Flow to a Well in Radially Nonuniform Media

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December, 1989

Kansas Geological Survey
Open-File Report 89-36

ABSTRACT

A semianalytical solution for flow in response to pumping at a central well has been derived using the Laplace-transformation technique for the case of an aquifer whose properties vary arbitrarily in the radial direction. The complexity of the solution in Laplace space, however, makes it difficult to gain much insight into the nature of flow to a pumping well in radially nonuniform media. An approximate analytical solution is obtained using the Boltzmann transformation to maximize physical understanding of flow to a well in this configuration. The form of the approximate solution allows considerable insight to be obtained concerning the nature of observation-well drawdown during a pumping test. Pumping-induced drawdown at an observation well can be shown to consist of a component dependent on near-well properties and a component dependent on the properties of more distant areas. The aquifer volume controlling this second component of drawdown increases considerably with duration of pumpage, making it difficult to characterize moderate-and high-frequency nonuniformities at a distance from the pumping well. The Thiem equation, when applied to a radially nonuniform aquifer, yields a transmissivity that is a distance-weighted harmonic average of the transmissivity between two observation wells. Drawdown behavior during the period when the front of the cone of depression is passing across an interface between materials of differing properties is a function of the relative size and direction of the variations in flow properties across the boundary. In most cases, it is difficult to recognize changes in storativity across a boundary from simple inspection of a semilog drawdown plot. Although the configuration examined here is an admitted simplification of reality, the insight developed from this work should have considerable value for the case of pumping tests in more complex media.

INTRODUCTION

Analytical solutions for drawdown in response to pumping at a central well are the mainstays of conventional approaches for pumping-test analysis. Despite the great amount of effort that has been expended in the development of these solutions, some basic questions concerning the nature of pumping-induced drawdown in nonuniform aquifers remain unanswered. Recently, Butler and McElwee (in press) have employed numerically derived sensitivity coefficients to provide some insight into pumping-induced drawdown in radially nonuniform aquifers. Their conclusions, however, are incomplete due to the inherent limitations of numerical approaches. More insight into the factors controlling drawdown during a pumping test in a nonuniform aquifer can be obtained through the use of an analytical solution that incorporates variations in flow properties. The derivation of such a solution is a primary focus of this work.

An analytical solution is outlined here for the case of arbitrary variations in flow properties in the radial direction. Mathematical complexities make the derivation of a more general analytical solution incorporating variations in all directions considerably more difficult. Butler (1986), however, has demonstrated in a series of numerical experiments that variations in flow properties in the radial direction can be a significant influence on drawdown during a pumping test. Past analytical approaches that have considered radial variations in flow properties have been limited to the case of pumping from within a circular disk embedded in a matrix of differing properties. Butler (1988) and Karasaki (1986) summarize the majority of this work. The research reported here extends the analysis from two zones of differing properties to any number of zones. Several reasons can be proposed to justify the usefulness of this extension. First, such an extension may enable us to improve our present level of understanding concerning drawdown in response to pumping in a nonuniform aquifer. A better understanding of drawdown behavior can then serve as the basis for improved methods of aquifer characterization. Second, this analytical solution may be of use in the verification of numerical models, as it enables assessment of the capability of a numerical model to simulate flow behavior at the interface between materials of differing properties. Finally, the solution may also prove useful in developing type curves for particular flow conditions.

This article employs the Laplace-transformation technique to examine flow in radially nonuniform aquifers. One of the serious limitations of the Laplace-transform approach for well-hydraulics applications is the complexity of the inversion back from Laplace space. Most workers have employed a numerical approximation to perform the back transformation, thus preventing a closed-form analytical expression from being obtained in untransformed space. Even in the case when the back transformation is performed analytically (e.g., Karasaki (1986)), however, the resulting solution is normally so complex that relatively little insight can be gained from consideration of the form of the expression. Ramey (1970), recognizing that little insight can be derived from the Laplace-transform analytical solution for a two-zone nonuniform aquifer, proposed an approximate analytical solution based on the Boltzmann transformation for the purpose of improving physical understanding. The Boltzmann transformation is employed here for the case of an arbitrary number of zones in order to augment the insight that can be gained from the Laplace-transform solution.

PROBLEM STATEMENT

The problem of interest is that of pumping from a fully penetrating well at the center of a circular disk of radius R_1, around which are a series of concentric rings (jth ring has outer radius of R_j). As illustrated in Figure 1, the final concentric ring is considered to extend out to infinity. Confined flow is assumed and, save for the incorporation of radial variations in flow properties, the standard suite of assumptions that accompany analytical solutions for flow in confined units is adopted (see Theis (1935)).

The approach employed here assumes that flow properties are uniform within each ring, with step-wise variations being allowed across ring boundaries. Flow in each ring can therefore be represented by the conventional radial-flow equation:

$$\frac{\partial^2 s_i(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial s_i(r,t)}{\partial r} = \frac{S_i}{T_i}\frac{\partial s_i(r,t)}{\partial t}, \ i = 1, NZ \qquad (1)$$

where

T_i= transmissivity in zone i, $[L^2/T]$;
S_i= storage coefficient (storativity) in zone i, dimensionless;
$s_i(r,t)$= drawdown in zone i at location r and time t, [L];
r= radial direction, $R_{i-1} \leq r \leq R_i$, [L];
R_i = outer radius of zone i, [L];
NZ = total number of concentric rings;
t= time, [T].

Initial conditions are the same in each ring,

$$s_i(r,0) = 0, \ i = 1, NZ; \ r_w \leq r < \infty \qquad (2)$$

where
r_w= radius of pumping well, [L].

Boundary conditions, however, do depend on ring position within the flow system:

$$\lim_{r_w \to 0} 2\pi r_w T_1 \frac{\partial s_1}{\partial r}(r_w, t) = -Q, \ t > 0 \qquad (3)$$

$$s_i(R_i,t) = s_{i+1}(R_i,t), \ i=1, NZ-1 \qquad (4)$$

$$T_i \frac{\partial s_i}{\partial r}(R_i,t) = T_{i+1}\frac{\partial s_{i+1}}{\partial r}(R_i,t), \ i = 1, NZ - 1 \qquad (5)$$

$$s_{NZ}(\infty,t) = 0 \qquad (6)$$

where
Q = pumpage from well, $[L^3/T]$.

DERIVATION OF SOLUTION VIA THE LAPLACE-TRANSFORMATION METHOD

Equations (1) - (6) describe the flow conditions of interest here. In order to find the function s_i that satisfies these conditions for a given set of zones and zonal properties, the Laplace-transformation approach can be applied. The Laplace-transformation approach (Carslaw and

Jaeger, 1959; Doetsch, 1961) enables a system of partial differential equations in r and t to be rewritten as a system of ordinary differential equations in r. The Laplace-transform equivalents of (1) - (6) are:

$$\frac{\partial^2 \overline{s}_i}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{s}_i}{\partial r} = \frac{S_i}{T_i} p \overline{s}_i \, , \; i = 1, NZ \tag{7}$$

$$\lim_{r_w \to 0} 2\pi r_w T_1 \frac{\partial \overline{s}_1}{\partial r}(r_w, t) = - Q/p \tag{8}$$

$$\overline{s}_i(R_i, p) = \overline{s}_{i+1}(R_i, p), \; i = 1, \; NZ - 1 \tag{9}$$

$$T_i \frac{\partial \overline{s}_i}{\partial r}(R_i, p) = T_{i+1} \frac{\partial \overline{s}_{i+1}}{\partial r}(R_i, p), \; i = 1, NZ - 1 \tag{10}$$

$$\overline{s}_{NZ}(\infty, p) = 0 \tag{11}$$

where

$\overline{s}_i$ = the Laplace transform of s_i;
p = the Laplace-transform variable.

The solution to this system of transformed equations is straightforward as (7) is simply a form of the modified Bessel equation (Bronshtein and Semendyayev, 1985). A solution can therefore be proposed in the form:

$$\overline{s}_i = C_{1i} K_0(N_i r) + C_{2i} I_0(N_i r), \quad i = 1, \; NZ \tag{12}$$

where:

I_0 = modified Bessel function of the first kind of order 0;
K_0 = modified Bessel function of the second kind of order 0;
$N_i = \sqrt{S_i p / T_i}$;
C_{1i}, C_{2i} = constants for zone i eqn.

The constants in the system of equations given in (12) can be evaluated using the boundary conditions (8) - (11). Application of (8) to (12) allows C_{11} to be evaluated as follows:

$$C_{11} = \frac{Q}{2\pi T_1 p} \qquad (13)$$

Application of (11) to (12) allows C_{2NZ} to be evaluated as:

$$C_{2NZ} = 0 \qquad (14)$$

Using boundary conditions (9) and (10) produces a system of 2(NZ-1) equations with 2(NZ-1) unknown constants (C_{21}, C_{1i}, C_{2i}, i=2, NZ-1, C_{1NZ}). Solution of this system of equations is straightforward, although quite tedious, for NZ > 2. The form of the expressions for C_{1i} and C_{2i} in the general case is:

$$C_{1i} = C_{11} * \left(\prod_{jj=1}^{i-1} KI(jj) \right) \frac{NUM\ 1}{DENOM}; \qquad (15)$$

$$C_{2i} = C_{11} * \left(\prod_{jj=1}^{i-1} KI(jj) \right) \frac{NUM\ 2}{DENOM}; \qquad (16)$$

where:

$$NUM\ 1 = \sum_{\substack{m=0 \\ m \neq odd}}^{NZ-i} \; \sum_{j=1}^{\binom{NZ-i}{m}} \left[\left\{ \prod_{k=1}^{m} BB(Z(j,k,m) + i - 1) \right\} * \left\{ \prod_{\substack{jj=i \\ jj \neq [Z(j,k,m)+i-1,\, k=1,2,3,\ldots,m]}}^{NZ-1} FF(jj) \right\} \right];$$

$$NUM\ 2 = \sum_{\substack{m=1 \\ m \neq even}}^{NZ-i} \; \sum_{j=1}^{\binom{NZ-i}{m}} \left[\left\{ \prod_{k=1}^{m} DD(Z(j,k,m) + i - 1) \right\} * \left\{ \prod_{\substack{jj=i \\ jj \neq [Z(j,k,m)+i-1,\, k=1,2,3,\ldots,m]}}^{NZ-1} CC(jj) \right\} \right];$$

$$\mathrm{DENOM} = \sum_{\substack{m=0 \\ m \neq odd}}^{NZ-1} \sum_{j=1}^{\binom{NZ-1}{m}} \left[\left\{ \prod_{k=1}^{m} BB(Z(j,k,m)) \right\} * \left\{ \prod_{\substack{jj=1 \\ jj \neq [Z(j,k,m), k=1,2,\ldots,m]}}^{NZ-1} FF(jj) \right\} \right];$$

$KK(i) = K_0(N_{i+1} R_i)\, K_1(N_i R_i) - A(i)\, K_0(N_i R_i)\, K_1(N_{i+1} R_i)$
$II(i) = I_0(N_{i+1} R_i)\, I_1(N_i R_i) - A(i)\, I_0(N_i R_i)\, I_1(N_{i+1} R_i)$
$KI(i) = K_0(N_i R_i)\, I_1(N_i R_i) + K_1(N_i R_i)\, I_0(N_i R_i)$
$KI0(i) = K_0(N_{i+1} R_i)\, I_1(N_i R_i) + A(i)\, K_1(N_{i+1} R_i)\, I_0(N_i R_i)$
$KI1(i) = K_1(N_i R_i)\, I_0(N_{i+1} R_i) + A(i)\, K_0(N_i R_i)\, I_1(N_{i+1} R_i)$
I_1 = modified Bessel function of the first kind of order 1;
K_1 = modified Bessel function of the second kind of order 1;
$A(i) = \sqrt{\frac{S_{i+1} T_{i+1}}{S_i T_i}}$
m = the total number of KK and II expressions appearing in each product term;
j = counter for the number of product terms with m number of KK and II expressions;
BB(Z(j,k,m)) = II(Z(j,k,m)) if k = odd,
KK(Z(j,k,m)) if k = even;
Z(j,k,m) = a matrix used to calculate the coefficients of the II and KK terms (see Appendix B for details);
CC(jj) = KI1(jj) if jj < IMIN,
KI0(jj) if jj > IMIN;
IMIN = Min [Z(j,k,m), k = 1,2,....,m],
0, if m = 0;
IMAX = Max [Z(j,k,m), k = 1,2,....,m],
0, if m = 0;
DD(Z(j,k,m)) = KK(Z(j,k,m)) if k = odd,
II (Z(j,k,m)) if k = even;
FF(jj) = KI0(jj) if jj < IMIN or jj > IMAX,
KI1(jj) if IMIN < jj < IMAX.

Note that when the uniform-aquifer case is considered (NZ=1), (12) reduces to:

$$\bar{s}_1 = \left(\frac{Q}{2\pi T}\right) \frac{K_0\left(\sqrt{\frac{Sp}{T}}\; r\right)}{p} \qquad (17)$$

which is simply the Laplace-space form of the solution of Theis (1935). Note further that when

NZ = 2, (12) reduces to the following two expressions:

$$\bar{s}_1 = \left(\frac{Q}{2\pi T_1}\right)\frac{K_0(N_1 r)}{p} + \left(\frac{Q}{2\pi T_1}\right)\frac{KK(1)}{KI0(1)}\ \frac{I_0(N_1 r)}{p} \tag{18}$$

$$\bar{s}_2 = \left(\frac{Q}{2\pi T_1}\right)\frac{KI(1)}{KI0(1)}\ \frac{K_0(N_2 r)}{p} \tag{19}$$

which are the Laplace-space form of the solution of Butler (1988) rewritten using the notation of this paper. A more general check to verify (12)-(16) can be performed by demonstrating that the solution satisfies the governing equation and boundary conditions given in (7) - (11). This has been done for the case of NZ = 5. It is assumed that the solution is valid for the case of NZ > 5, although it has not been rigorously demonstrated. The comparison of (12)-(16) with a finite-difference radial flow model is discussed in a subsequent paragraph.

The final step of the Laplace-transformation approach is to invert the Laplace-space solution back to untransformed space. For a solution of the complexity of (12)-(16), the analytical back transformation is extremely tedious. Therefore, as discussed in the introduction, numerical approximations of this back transformation are normally employed. Several approaches have been proposed (e.g., Stehfest, 1970; Talbot, 1979) for the numerical inversion of Laplace-space expressions. The approach of Stehfest (1970), which has been commonly used for well-hydraulics applications (Moench and Ogata, 1984), is employed here. Karasaki (1986) has demonstrated, for the case of radial flow to a pumping well (NZ=2), that the Stehfest algorithm provides a solution within 1% of the analytical expression. The analytical expression ((B.9a) of Karasaki (1986)) is a complicated semi-infinite integral involving Bessel functions of the first and second kind. Since Bessel functions of the first and second kind are of an oscillatory nature (Abramowitz and Stegun, 1965), the evaluation of this integral expression takes considerably more

CPU time than does the numerical approach. Given its greater efficiency and acceptable accuracy, the Stehfest algorithm is used for inverting the Laplace-space expressions of this work.

An important use of the solution presented here is in verifying the capabilities of numerical models to represent flow to a pumping well in a nonuniform aquifer. Figure 2 displays the results of a comparison of drawdown predicted by the semianalytical approach given here with that predicted by the finite-difference radial flow model employed by Butler and McElwee (in press) for the six-ring case portrayed in Table 1. Note that differences in aquifer properties of four orders of magnitude were assumed across zonal interfaces in order to assess performance in highly variable systems. The favorable agreement between these two independently derived approaches lends further credence to both models. Similar comparisons using a greater number of zones have been employed to demonstrate the close agreement between the two models at a large NZ.

DRAWDOWN IN A RADIAL FLOW FIELD

As stated at the outset, a primary motivation for examining flow in the radially nonuniform case is to develop greater insight into the factors controlling drawdown during a pumping test. An initial question that can be asked is what portions of an aquifer control drawdown at different times during a pumping test. Butler (1988) was able to develop some insight into the controls on drawdown for the NZ=2 case by analytically back transforming large-time expressions for drawdown. The complexity of (12)-(16) precludes a similar approach from being employed here. Instead, an approximate analytical solution based on the Boltzmann-transformation approach of Ramey (1970) is employed to address this issue. In Appendix A, the approximate solution for the general case is derived and the bounds of its applicability are discussed. In all cases, the approximate form is within 1% of the actual solution for the same conditions at which the analytical back transformation of Butler (1988) is applicable, i.e. the dimensionless times with respect to a zonal boundary ($4T_{i+1}\,t/(S_{i+1}\,R_i^2)$ or $4T_i\,t/(S_i\,R_i^2)$) are greater than or equal to 100. When the variation in diffusivity across a zonal boundary is small (i.e. less than an order of magnitude), the approximation is valid at considerably smaller dimensionless times.

Given the above caveats concerning the validity of the Boltzmann-transformation solution, the question of which portions of an aquifer control drawdown at different times during a pumping test can be addressed by rewriting (A.18) in the following manner:

$$s_i\,(r_o,t) = s_{i+1}(R_i,t) + \Delta s_i\,(r_o,t) \tag{20}$$

where

$s_{i+1}(R_i,t)$= drawdown at the boundary between zones i and i+1=

$$\frac{Q}{4\pi}\left[\sum_{j=i+1}^{NZ-1}\left(\frac{P_j}{T_j}(W(U_j(R_{j-1})) - W(U_j(R_j)))\right) + \frac{P_{NZ}}{T_{NZ}}W(U_{NZ}(R_{NZ-1}))\right]$$

$\Delta s_i(r_o,t)$= drawdown at radial distance r_o that is due to the properties of zone i =

$$\frac{Q}{4\pi}\frac{P_i}{T_i}[W(U_i(r_o)) - W(U_i(R_i))]$$

Note that expressing the drawdown in a radial flow field in this manner is in keeping with the assertion of Butler (1986) that drawdown in a radial flow field can be considered to consist of a component dependent on near-well materials (Δs_i) and a component dependent on more distant properties (s_{i+1}). Equation (20) can be developed further by replacing the s_{i+1} term by a s_{i+2} term and a Δs_{i+1} term and so on, eventually leading to the following equation:

$$s_i(r_o,t) = \Delta s_i(r_o,t) + \sum_{j=i+1}^{NZ-1}\Delta s_j + \frac{Q}{4\pi}\frac{P_{NZ}}{T_{NZ}}W(U_{NZ}(R_{NZ-1})) \tag{21}$$

where

Δs_j = drawdown due to the properties of the jth zone =

$$\frac{Q}{4\pi}\frac{P_j}{T_j}[W(U_j(R_{j-1})) - W(U_j(R_j))]$$

The form of (21) clearly demonstrates how drawdown is related to the properties of each portion of the flow system. Figure 3 is a plot of the dimensionless Δs_i term $(4\pi T_i \Delta s_i / Q)$ versus dimensionless time $(1/U_i\,(R_i))$ indicating that the Δs_i term can certainly be considered constant beyond a dimensionless time of 100. After this time, further changes in drawdown will be effectively independent of the properties in the ith zone. Note that this finding also corroborates the numerical sensitivity analysis of Butler and McElwee (in press). The Δs_j terms of (21) can be plotted in a similar manner, as shown in Figure 4, graphically demonstrating the time interval during which the properties of each zone influence changes in drawdown in zone i. Note that the abscissa of Figure 4 has the same dimensionless-time scale as in Figure 3. If the dimensionless-time scale was with respect to either zones 3 or 5, a flattening of the drawdown curve for the respective zone would occur prior to a dimensionless time of 100. Figures 3 and 4 indicate that once the front of the cone of depression has passed out of a given zone, the contribution of that zone to observation-well drawdown will remain unchanged, unless the pumping rate is varied. Increasing the influence of a given zone through pumping rate changes is the basis of a recently proposed pumping-test approach (Butler and McElwee, in press).

For all zones in which the dimensionless time of pumpage exceeds 100, a Thiem analysis (i.e. a Cooper-Jacob analysis through space) can be applied to develop expressions for the transmissivity within (22) and between zones (23):

$$T_k = \frac{Q}{2\pi\Delta s}\ln\left(\frac{r_b}{r_a}\right) \qquad (22)$$

where:

r_a, r_b= radial end points of interval over which Thiem analysis is applied, $r_a < r_b$,[L];
Δs = drawdown between r_a and r_b, [L].

$$T = \ln\left(\frac{r_b}{r_a}\right)\left(\frac{1}{T_k}\ln\left(\frac{R_k}{r_a}\right) + \sum_{j=k+1}^{n}\left[\frac{1}{T_j}\ln\left(\frac{R_j}{R_{j-1}}\right)\right] + \frac{1}{T_{n+1}}\ln\left(\frac{r_b}{R_n}\right)\right)^{-1} = \frac{Q}{2\pi\,\Delta s}\ln\left(\frac{r_b}{r_a}\right) \tag{23}$$

where

$R_{k-1} \leq r_a \leq R_k$;
$R_n \leq r_b \leq R_{n+1}$;
T = interzonal transmissivity between r_a and r_b, [L^2/T].

Equation (22) is simply the conventional Thiem equation (Bear, 1979) for a uniform aquifer. The corresponding nonuniform-aquifer expression (23) indicates that the interzonal transmissivity is in the form of a weighted harmonic mean of the transmissivities of the zones between the two radial observation points. The weighting is such that a zone closer to the pumping well has a greater influence than a zone of a similar width at a larger radial distance, as would be expected from the convergent nature of flow in a radial flow field. Note that this distance-weighted harmonic average is analogous to the time-weighted harmonic average proposed by Butler (1986) for the interzonal transmissivity arising from a Cooper-Jacob analysis through time in a nonuniform aquifer.

An issue of considerable interest for the design of pumping tests in nonuniform aquifers is the size of a zone and its implications for pumping-test analysis. Note that the Δs of (22) is the large-time form of the Δs_i and Δs_j terms of (21). Thus, the Thiem equation for a single zone (22) can be used to approximate the contribution of a given zone to drawdown at an observation well closer to the pumping well. This equation can also be used to examine how the contribution of a given zone to observation-well drawdown (i.e. Δs) changes with the radial distance between the zone and the pumping well. Consider a zone with inner and outer radial boundaries, r_a and r_b, respectively. The area of this zone can be written as:

$$\text{Area} = \pi\left(r_b^2 - r_a^2\right) = \pi r_a^2\left(\left(\frac{r_b}{r_a}\right)^2 - 1\right) \tag{24}$$

Assuming that the area of the zone remains constant (Area = C), (24) can be rewritten:

$$\left(\frac{r_b}{r_a}\right) = \left[\frac{C}{\pi r_a^2} + 1\right]^{\frac{1}{2}} \tag{25}$$

By substituting (25) into (22), we can obtain an expression that will allow us to assess how Δs varies with r_a:

$$\Delta s = \frac{Q}{2\pi T_k}\ln\left(\frac{r_b}{r_a}\right) = \frac{Q}{2\pi T_k}\ln\left[\frac{C}{\pi r_a^2} + 1\right]^{\frac{1}{2}} \tag{26}$$

By taking the limit of this expression, we can examine the behavior of Δs as r_a goes to extreme values:

$$\lim_{r_a \to \infty} \Delta s = \frac{Q}{2\pi T_k}\left(\tfrac{1}{2}\right)\ln(1) = 0 \tag{27}$$

$$\lim_{r_a \to 0} \Delta s = \frac{Q}{2\pi T_k}\left(\tfrac{1}{2}\right)\ln\left(\frac{C}{\pi r_a^2}\right) \to \infty \tag{28}$$

Figure 5, a plot of $\frac{2\pi T_k \Delta s}{Q}$ versus ln r_a, displays how the contribution of a zone to observation-well drawdown (Δs) decreases as the distance between the zone and the pumping well increases. It can be shown that the slope of this plot is -1 near $r_a = 0$ and goes to 0 as r_a gets large. The implication is that a zone at a sizable distance from the pumping well may be undetectable during a pumping test. This effect, which is the basis for the composite model of Karasaki (1986) for analysis of pumping tests in fractured media, makes it difficult to characterize moderate- and high-frequency nonuniformities at a distance from the pumping well, regardless of the position of the observation well. Note that (22) clearly shows that the transmissivity of a zone is an important determinant of the magnitude of the contribution of that zone to observation-well drawdown, and thus whether or not a pumping test will be an effective approach for estimation of zonal properties. This finding is again in keeping with the results of the numerical sensitivity analysis of Butler and McElwee (in press).

The drawdown response produced by the front of the cone of depression (cone front) crossing an interface between materials of differing properties is another area for which (20) can provide considerable insight. If we confine our interest to the drawdown in response to the cone front moving across a single radial boundary, we can rewrite (20) for the case of NZ=2:

$$s_1(r_o,t) = \frac{Q}{4\pi}\left(\frac{(W(U_1(r_o)) - W(U_1(R_1)))}{T_1} + P_2\frac{W(U_2(R_1))}{T_2}\right) \qquad (29)$$

where

$$r_o \leq R_1.$$

Note that $P_i = 1$ when i=1 or the dimensionless time of pumpage is large (see (A.14)). As is discussed in Appendix A, the Boltzmann-transformation solution may be a poor approximation during the transition period when the cone front is actually moving across a zonal boundary. The large-time form of (29), however, can be used to gain some insight concerning the controls on drawdown during this period. Using the truncated series approximation of the well function suggested by Cooper and Jacob (1946) and the fact that P_2 goes to one at large times of pumpage, a large-time form of (29) can be written as:

$$s_1(r_o,t) = \frac{Q}{4\pi T_1}\left(-\ln\left(\frac{r_o^2 S_1}{4T_1 t}\right) + \ln\left(\frac{R_1^2 S_1}{4T_1 t}\right) - \left(\frac{T_1}{T_2}\right)\ln\left(\frac{R_1^2 S_2}{4T_2 t}\right) - .5772\right) \qquad (30)$$

Past workers (e.g., Fenske, 1984; Butler, 1988) have attributed drawdown behavior during the period when the cone front is passing across a zonal interface to be primarily a function of the diffusivity contrast across the boundary. Actually, the behavior can be better viewed as a function of independent variations in transmissivity and storativity across the boundary. These two properties have distinctly different effects on drawdown during the transition period, which are best considered separately.

If only transmissivity varies across a zonal boundary, (30) can be rewritten in the following manner:

$$\frac{4\pi T_1 s_1}{Q} = n \ln \frac{4T_1 t}{r_o^2 S_1} - n \ln n - 2(n-1) \ln \frac{R_1}{r_o} - .5772 \qquad (31)$$

where

$n = T_1/T_2$

Note that a dimensionless time-versus-drawdown plot for this configuration has a slope of n (n=1 in a uniform aquifer) at large times. Thus, the movement of the cone front across a zonal interface results in a permanent change in the slope of a semilog drawdown plot. Figure 6 is a dimensionless time-versus-drawdown plot produced using the semianalytical Laplace-transform solution. The curves of Figure 6a, in conjunction with additional numerical experiments, indicate that a semilog drawdown-versus-time plot will always display a monotonic change in slope as the interface between material of two different transmissivities is crossed as long as the storativity remains unchanged. Note that Figure 6a indicates that large increases or decreases in transmissivity across a zonal boundary may be easily mistaken for recharge or discharge boundaries, respectively.

If only storativity varies across the zonal boundary, (30) can be rewritten as:

$$\frac{4\pi T_1 s_1}{Q} = \ln\left(\frac{4T_1 t}{r_o^2 S_1}\right) + \ln\left(\frac{S_1}{S_2}\right) - .5772 \qquad (32)$$

Equation (32) indicates that a change in storage properties across an interface will offset the second limb of a semilog drawdown plot with respect to the first limb, although the slope will remain unchanged. The nature of the offset is such that an increase in storativity across the boundary will produce a negative offset. Figure 6b, in conjunction with additional numerical experiments, indicates that a semilog drawdown-versus-time plot will display a nonmonotonic change in slope as the interface between material of two different storage properties is crossed. Note that the drawdown behavior illustrated here is the same as that seen in double-porosity fractured aquifers.

In cases when both transmissivity and storativity vary across a boundary, numerical experiments show that the nature of the drawdown behavior during the transition period is a function of whether transmissivity and storativity are changing in the same direction. If transmissivity and storativity variations across an interface are in opposite directions, a semilog drawdown plot will normally display a monotonic change in slope as the cone front passes across the interface. Only when the change in transmissivity is small and that in storativity very large will nonmonotonic behavior be observed. When transmissivity and storativity variations across an interface are in the same direction, a semilog drawdown plot will usually display a slightly nonmonotonic change in slope during the transition period. As the magnitude of the variation in storativity with respect to that in transmissivity increases, the nonmonotonic behavior becomes more evident. These results imply that the effect of transmissivity variations across a boundary is usually much stronger than that of storativity variations.

In summary, the existence of a variation in zonal transmissivity across an interface will be clear from the permanent change in slope of the straight-line portions of the semilog drawdown plot. Even when the size of a zone is too small to produce a straight line on a semilog plot, the break in the plot will still be evident. A variation in storage properties across an interface will not be as clear, however, except when the relative change in storativity is large with respect to that in transmissivity, and both are in the same direction.

The preceding paragraphs have focussed on the relationship between drawdown and the properties of material lying radially outward from the observation well, neglecting the influence of material between the pumping and observation wells. The major reason for this is that the influence of these interwell materials becomes negligible rather rapidly under most conditions.

Unlike the Δs_i and Δs_j components of (21), which can be assumed constant after a dimensionless time of 100, the contribution of interwell material goes to zero with increased duration of pumpage (Bütler, 1988). Thus, drawdown at an observation well in a radial flow field is of limited use in characterizing materials between the observation and pumping wells. Only a hydraulic test that emphasizes the early-time drawdown behavior (e.g., a pulse test (Johnson et al., 1966)) is able to provide considerable information concerning the interwell region. Even in the case of pulse testing, however, only the time lag between the pumping and observation wells is a strong function of interwell materials.

CONCLUSIONS

A semianalytical solution was derived here for the case of flow to a pumping well in an aquifer whose properties vary arbitrarily in the radial direction. The complexity of the solution, however, required that an approximate analytical solution be employed in order to develop insight into flow behavior in this configuration. The insight developed with respect to the nature of observation-well drawdown during a pumping test is most relevant, as it leads to several important conclusions concerning observation-well placement in nonuniform aquifers. First, as shown by (20) and (21), an observation well is most effective in characterizing aquifer properties in zones that lie radially outward from the observation well. Observation-well drawdown during a conventional pumping test is of little use for characterizing the properties of the interwell region. Second, the volume of the aquifer that is controlling observation-well drawdown during a given time interval increases considerably with the duration of pumpage. The increasing size of this volume with duration of pumpage makes it difficult to characterize moderate- and high-frequency nonuniformities at a distance from the pumping well. Observation wells respond to such variations in a manner analogous to a uniform aquifer whose transmissivity and storativity parameters are some sort of average of the actual spatially varying properties. Even drawdown at an observation well very close to the pumping well only displays the effects of relatively high-frequency nonuniformities at early times of pumpage. Later-time drawdown is essentially unaffected by such nonuniformities, as in the case of more distant observation wells. A Thiem analysis ((22) and (23)), using the change in drawdown between two closely spaced wells, is probably the most appropriate approach for characterizing moderate- to high-frequency transmissivity variations in an aquifer when conventional pumping-test procedures are employed. This approach, however, requires a large number of wells for detailed description of property variations. Finally, it should be noted that (22) indicates that the magnitude of the contribution of a given zone to observation-well drawdown (Δs) is an inverse function of zonal transmissivity. Thus, the more permeable the aquifer the more difficult it may be to characterize variations in aquifer properties.

The above conclusions are based on an examination of pumping-induced drawdown in the ideal case when aquifer properties vary solely in the radial direction. Although this configuration is an admitted simplification of reality, the insight developed from this work should have considerable value for the case of pumping tests in nonuniform aquifers. Further work, however, is required to fully assess the general applicability of these concepts to complex natural systems.

APPENDIX A

In this section, an approximate analytical solution to (1)-(6) is derived using the Boltzmann transformation. The derivation is similar to that in Sen (1987), although Sen apparently did not recognize the approximate nature of the method. Following the derivation, the range over which the approach is a close approximation of the Laplace-transform solution is discussed.

Due to the nature of the approach employed here, the series of equations represented by (1) must be rewritten in terms of specific discharge:

$$\frac{\partial q_i(r,t)}{\partial r} + \frac{1}{r} q_i(r,t) = \frac{S_i}{B_i} \frac{\partial s_i(r,t)}{\partial t}, \quad i = 1, NZ \qquad (A.1)$$

where

$q_i(r,t) = K_i \dfrac{\partial s_i}{\partial r}$ = specific discharge in zone i at location r and time t, [L/T];
B_i = aquifer thickness in zone i, [L].

This series of partial differential equations (pdes) can be transformed into a series of ordinary differential equations (odes) in q_i using the Boltzmann similarity transformation. The series of odes can be solved using integration factors. Substitution of Darcy's Law for q_i and integration in the radial direction then yields an expression in terms of pumping-induced drawdown. The Boltzmann similarity transformation is a well-known approach for transforming pdes into odes that has been employed frequently in the unsaturated flow literature (e.g., Remson et al., 1971). The transformation for the case of (A.1) can be written as

$$\eta = \frac{r^2}{4t} \qquad (A.2)$$

where:

η = similarity variable, [L^2/T].

Taking the spatial and temporal derivatives of the similarity variable and employing the chain rule for derivatives allows (A.1) to be transformed into the form given below:

$$2\eta\frac{d\,q_i(\eta)}{d\eta}+q_i(\eta)=-\frac{S_i}{B_i}\frac{r\eta}{t}\frac{d\,s_i(\eta)}{d\eta},\quad i=1,NZ \tag{A.3}$$

In order to rewrite (A.3) in terms of a single dependent variable, Darcy's Law, written in terms of the similarity variable, is employed to substitute q_i for the drawdown term. This substitution results in the following series of odes written in terms of q_i:

$$\frac{d\,q_i(\eta)}{d\eta}+\left(\frac{1}{2\eta}+\frac{S_i}{T_i}\right)q_i(\eta)=0,\quad i=1,\,NZ \tag{A.4}$$

The relevant boundary and initial conditions can be similarly transformed as follows:

$$\lim_{\eta\to 0}4\pi\sqrt{\eta t}\;B_1q_1(\eta)\;=-Q \tag{A.5}$$

$$B_iq_i(\eta)_{R_i}=B_{i+1}\,q_{i+1}\,(\eta)_{R_i},i=1,NZ-1 \tag{A.6}$$

$$q_{NZ}(\infty)=0, \tag{A.7}$$

The solution of the transformed series of equations given in (A.4) can be obtained in a straightforward manner using integration factors (Boyce and DiPrima, 1986), resulting in the following expression for q_i:

$$q_i(\eta)=C_i\exp\left(-S_i\eta/T_i\right)/\sqrt{\eta},\quad i=1,\,NZ \tag{A.8}$$

where

$$\eta=\frac{r^2}{4t},\quad R_{i-1}\le r\le R_i,[L^2/T];$$

C_i = constant in specific discharge expression for ith concentric ring, $[L^2/T^{1.5}]$.

Applying boundary condition (A.5) enables C_1 to be evaluated:

$$C_1=-Q/(4\pi B_1\sqrt{t}) \tag{A.9}$$

The expression for specific discharge in zone 1 can then be written as:

$$q_1(\eta)=\frac{-Q}{2\pi rB_1}\exp\left(-\;S_1\eta/T_1\right) \tag{A.10}$$

The continuity condition for flow between rings (A.6) can then be employed to yield C_2:

$$C_2 = \frac{-Q}{4\pi B_2\sqrt{t}}\,\frac{\exp\left(-S_1\eta_1/T_1\right)}{\exp\left(-S_2\eta_1/T_2\right)} \tag{A.11}$$

where:

$$\eta_1 = \frac{R_1^2}{4t}, [L^2/T].$$

By repeated application of this approach, a series of expressions for C_i and q_i can be obtained:

$$C_i = \frac{-Q}{4\pi B_i\sqrt{t}}\left(\prod_{j=1}^{i-1}\frac{\exp\left(-S_j\eta_j/T_j\right)}{\exp\left(-S_{j+1}\eta_j/T_{j+1}\right)}\right) \tag{A. 12}$$

$$q_i(\eta) = \frac{-Q}{2\pi B_i r}\left(\prod_{j=1}^{i-1}\frac{\exp\left(-S_j\eta_j/T_j\right)}{\exp\left(-S_{j+1}\eta_j/T_{j+1}\right)}\right)\exp\left(\frac{-S_i\eta}{T_i}\right) \tag{A. 13}$$

Defining P_i as in (A.14) and rewriting (A.13) in terms of radial distance and time

$$P_i = \begin{cases} 1 & , i=1 \\ \displaystyle\prod_{j=1}^{i-1}\frac{\exp\left(-\eta_j S_j/T_j\right)}{\exp\left(-\eta_j S_{j+1}/T_{j+1}\right)} & , i=2,NZ \end{cases} \tag{A.14}$$

allows a general expression that represents the specific discharge in any zone to be obtained:

$$q_i(r,t) = \frac{-Q}{2\pi B_i r} P_i \exp\left(-r^2 S_i/4tT_i\right), \quad i=1,NZ \tag{A.15}$$

The end goal of this exercise, however, is not to obtain an expression for specific discharge. Rather, the purpose is to obtain an expression written in terms of drawdown. The series of equations given in (A.15) can be rewritten in terms of drawdown by substituting Darcy's Law for specific discharge and then integrating in the radial direction from the observation well to infinity. In order to simplify the integration, it can be written as a sum of integrals (A.16), where each integral represents the contribution of a single concentric ring of uniform properties to observation-well drawdown.

$$\int_{r_o}^{\infty} ds = -\frac{Q}{2\pi}\Big(\frac{P_i}{T_i}\int_{r_o}^{R_i}\frac{\exp\left(-r^2S_i/4tT_i\right)}{r}dr + \sum_{j=i+1}^{NZ-1}\Big(\frac{P_j}{T_j}\int_{R_{j-1}}^{R_j}\frac{\exp\left(-r^2S_j/4tT_j\right)}{r}dr\Big)$$
$$+\frac{P_{NZ}}{T_{NZ}}\int_{R_{NZ-1}}^{\infty}\frac{\exp\left(-r^2S_{NZ}/4tT_{NZ}\right)}{r}dr\Big) \qquad \text{(A.16)}$$

where

r_o = radial location of observation well, $R_{i-1} \leq r_o \leq R_i$, [L].

Each individual integral can be rewritten using the U variable of Theis (1935) as follows:

$$\int_{R_{j-1}}^{R_j}\frac{\exp\left(-r^2S_j/4tT_j\right)}{r}dr = \frac{1}{2}\int_{U_j(R_{j-1})}^{U_j(R_j)}\frac{\exp(-\xi)}{\xi}d\xi$$

where

$$U_j(r) = \frac{r^2S_j}{4T_jt}$$

Using the standard W(U) notation to represent the integration from U to infinity, (A.16) can be rewritten in the following manner:

$$s_i(r_o,t) = \frac{Q}{4\pi}\Big(\frac{P_i}{T_i}\left(W(U_i(r_o)) - W(U_i(R_i))\right)$$
$$+\sum_{j=i+1}^{NZ-1}\left(\frac{P_j}{T_j}\left(W(U_j(R_{j-1})) - W(U_j(R_j))\right)\right)$$
$$+\frac{P_{NZ}}{T_{NZ}}W(U_{NZ}(R_{NZ-1}))\Big) \qquad \text{(A.18)}$$

This expression enables us to evaluate the drawdown, at any point in a series of concentric rings of differing properties, in response to constant pumpage at a central well.

The attractiveness of the Boltzmann-transformation solution given in (A.18) is that the expression is much simpler than the corresponding Laplace-transformation form. The problem is that (A.18) is only an approximate solution to (1)-(6). The approximate nature of the approach is clearly revealed when considering (A.11) and (A.12), as the supposed constant coefficients are actually functions of η. The reason for this is the boundary condition given in (A.6), for which η is not a constant. As Ramey (1970) points out, the Boltzmann transformation only strictly holds true for this configuration when the boundary is moving such that η remains constant in (A.6).

Although not strictly constant, the coefficients of (A.8) can be considered approximately constant for a broad range of conditions. The range of conditions can be defined as that over which the P_i term (A.14) is approximately equal to one. If we just consider the two-zone case, (A.14) can be written as:

$$P_2 = \frac{\exp\left(-R_1^2 S_1/4tT_1\right)}{\exp\left(-R_1^2 S_2/4tT_2\right)} = \exp\left(\frac{R_1^2}{4t}\left(\frac{S_2}{T_2} - \frac{S_1}{T_1}\right)\right) \tag{A.19}$$

P_2 will be within 1% of 1.0 when

$$\left|\frac{R^2}{4t}\left(\frac{S_2}{T_2} - \frac{S_1}{T_1}\right)\right| \le 0.01 \tag{A.20}$$

Rearranging and rewriting (A.20) in terms of dimensionless time produces the following condition:

$$\frac{4T_1 t}{S_1 R_1^2} \ge \left|100\left(\frac{D_1}{D_2} - 1\right)\right| \tag{A.21}$$

where

$D_i = T_i/S_i$ = diffusivity of zone i, $[L^2/T]$.

Expression (A.21) enables the dimensionless time at which P_2 can be considered approximately one to be defined in terms of the diffusivity contrast across the interface. When the diffusivity does not change across an interface, the Boltzmann-transformation approximation should be valid at all times. As the magnitude of the diffusivity variation increases, the dimensionless time at which the approximation is valid increases. It is at the early dimensionless times that the approximation is least appropriate; the period during which the front of the cone of depression is crossing an interface. Once the cone front has passed across an interface, the approximation is quite good. Figure 7 displays two dimensionless time-versus-drawdown plots comparing the Laplace- and Boltzmann-transformation solutions for different diffusivity ratios. Figure 7a displays the comparison for a diffusivity contrast of an order in magnitude in the six-ring configuration of Table 1. Note that the largest difference between the two solutions is during the period the cone front is passing across a zonal interface. Figure 7b displays the comparison for a diffusivity contrast of a factor of two in the same configuration. In this case, only a slight difference is seen as the cone front passes across a zonal interface. Thus, for configurations in which there is a relatively small contrast in diffusivity across a zonal interface, the Boltzmann-transformation expression (A.18) is an excellent approximation of the Laplace-transform solution. The type curves of Sen (1987) would be appropriate in this situation.

APPENDIX B

The matrix Z(j,k,m) is used to calculate the coefficients of the II and KK terms of (15) and (16). A brief description of the form of this matrix is given in this section.

The II and KK terms of (15) and (16) are elements in a double summation of products. Each product consists of m II and KK terms with additional KI0 and KI1 terms. Each inner summation is over all possible combinations (without replacement) of m terms given NZ zones and an observation well in zone i, i.e. $\binom{NZ - i}{m}$. The outer summation is over the possible values for m ($0 \leq m \leq NZ - i$). Individual elements of the Z array are determined by considering the $\binom{NZ - i}{m}$ possible combinations of m terms. The j indice of the Z array is the counter for the number of possible combinations, i.e. the number of products with m KK and II terms. The j designation for a given combination is arbitrary since all products are summed together. The k indice is the counter for the elements within each combination ($1 \leq k \leq m$), once again the order is arbitrary since the terms are being multiplied together. For example, if NZ=5, i=1, and m=3, there will be four combinations of three terms each. The elements of the Z matrix corresponding to these combinations can be written as follows:

$$
\begin{aligned}
&Z(1,k,3) = 1,2,3 \text{ for } k=1,2,3, \text{ respectively;} \\
&Z(2,k,3) = 1,2,4, \text{ for } k=1,2,3, \text{ respectively;} \qquad \text{(B.1)} \\
&Z(3,k,3) = 1,3,4, \text{ for } k=1,2,3, \text{ respectively;} \\
&Z(4,k,3) = 2,3,4, \text{ for } k=1,2,3, \text{ respectively.}
\end{aligned}
$$

The form of the Z matrix allows the calculation of individual elements of the array to be readily programmed. Note that the number of products in each summation will become very large with increases in NZ. Thus, the solution given in (12)-(16) is not practical for cases where NZ is large.

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Zone	R_{i-1}(m)	R_i(m)
1	.05	4.456
2	4.456	24.00
3	24.00	129.3
4	129.3	696.2
5	696.2	3750.
6	3750.	∞

R_{i-1} = radius of inner boundary of zone i,

R_i = radius of outer boundary of zone i.

Table 1 -Radial boundaries for the zones of the hypothetical aquifer of Figure 1.

FIGURE CAPTIONS

Figure 1 Hypothetical aquifer in which radial variations in flow properties are represented by a series of concentric rings of differing properties. Pumping well is located at the center of zone 1

Figure 2 Comparison of drawdown predicted by the semianalytical Laplace-transform solution of this work with that predicted by the finite-difference radial flow model of Butler and McElwee (in press) for the six-ring configuration of Table 1 (r_o=1.45 m, Q = 1000. m^3/d):

a) Results for the variable transmissivity case ($T_1 = T_3 = T_5 = 1.$ m^2/d, $T_2 = T_4 = T_6 = 10{,}000.$ m^2/d, $S_i = .0001$, i=1,6);

b) Results for the variable storativity case ($S_1 = S_3 = S_5 = .001$, $S_2 = S_4 = S_6 = 1.0x10^{-7}$, $T_i = 100$ m^2/d, i=1,6), note that zonal boundaries in this case are slightly smaller than those of Table 1 due to the finite-difference model handling transmissivity and storativity variations in a different manner.

Figure 3 Dimensionless plot of Δs_i (i=1) versus time for the six-ring configuration of Table 1 and Figure 2a).

Figure 4 Dimensionless plot of Δs_j (j= 3 and 5) versus time for the six-ring configuration of Table 1 and Figure 2a).

Figure 5 Plot of dimensionless Δs ($2\pi T_k \Delta s/Q$) versus radial distance to inner boundary of zone in which Δs is measured (r_a); zone area remains a constant equal to the area of zone two in Table 1.

Figure 6 Dimensionless drawdown ($4\pi s_1T_1/Q$) versus dimensionless time ($4T_1t/S_1r^2_o$) plots illustrating behavior during the transition period when the cone front is passing across a zonal interface ($R_1/r_o = 15$):

a) Results for the variable transmissivity case ($(T_2/T_1)_A = .1$, $(T_2/T_1)_B = 1.$, $(T_2/T_1)_C = 10.$, $S_2/S_1 = 1.0$);

b) Results for the variable storativity case ($T_2/T_1 = 1.$, $(S_2/S_1)_A = .01$, $(S_2/S_1)_B = 1.$, $(S_2/S_1)_C = 100.$).

Figure 7 Comparison of drawdown predicted by the Laplace-transform solution with that predicted by the approximate Boltzmann-transform solution for the six-ring configuration of Table 1 (r_o=1.45 m, Q=1000. m^3/d, $D_1 = D_3 = D_5$, $D_2 = D_4 = D_6$):

a) Results for the case of an order of magnitude variation in diffusivity ($D_2/D_1 = 10.$);

b) Results for the case of a factor of two variation in diffusivity ($D_2/D_1 = 2.$).

Figure 1

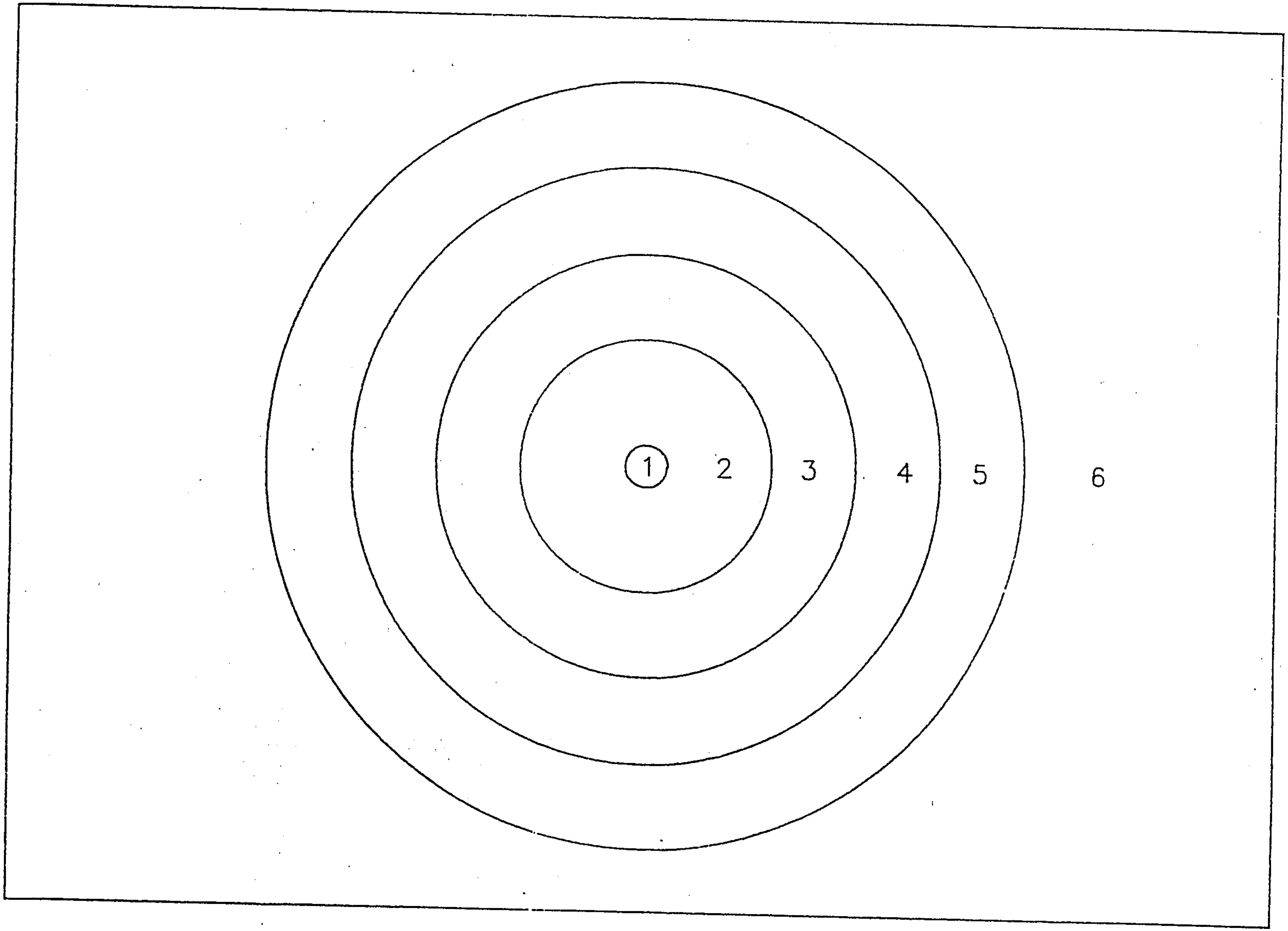
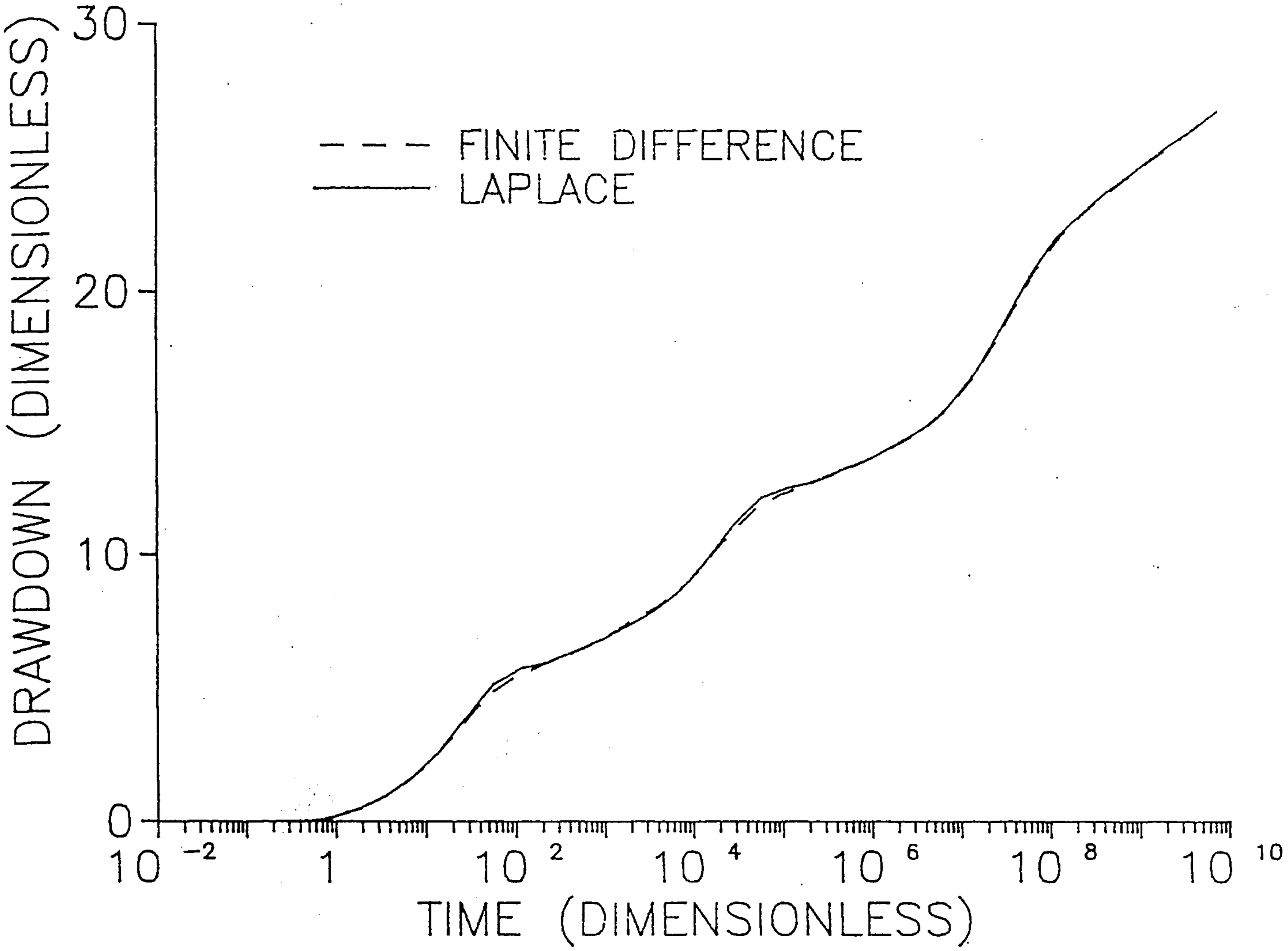
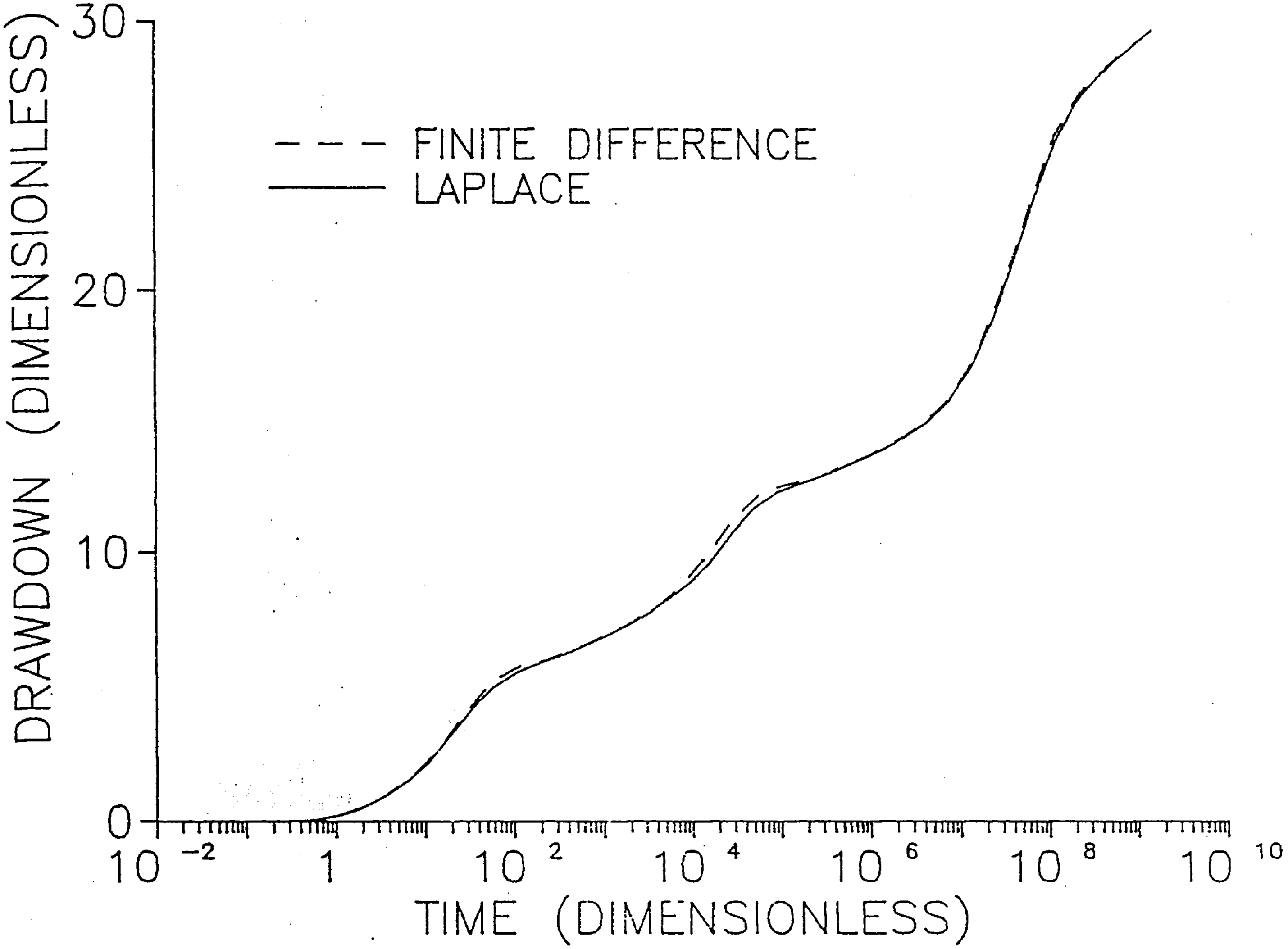


Figure 2a



DRAWDOWN (DIMENSIONLESS)
30
20
10
0
FINITE DIFFERENCE
LAPLACE
10^{-2}
1
10^{2}
10^{4}
10^{6}
10^{8}
10^{10}
TIME (DIMENSIONLESS)

Figure 2b



FINITE DIFFERENCE
LAPLACE
DRAWDOWN (DIMENSIONLESS)
30
20
10
0
10^{-2}
1
10^{2}
10^{4}
10^{6}
10^{8}
10^{10}
TIME (DIMENSIONLESS)

Figure 3

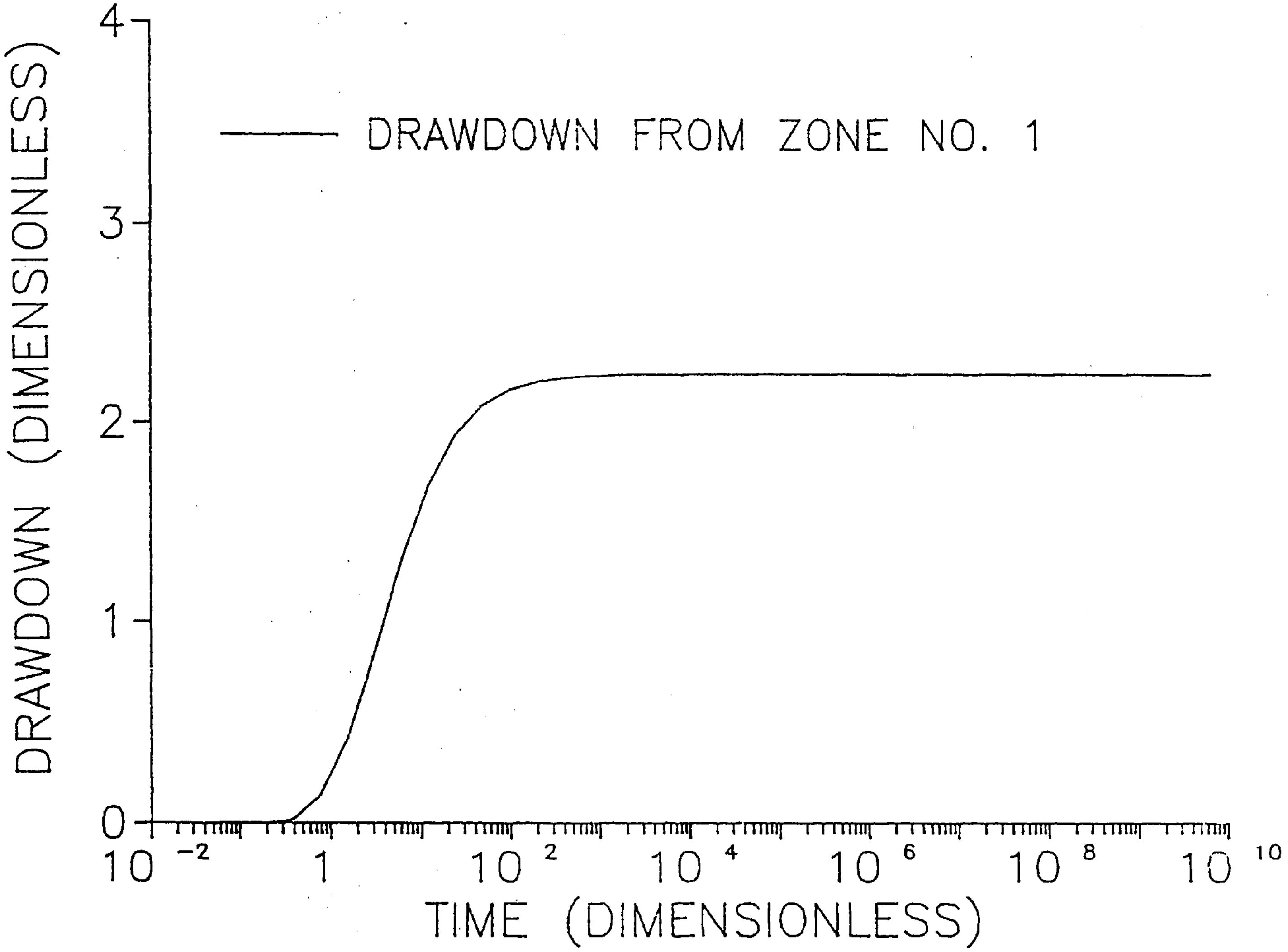
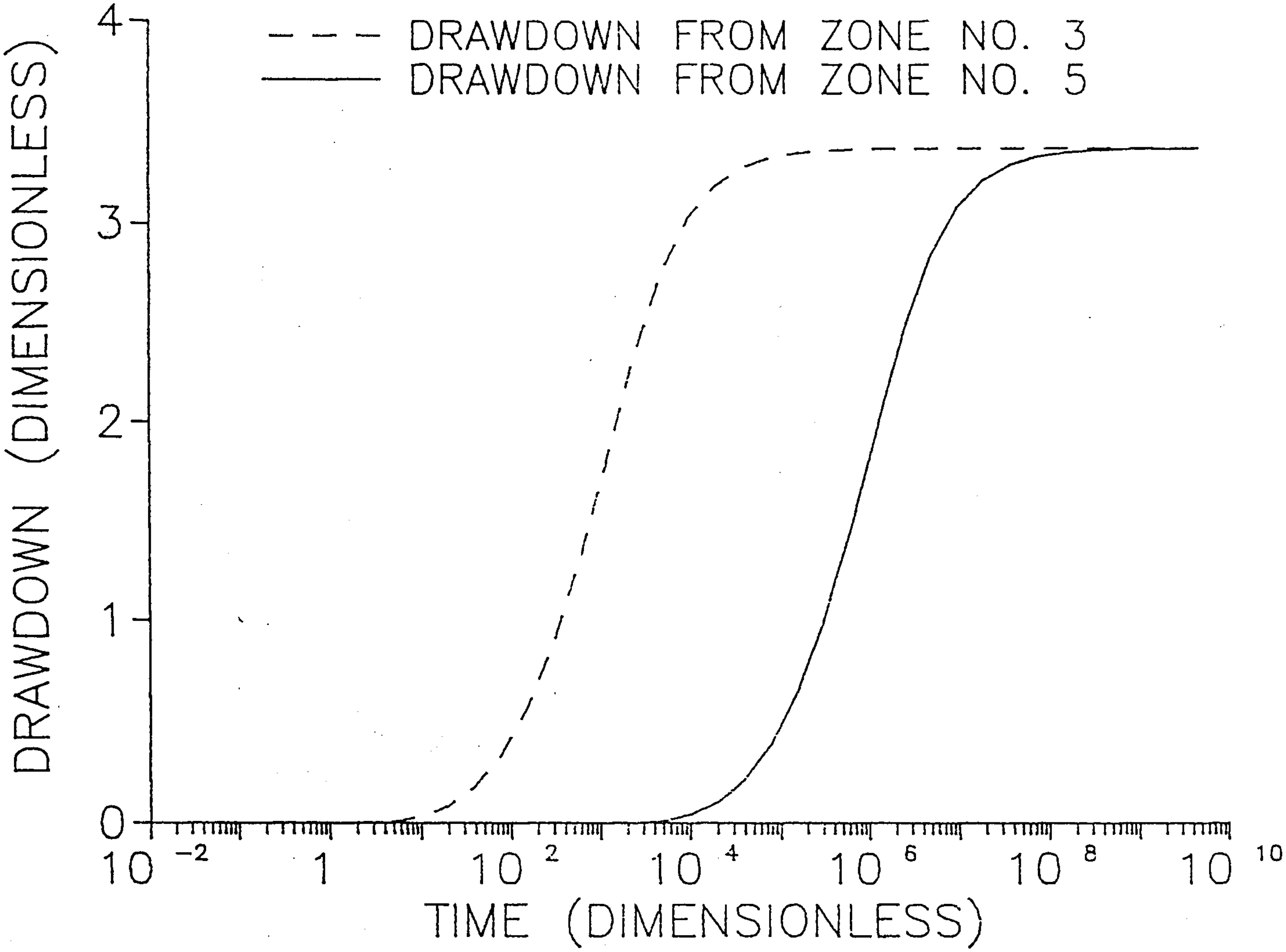

DRAWDOWN (DIMENSIONLESS)
4
3
2
1
0
DRAWDOWN FROM ZONE NO. 1
10^{-2}
1
10^{2}
10^{4}
10^{6}
10^{8}
10^{10}
TIME (DIMENSIONLESS)

Figure 4



DRAWDOWN FROM ZONE NO. 3
DRAWDOWN FROM ZONE NO. 5
DRAWDOWN (DIMENSIONLESS)
TIME (DIMENSIONLESS)
0
1
2
3
4
10 −2
1
10 2
10 4
10 6
10 8
10 10

Figure 5

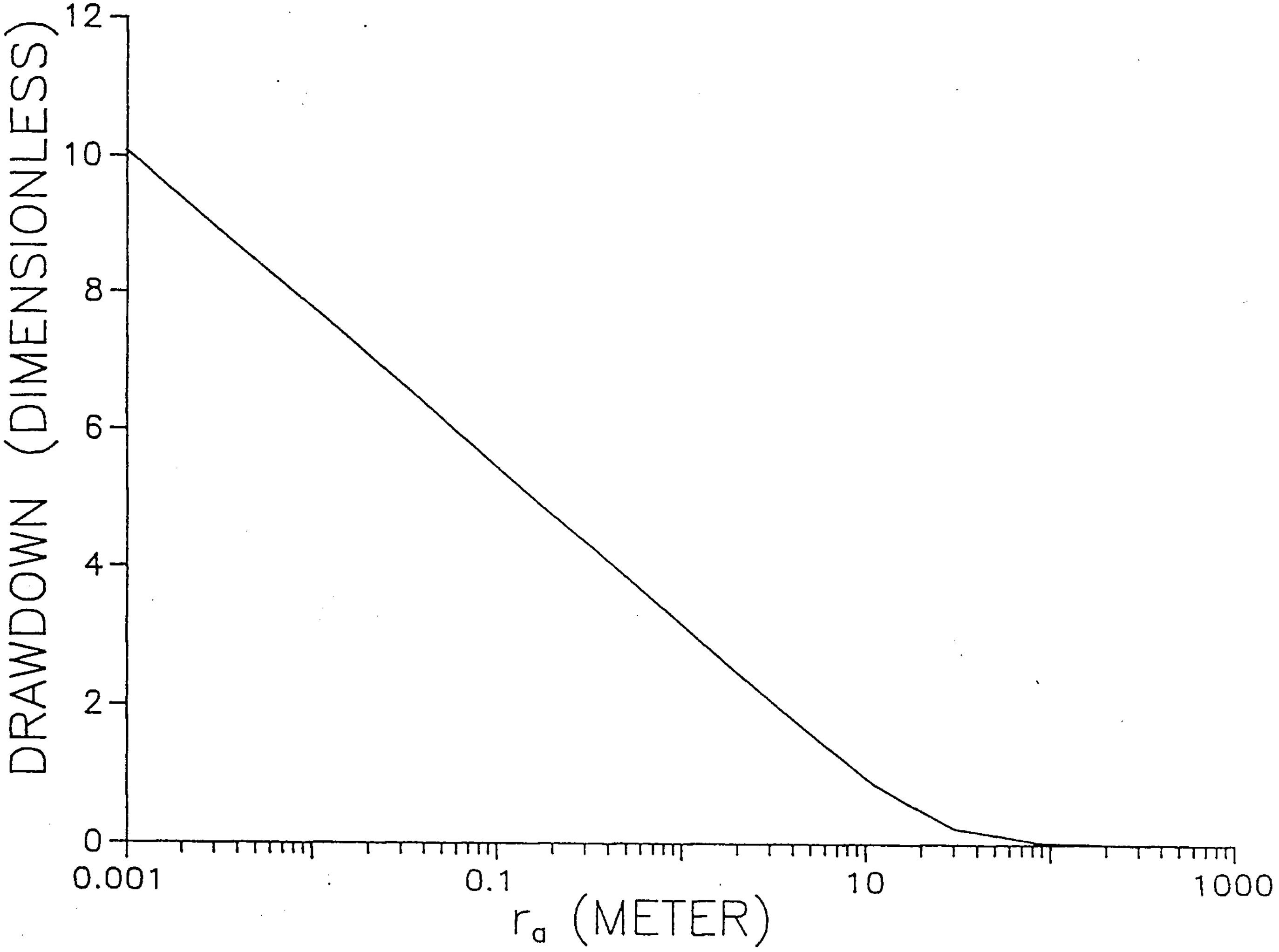
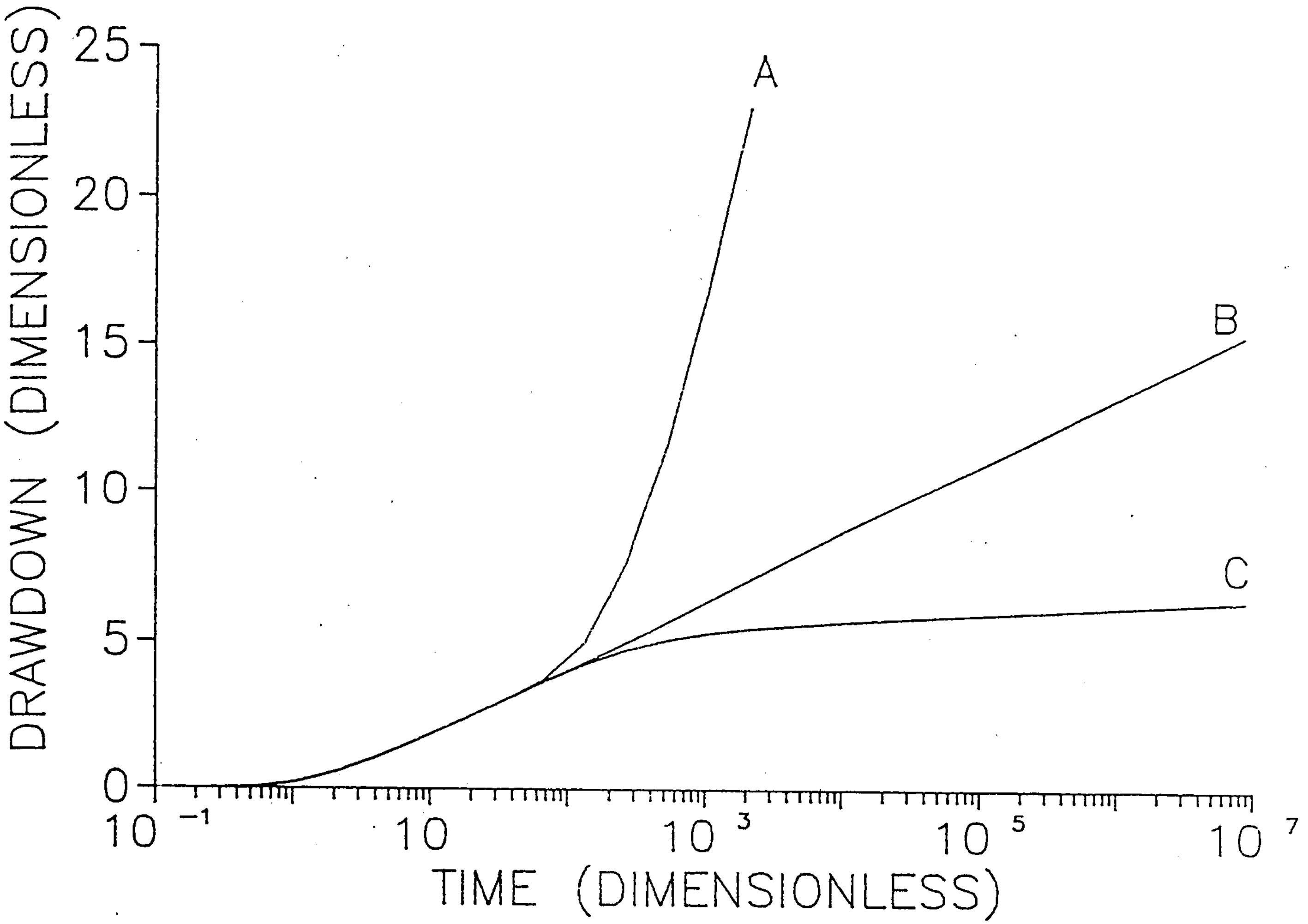
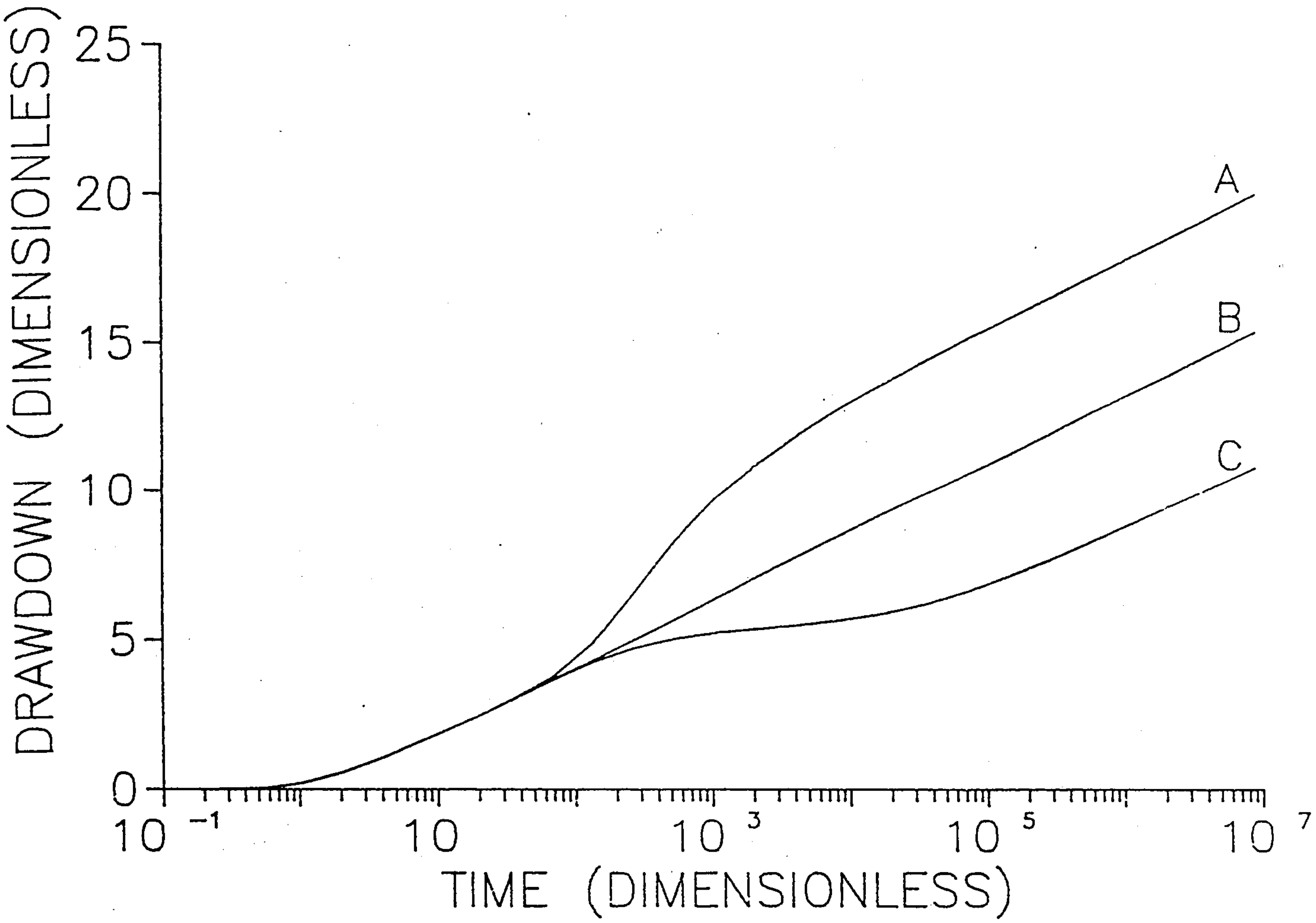


Figure 6a



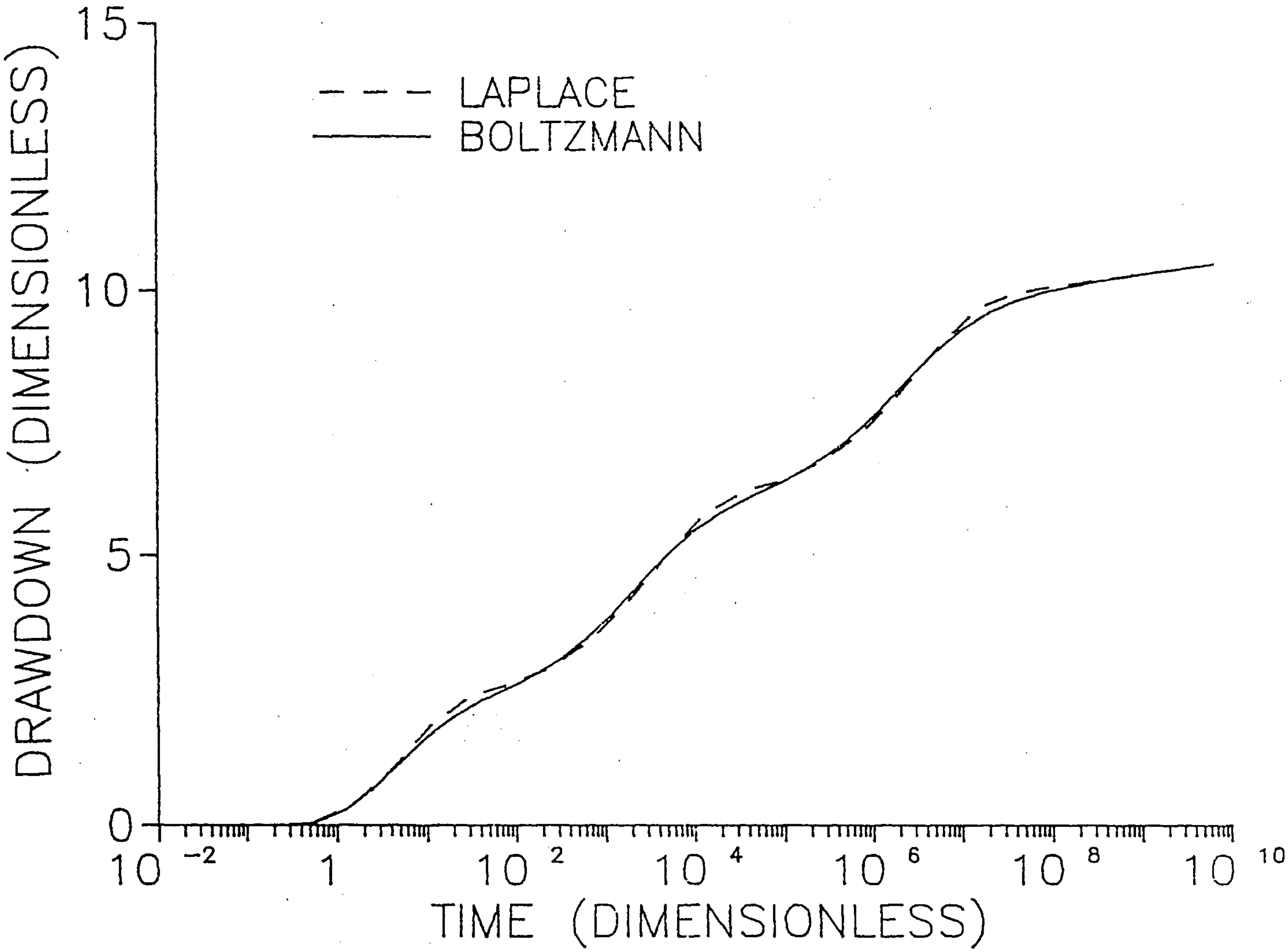
DRAWDOWN (DIMENSIONLESS)
25
20
15
10
5
0
10^{-1}
10
10^{3}
10^{5}
10^{7}
TIME (DIMENSIONLESS)
A
B
C

Figure 6b



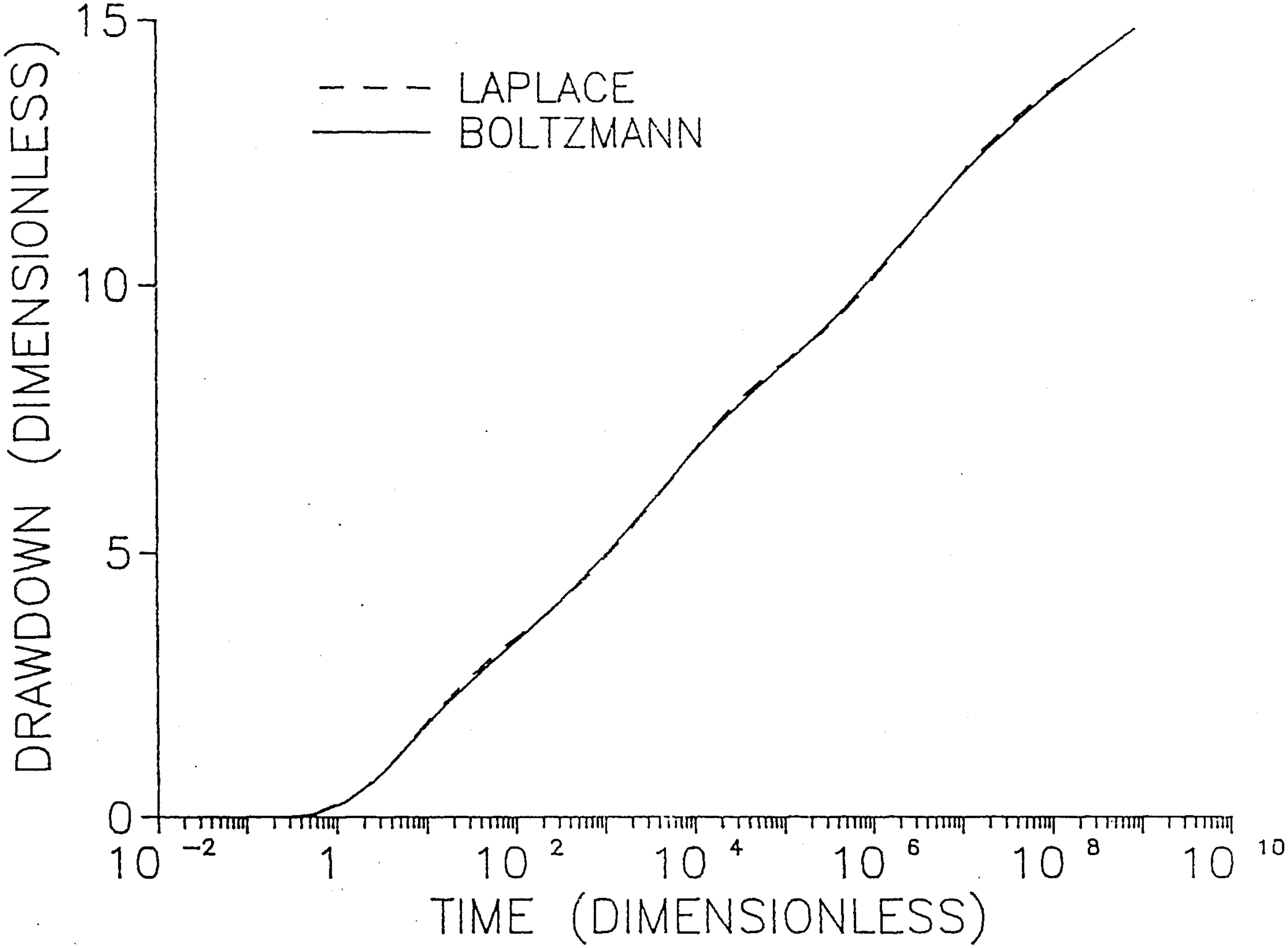
DRAWDOWN (DIMENSIONLESS)
25
20
15
10
5
0
A
B
C
10^{-1}
10
10^{3}
10^{5}
10^{7}
TIME (DIMENSIONLESS)

Figure 7a



15
10
5
0
DRAWDOWN (DIMENSIONLESS)
LAPLACE
BOLTZMANN
10^{-2}
1
10^{2}
10^{4}
10^{6}
10^{8}
10^{10}
TIME (DIMENSIONLESS)

Figure 7b



15
10
5
0
DRAWDOWN (DIMENSIONLESS)
LAPLACE
BOLTZMANN
10 −2
1
10 2
10 4
10 6
10 8
10 10
TIME (DIMENSIONLESS)