PUMPING TEST DESIGN AND ANALYSIS USING SENSITIVITY COEFFICIENTS

by

G.C. Bohling
Carl D. McElwee

Kansas Geological Survey
Open-file Report 89-34

PUMPING TEST DESIGN AND ANALYSIS USING SENSITIVITY COEFFICIENTS

G. C. Bohling and C. D. McElwee
Kansas Geological Survey
1930 Constant Ave., Campus West
Lawrence, KS 66047

Prepared for presentation at
The American Geophysical Union
Fall Meeting in San Francisco, CA
Dec. 4, 1989

KGS Open File Report #89-34

I. INTRODUCTION

Sensitivity coefficients describe how a system response, such as drawdown, varies with an incremental change in a system parameter, such as transmissivity. Sensitivity coefficients are used fairly often in groundwater studies. They are calculated in automated parameter estimation procedures and they are important in algorithms that determine experimental designs which minimize parameter variance for a given model. We have found that it is useful to study the behavior of the sensitivities themselves, since they have a physical meaning for physically based models.

II. PROGRAM FLOW CHART

At the Kansas Geological Survey, we are developing a program intended to aid in pumping test design and analysis. In design mode, the program calculates and allows the user to plot drawdown and the sensitivities to different parameters for a given pumping test configuration. In analysis mode, the program determines optimal parameter estimates based on drawdown data from an actual test.

The flow chart shown in Figure 1 summarizes the major features of the program. The various INPUT subroutines allow the user to specify pumping histories and drawdown data for a number of pumping wells and observation wells, determine which well function to use, and specify boundary configurations and anisotropy parameters.

The FORWARD SOLUTION module calculates the drawdowns and sensitivities at all the observation points, based on current parameter estimates. This module handles the convolution and superposition of the effects of multiple pumping wells with multiple pumping rates, the reflection of pumping wells across boundaries, and the transformation of X,Y coordinates based on the current anisotropy parameters. The well functions are contained in a separate module and are written as functions of a single observation time and radius for a single well pumping at a constant rate. The well functions are called after the forward solution module has isolated a single observation well-pumping well pair and taken anisotropy into account. This modular structure allows new well functions to be easily added, since most of the code is independent of which well function is used.

In analysis mode, the PARAMETER UPDATING module will determine improved parameter estimates for the current iteration of the fitting process. The parameter updating module is entirely independent of the specific pumping test configuration. If the system is anisotropic, the anisotropy parameters are updated along with the well function parameters.

A number of graphics options are available, including plots of drawdowns and sensitivities in both time and space.

III. TAYLOR EXPANSION

A Taylor's series expansion about the current value of calculated head, hm, forms the basis for the fitting process. In Figure 2, the Taylor expansion is written for the Theis function, as an example. This expression is an approximation, since we have dropped the non-linear terms from the expansion. h* is a vector of heads calculated from the (theoretical) true parameters, T*, S*. m is an iteration index. Ut and Us are the sensitivities of the current values of calculated head to transmissivity and storage coefficient, respectively. They are simply the first derivatives of head with respect to the parameters. DELTA-T and DELTA-S are unknown perturbations in the parameter values, to be determined at each step of the iteration process.

IV. PARAMETER ESTIMATION USING LINEARIZATION

The objective of the fitting process is to determine parameter values which minimize the sum squared difference between the observed and calculated heads, as shown in Figure 3. The index "i" used here runs over all observation locations and times. If we assume that the observed heads can be expressed as the sum of the vector of heads based on the true parameters and an error vector, epsilon, and substitute into the Taylor's series expansion shown in Figure 2, then we obtain the expression shown here. Here, the left-hand side is a vector of differences between the observed and calculated heads. We can now solve for the scalar values, DELTA-T and DELTA-S, using linear least squares regression. We add DELTA-T and DELTA-S to the current parameter estimates to get the estimates for the start of the next iteration. Repeatedly applying this process will (hopefully) bring the calculated heads closer to the observed heads.

V. NORMALIZED SENSITIVITIES

We have found it useful to reformulate the problem in the fashion shown in Figure 4. Each term on the right hand side of the regression equation can be multiplied and divided by its respective parameter value, producing a new equation written in terms of normalized sensitivities, Ut' and Us'. The unknowns to be solved for are now relative, rather than absolute, parameter variations, represented here by DELTA-T' and DELTA-S'. Notice that the normalized sensitivities can be written with the parameter dividing the differential in the denominator. This form is instructive: It shows that the normalized sensitivity to transmissivity, for example, is the amount that head will change with a unit ratio change in transmissivity.

Although the normalized sensitivity formulation is mathematically equivalent to the raw sensitivity formulation, the two problems have different numerical properties. In the actual fitting process, The sensitivities to the parameters are collected into a sensitivity matrix, which is then pre-multiplied by its transpose to form the sensitivity cross-products matrix. The sensitivity cross-products matrix is commonly called the design matrix. The solution of the regression problem involves inverting the design matrix, and the nature of this matrix determines the conditioning of the fitting procedure.

The condition number of the design matrix is a measure of the extent to which small errors in the calculated values of the sensitivities and in the calculated values of head differences will be magnified in the process of inverting the design matrix and multiplying by that inverse. We have found reductions of 4 to 6 orders of magnitude in condition number of the design matrix between a problem using raw sensitivities and the equivalent problem using normalized sensitivities. This reduction is due to the fact that the sensitivities to ratio changes in different parameters are comparable in magnitude, whereas raw sensitivities may differ by several orders of magnitude.

VI. THEIS SENSITIVITY TYPE CURVE GRAPH

Another advantage of using normalized sensitivities is that normalized sensitivities to different parameters can readily be plotted together and compared. A plot of sensitivities can help the user determine times at which to sample different observation wells in order to maximize the sensitivity to each parameter and minimize the correlation between sensitivities. Achieving these two objectives will minimize the variance of the resulting parameter estimates.

Figure 5 shows the plots of the normalized sensitivities to transmissivity (Ut') and to storage coefficient (Us'), along with the drawdown itself, for the Theis function. Here, the normalized sensitivities have been non-dimensionalized by dividing by a reference drawdown, s0, the same factor used to convert between dimensionless and dimensional drawdown. 1/U is directly proportional to time, so these curves essentially show the time-dependence of the sensitivities. 1/U is inversely proportional to the squared radius, so the radial dependence of the sensitivities is less obvious from this plot.

The correlation between the two parameters will depend on how these curves are sampled in a given pumping test. The inherent correlation between these two parameters, determined from the 100 points used to construct these curves, is about -0.85.

VII. SENSITIVITY TYPE CURVE FOR LEAKY FUNCTION

Figure 6 shows the normalized sensitivities to the parameters of the leaky aquifer function. s0 here is given by the same expression as for the Theis function, as is U. z is a second dimensionless variable, given by the product of radius and a leakage coefficient, L. L is the inverse of B, the parameter which is more often used in the leaky function. Notice that the magnitude of the sensitivities to all parameters is lower for the higher value of leakage, represented by the solid line. The maximum values of sensitivities also occur later for the higher leakage.

In this case, the correlation between the parameters will depend on the value of z. The correlation between transmissivity and leakage coefficient is quite high, at -0.99 for $z = 0.2$ and -0.98 for $z = 0.1$.

VIII. WELL LOCATIONS

Figures 7 - 12 are from a hypothetical case study which shows how one might use knowledge of the behavior of parameter sensitivities to help develop a pumping test design. It is assumed that an initial pumping test has been performed in a leaky aquifer using the well configuration shown Figure 7. The pumping well was pumped at a constant rate of 50 cubic feet per minute for 4096 minutes. Drawdown measurements were made in the three observation wells shown. There is also a constant head boundary running parallel to the vertical axis, 1000 feet to the right of the pumping well. It was found in this case that the correlations among the fitted parameters were quite high, partly because of the presence of the constant head boundary. The goal is to design a pumping test which will lower parameter correlation. The program is used in design mode to test possible strategies to achieve this goal.

IX. PUMPING HISTORY

Figure 8 shows a pumping rate history which is intended to take advantage of the fact that, for the leaky aquifer function, the sensitivity to storage returns to zero after a certain time. Here, the pumping rate is increased every 1024 minutes, a period somewhat longer than the time required for sensitivity to storage to return to zero.

X. SENSITIVITIES AT OW1

Figure 9 is a plot of the normalized sensitivities to all three parameters at observation well 1, 70 feet to the right of the pumping well. The measurement times are shown by the plotted points. The fact that the sensitivity to storage coefficient returns to zero while the sensitivities to the other parameters continue to increase in magnitude with every increase in pumping rate greatly reduces the correlation between the storage coefficient and the other two parameters. The resulting correlation between the parameters storage coefficient and transmissivity, in this case, is -0.43. When the well is pumped at a constant rate for the entire 4096 minutes and the observation wells are sampled with a similar schedule, the correlation between transmissivity and storage coefficient is -0.82.

However, even with the stepped pumping history, the correlation between transmissivity and leakage coefficient is still quite high, at -.95.

XI. MAP OF SENS. TO TRANSMISSIVITY

Although the sensitivities to transmissivity and leakage coefficient are quite similar when plotted versus time, theses two sensitivities do exhibit distinctly different behavior in the spatial domain. Figure 10 shows a map of the sensitivity to transmissivity after 4096 minutes of constant pumping. The peak sensitivity is at the pumping well itself, while there is zero sensitivity along the constant head boundary.

XII. MAP OF SENS. TO LEAKAGE

Figure 11 shows a map of the sensitivity to leakage coefficient at the same time. It is clear that the peak sensitivity to leakage has moved to the left of the pumping well, away from the constant head boundary. Placing a fourth observation well to the left of the pumping well will help to reduce the correlation between transmissivity and leakage by taking advantage of this difference in spatial behavior.

XIII. PARAMETER CORRELATIONS

In design mode, the proposed fourth observation well can be added, in addition to the original three, and the resulting parameter correlations can be calculated. In Figure 12 the correlations are plotted as a function of the distance of the fourth observation well from the pumping well. The stepped pumping rate history shown earlier is employed and the drawdowns are measured according to the same schedule shown before. The minimum correlation between transmissivity and leakage coefficient, about -.86, occurs when the fourth observation well is located 600 feet away from the pumping well. Although this is still a fairly high correlation, it is a substantial improvement over -.95.

XIV. CONCLUSIONS

Thus, plotting and analyzing the behavior of sensitivities can be a powerful tool to aid in pumping test design. The use of normalized sensitivities not only improves the conditioning and reliability of automated parameter estimation processes, but also eases the graphical comparison of sensitivities to different parameters.

PROGRAM FLOW CHART

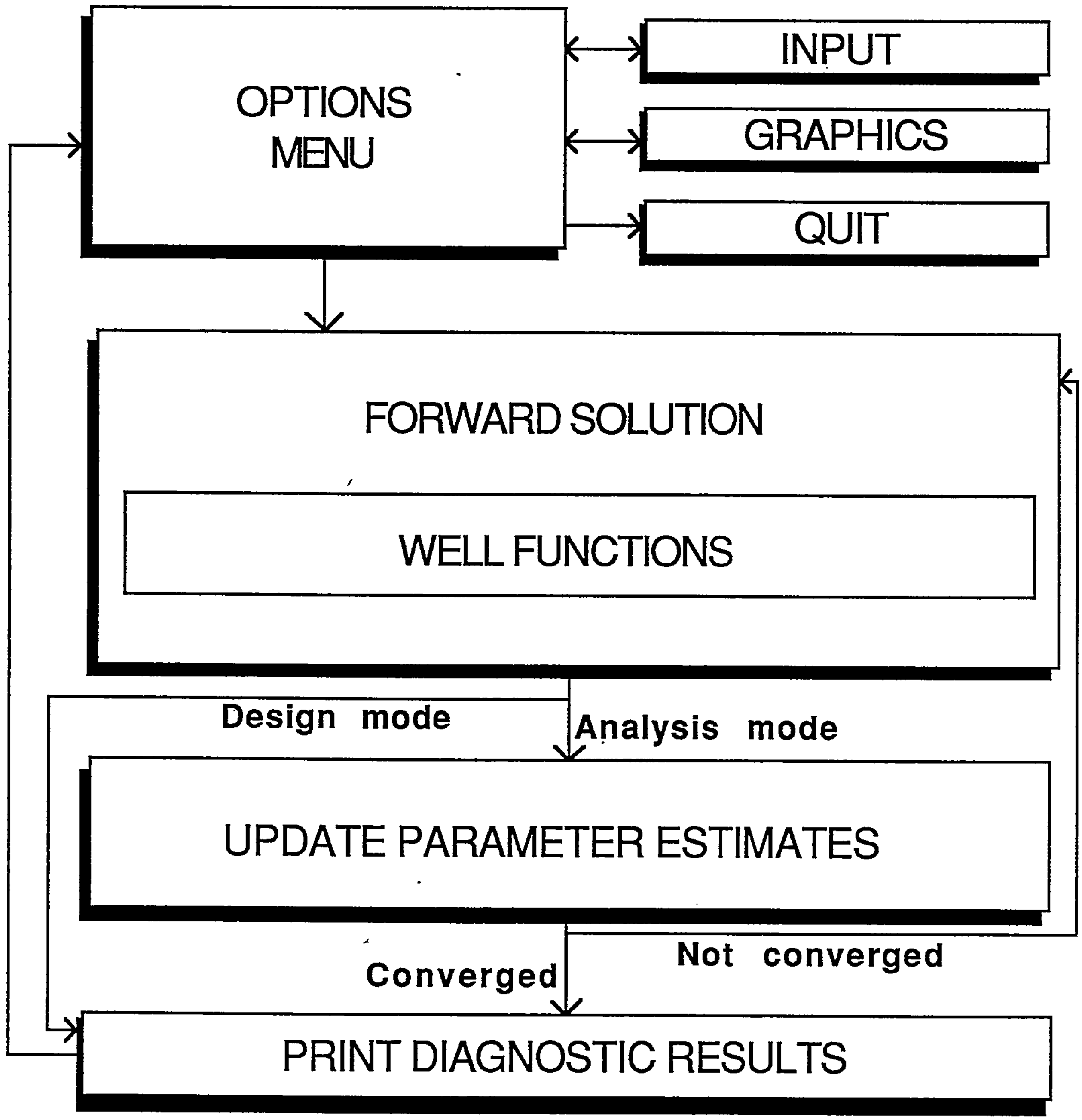


Figure 1

TAYLOR EXPANSION FOR THEIS FUNCTION

$$\mathbf{h}^* \cong \mathbf{h}^m + \mathbf{U}_T^m \Delta T^m + \mathbf{U}_S^m \Delta S^m$$

$\mathbf{h}^*$ = vector of heads based on true parameters T^*, S^*

$\mathbf{h}^m$ = vector of heads based on current estimates T^m, S^m

$\mathbf{U}_T^m = \dfrac{\partial \mathbf{h}^m}{\partial T^m}$ = vector of sensitivities to T^m

$\mathbf{U}_S^m = \dfrac{\partial \mathbf{h}^m}{\partial S^m}$ = vector of sensitivities to S^m

ΔT^m = unknown perturbation in transmissivity

ΔS^m = unknown perturbation in storage coefficient

Figure 2

PARAMETER ESTIMATION USING LINEARIZATION

OBJECTIVE: Minimize $E = \sum_i \left[h_i^e - h_i \right]^2$

h_i^e = observed head at index point i

h_i = calculated head at index point i

ASSUME: $\mathbf{h}^e = \mathbf{h}^* + \varepsilon$

ε = error vector

THUS: $\mathbf{h}^e - \mathbf{h}^m = \mathbf{U}_T^m \Delta T^m + \mathbf{U}_S^m \Delta S^m + \varepsilon$

$$T^{m+1} = T^m + \Delta T^m$$

$$S^{m+1} = S^m + \Delta S^m$$

Figure 3

NORMALIZED SENSITIVITIES

CAN WRITE: $\mathbf{h}^{e} - \mathbf{h} = T\,\mathbf{U}_T\left(\frac{\Delta T}{T}\right) + S\,\mathbf{U}_S\left(\frac{\Delta S}{S}\right)$

OR: $\mathbf{h}^{e} - \mathbf{h} = \mathbf{U}'_T \Delta T' + \mathbf{U}'_S \Delta S'$

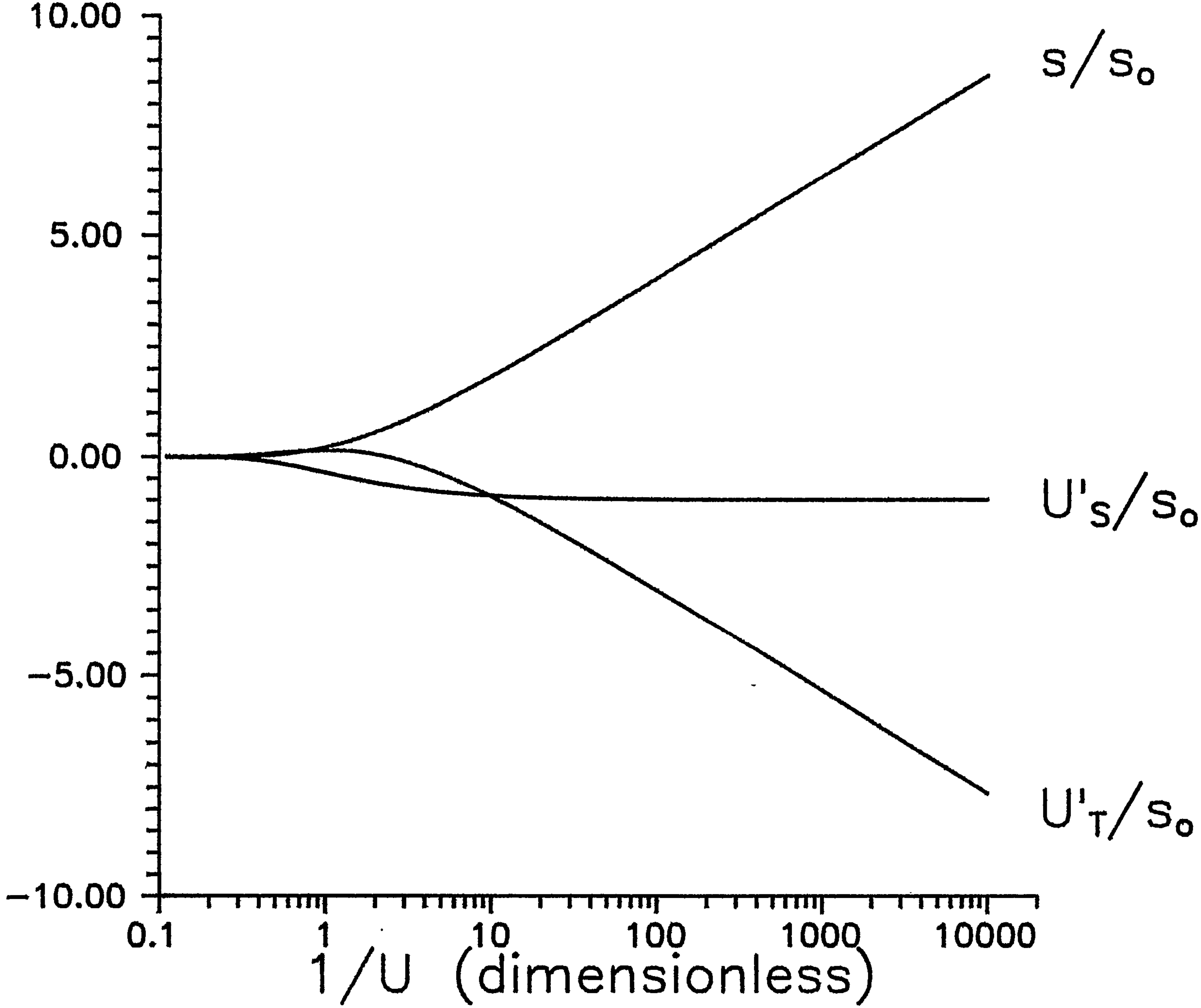
$$\mathbf{U}'_T = T\frac{\partial \mathbf{h}}{\partial T} = \frac{\partial \mathbf{h}}{\partial T\,/\,T} = \text{normalized sensitvity to } T$$

$$\mathbf{U}'_S = S\frac{\partial \mathbf{h}}{\partial S} = \frac{\partial \mathbf{h}}{\partial S\,/\,S} = \text{normalized sensitvity to } S$$

$$\Delta T' = \frac{\Delta T}{T}$$

$$\Delta S' = \frac{\Delta S}{S}$$

Figure 4



10.00
5.00
0.00
−5.00
−10.00
0.1
1
10
100
1000
10000
1/U (dimensionless)
S/So
U'S/So
U'T/So

Figure 5

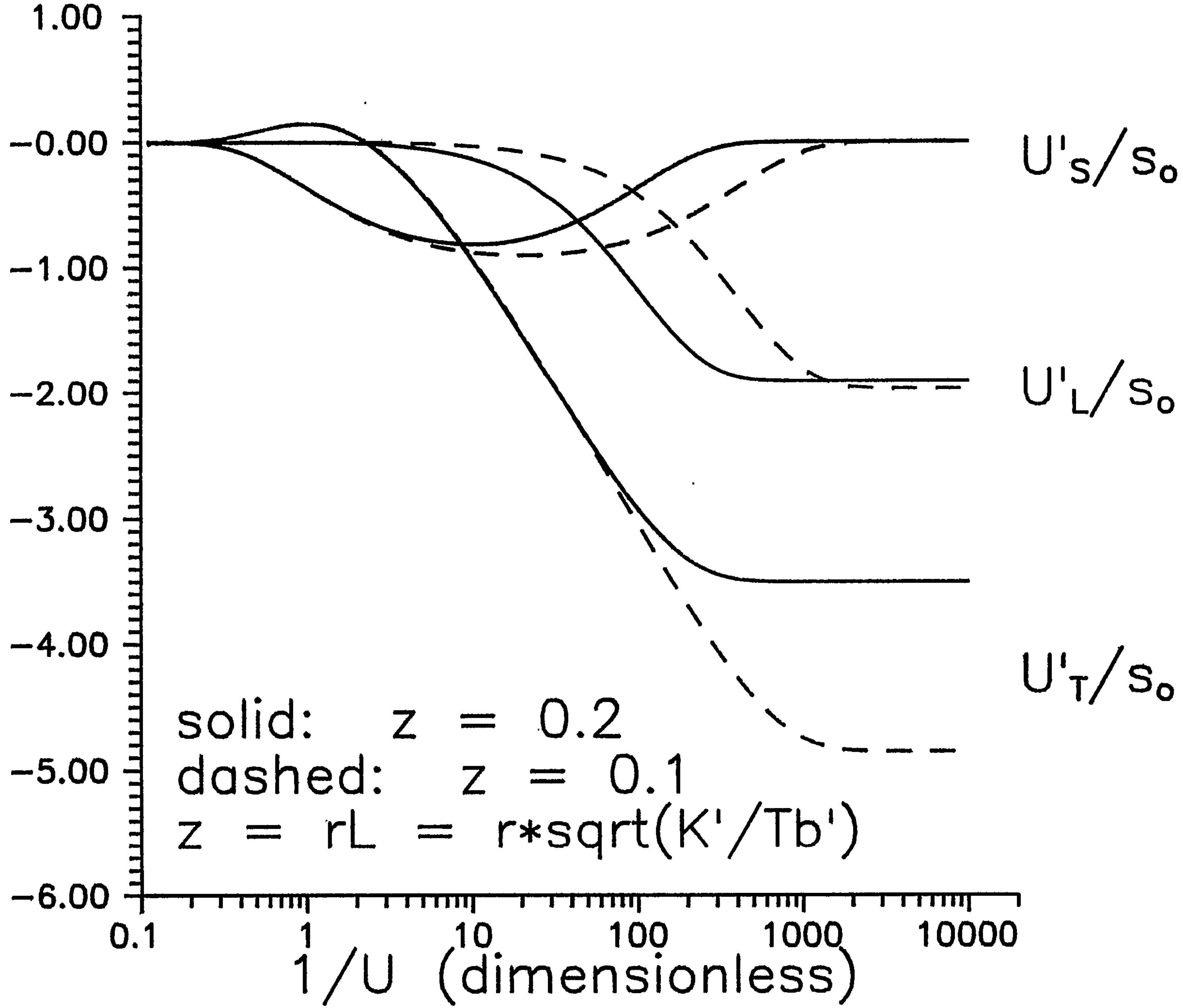
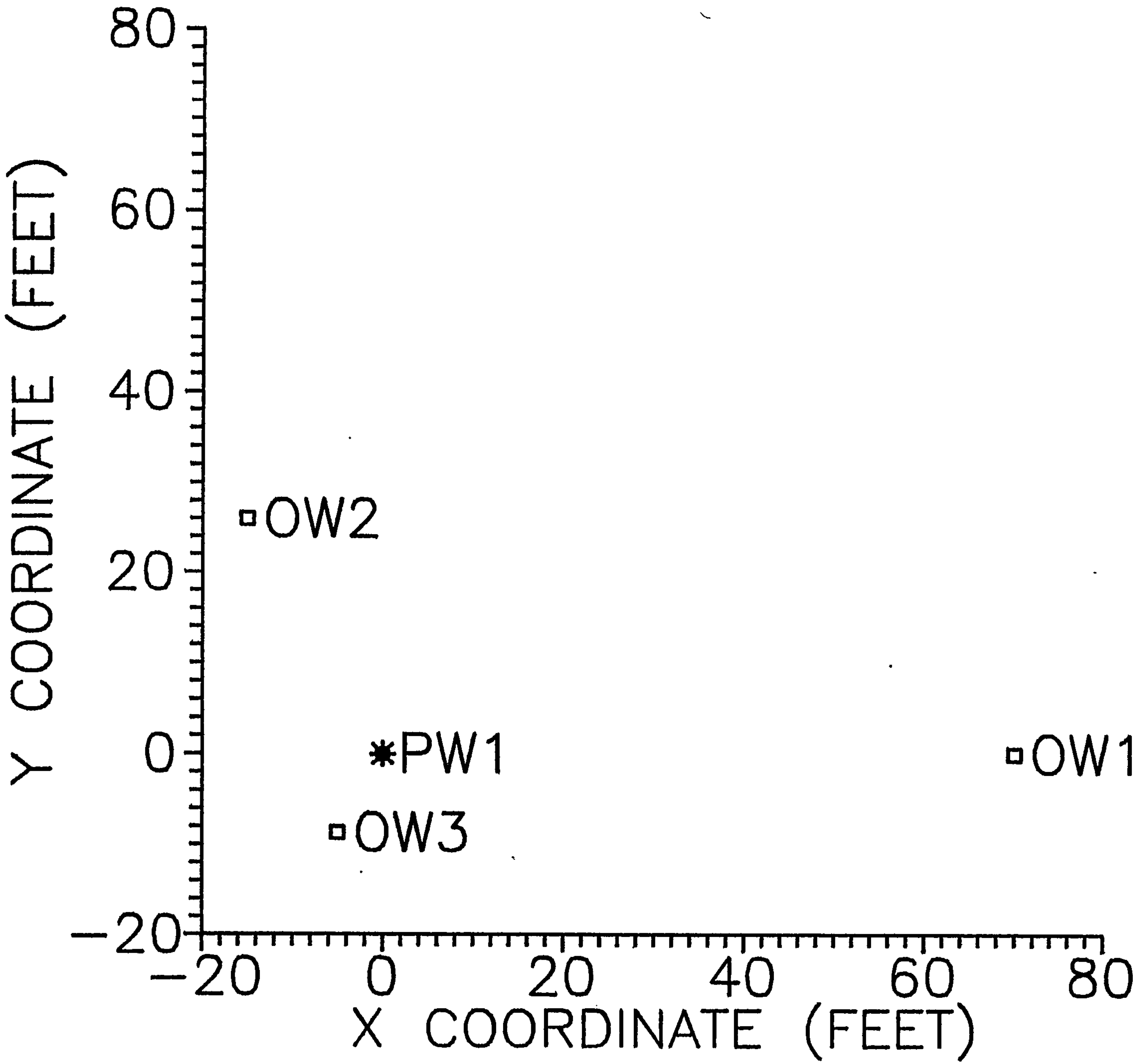
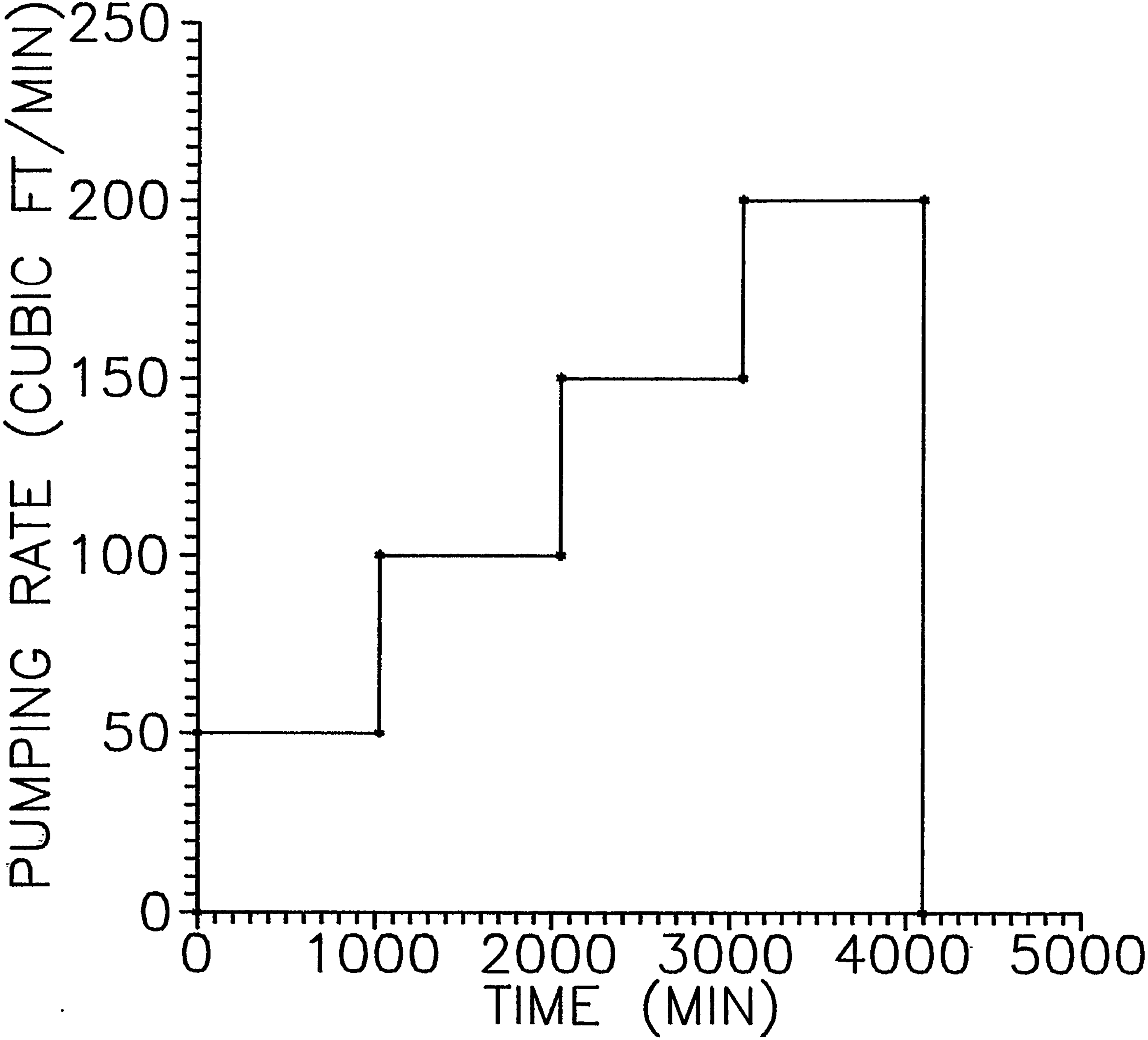

1.00
-0.00
-1.00
-2.00
-3.00
-4.00
-5.00
-6.00
0.1
1
10
100
1000
10000
1/U (dimensionless)
U'S/So
U'L/So
U'T/So
solid: z = 0.2
dashed: z = 0.1
z = rL = r*sqrt(K'/Tb')

Figure 6



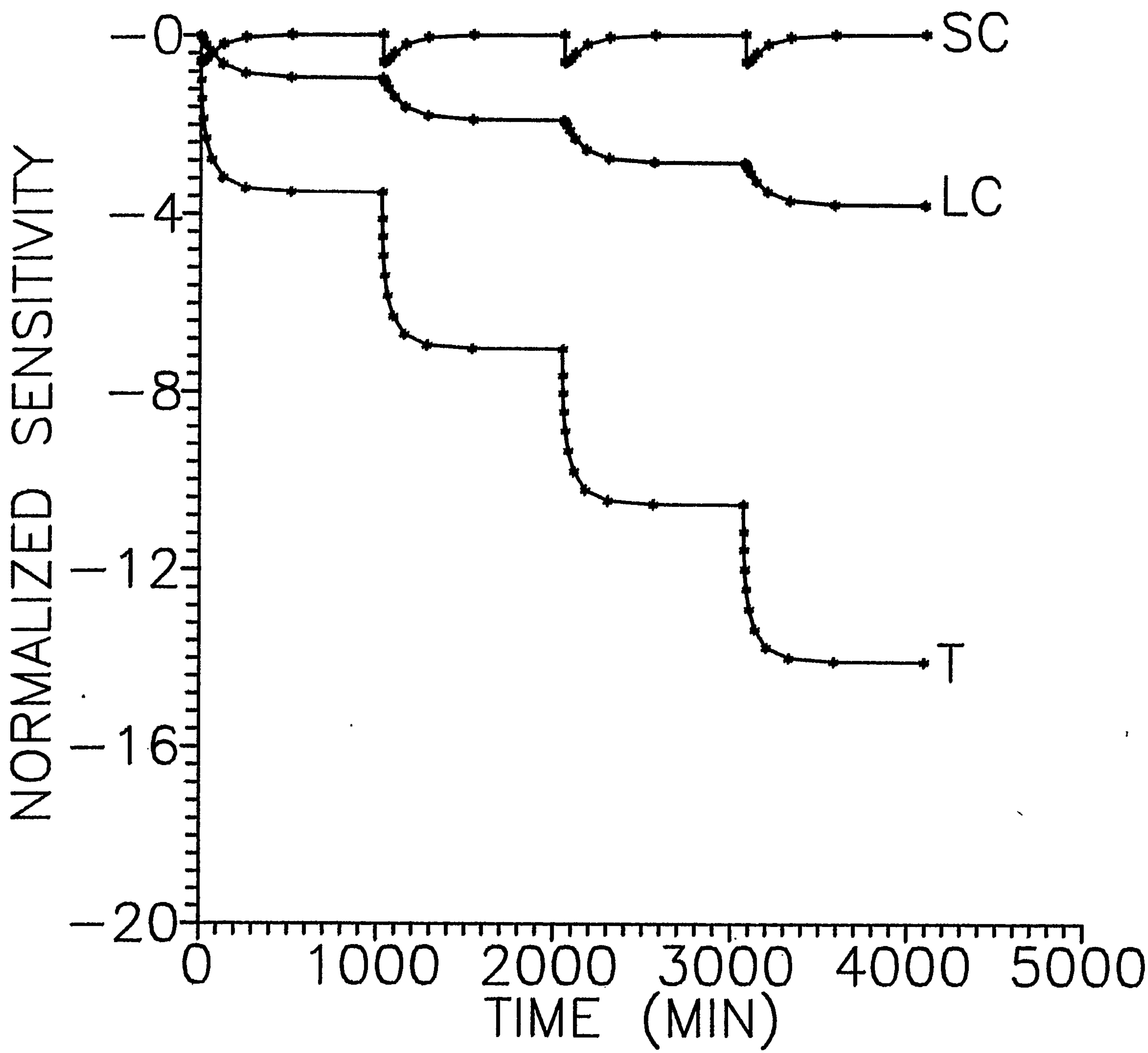
80
60
40
20
0
−20
Y COORDINATE (FEET)
OW2
PW1
OW1
OW3
−20
0
20
40
60
80
X COORDINATE (FEET)

Figure 7



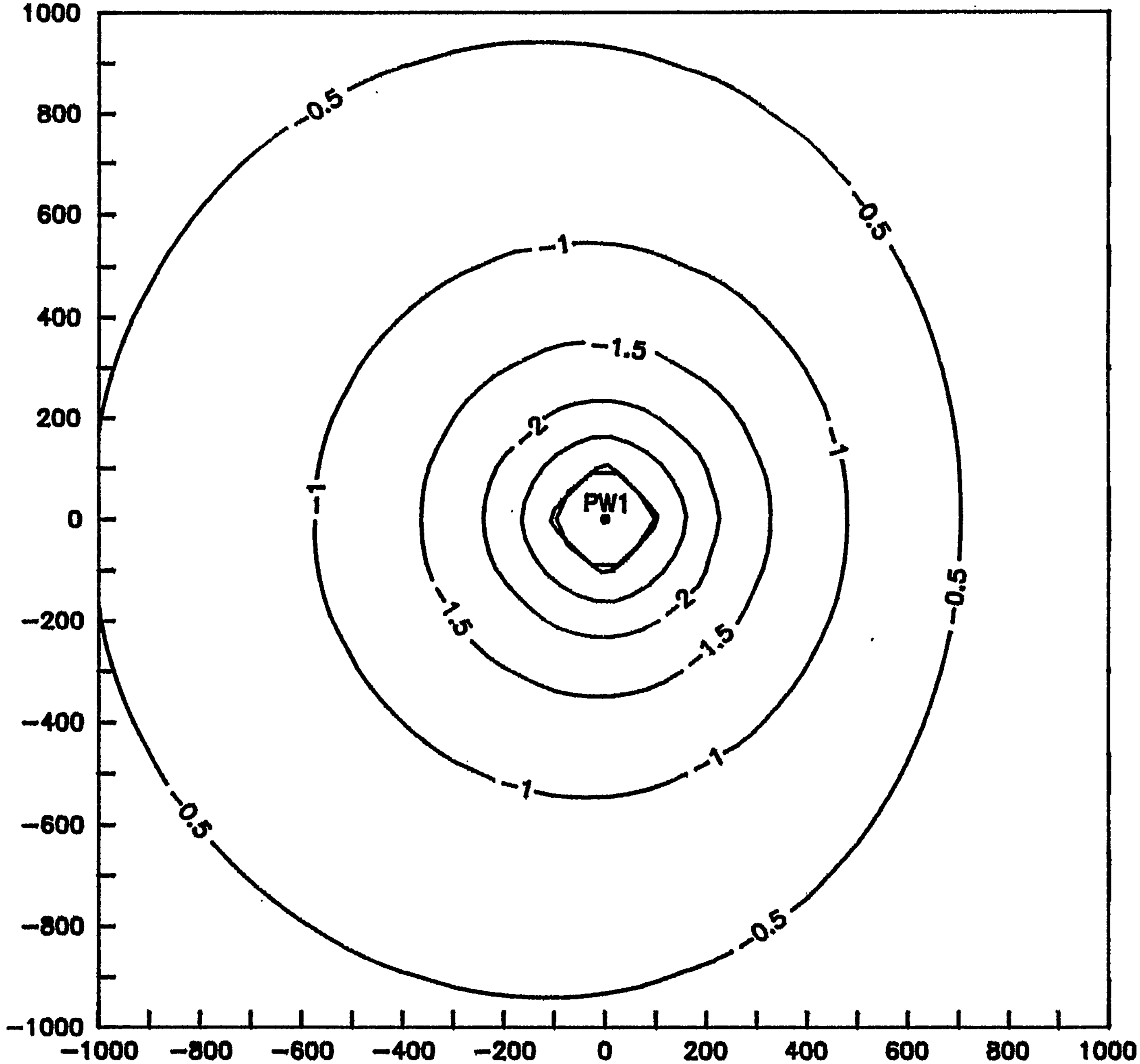
PUMPING RATE (CUBIC FT/MIN)
250
200
150
100
50
0
0
1000
2000
3000
4000
5000
TIME (MIN)

Figure 8



0
-4
-8
-12
-16
-20
NORMALIZED SENSITIVITY
0
1000
2000
3000
4000
5000
TIME (MIN)
SC
LC
T

Figure 9



1000
800
600
400
200
0
−200
−400
−600
−800
−1000
−1000
−800
−600
−400
−200
0
200
400
600
800
1000
-0.5
-1
-1.5
-2
PW1

Figure 10

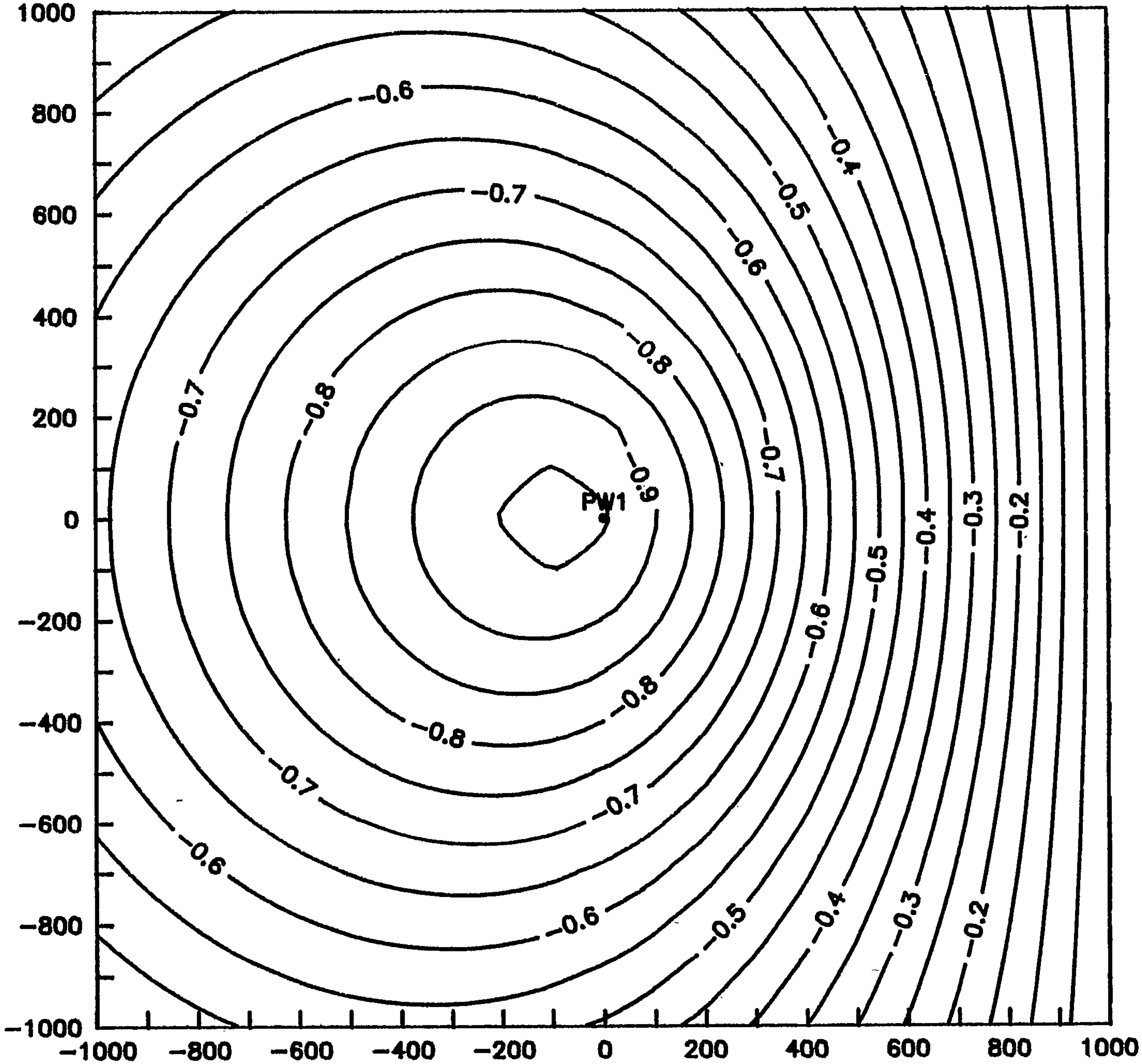

1000
800
600
400
200
0
−200
−400
−600
−800
−1000
−1000
−800
−600
−400
−200
0
200
400
600
800
1000
PW1
−0.9
−0.8
−0.7
−0.6
−0.5
−0.4
−0.3
−0.2

Figure 11

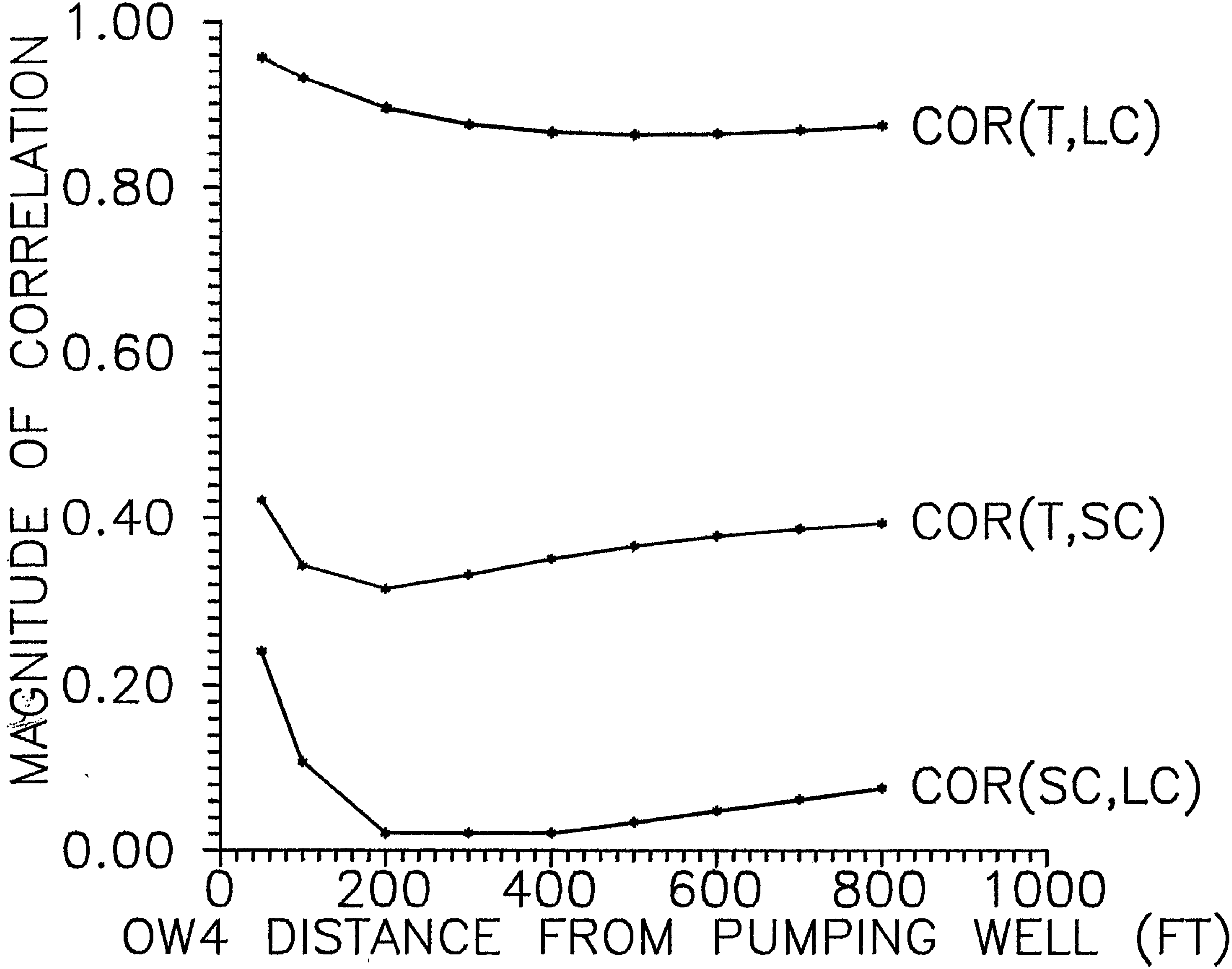

MAGNITUDE OF CORRELATION
1.00
0.80
0.60
0.40
0.20
0.00
0
200
400
600
800
1000
OW4 DISTANCE FROM PUMPING WELL (FT)
COR(T,LC)
COR(T,SC)
COR(SC,LC)

Figure 12