = \frac{\gamma/2}{(1+(\gamma/2)p\Omega^{*})}\Omega_i^{*} \tag{22}$$

where

$$\Omega_i^{*} = F_s^{-1}[F_s(\omega^{*})f_i]$$
$$= \frac{4}{\pi}\sum_{n=1}^{\infty}[1+(-1)^{n+1}]\frac{f_i(n)}{n}$$
$$*\sin(\frac{n\pi}{4\beta})\sin(\frac{n\pi(1+2\varsigma)}{4\beta})\sin(\frac{n\pi\eta}{2\beta})$$

Employing the transform-space analogue to (14), the application of an inverse modified Fourier sine transform to (21) produces a Laplace-space expression equivalent to (22), except that the Ω_i* expression is defined as

$$\Omega_i^{*} = \frac{4}{\pi\Delta\zeta}\sum_{n=1}^{\infty}[1+(-1)^{n+1}]\frac{f_i(n)}{n}$$
$$*\sin(\frac{n\pi}{4\beta})\sin(\frac{n\pi(1+2\varsigma)}{4\beta})\int_{\zeta_1}^{\zeta_2}\sin(\frac{n\pi\eta}{2\beta})d\eta \tag{23}$$

The integer in (23) can be rewritten as

$$\int_{\zeta_1}^{\zeta_2} \sin(\frac{n\pi\eta}{2\beta})d\eta = \frac{2\beta}{n\pi}[\cos(\frac{n\pi\zeta_1}{2\beta}) - \cos(\frac{n\pi\zeta_2}{2\beta})]$$
$$= \frac{4\beta}{n\pi}(\sin(\frac{n\pi(\zeta_1+\zeta_2)}{4\beta})\sin(\frac{n\pi\Delta\zeta}{4\beta})) \quad (24)$$

Substituting (24) into (23) produces the following expression for Ω_i*:

$$\Omega_i^* = \frac{16\beta}{\pi^2\Delta\zeta}\sum_{n=1}^{\infty}[1+(-1)^{n+1}]\frac{f_i(n)}{n^2}$$
$$*\sin(\frac{n\pi}{4\beta})\sin(\frac{n\pi(1+2\varsigma)}{4\beta})\sin(\frac{n\pi(\zeta_1+\zeta_2)}{4\beta})\sin(\frac{n\pi\Delta\zeta}{4\beta}) \quad (25)$$

In Liu and Butler (1995), equation (22) with the Ω_i* definition given in (25) is employed to calculate observation-well heads for the unconfined case. Note that in the homogeneous (no skin) unconfined case, the f_i terms are modified as in the confined case.

IV. References

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