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SENSITIVITY ANALYSIS OF SLUG TESTS

by

Carl D. McElwee
G.C. Bohling
J.J. Butler, Jr.

Kansas Geological Survey
1930 Constant Avenue
University of Kansas
Lawrence, KS 66047-3726

SENSITIVITY ANALYSIS

OF SLUG TESTS

Carl D. McElwee, G. C. Bohling,
and J. J. Butler, Jr.
Kansas Geological Survey
1930 Constant Ave.
Lawrence, KS 66047

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INTRODUCTION

The use of slug tests for site characterization is a common practice in hydrogeology due to the logistical and financial advantage of the procedure. Simple tests can be performed quickly and inexpensively with a minimum of equipment. However, to insure that the maximum amount of information is obtained, the tests should be planned carefully. A sensitivity analysis of slug-test responses can provide valuable information concerning optimal test design. Such an analysis allows issues regarding the quantity and timing of measurements, and the magnitude of the initial displacement to the addressed.

ANALYTICAL SOLUTION

The Cooper-Bredehoeft-Papadopulas analytical solution is shown in this slide. It shows that slug-test responses can be expressed as a function of three parameters: H_0, the initial head displacement; alpha, a parameter related to screen and casing radii and the storage coefficient; and beta, a dimensionless time involving transmissivity and the casing radius. Many times it is convenient to define relative head by dividing by H_0 as shown on this slide for alpha=.001. When plotted versus beta, the dimensionless time, it forms a generic type curve that is valid for any transmissivity with the same alpha. If alpha is allowed to vary, a family of type curves is generated. Graphical curve matching to this family of curves has been the traditional method of obtaining aquifer parameters. It is well known that it is difficult to obtain a unique match for alpha since the curves are very similar in shape.

SENTITIVITY ANALYSIS

Sensitivity analysis can be used to look at the effect of various parameters on the head. A first order Taylor series expansion can be performed and used to define sensitivity coefficients of the various parameters. In this case T, S, and H_0. The sensitivity coefficients are simply derivatives of the head with respect to the parameter, and may be evaluated analytically or numerically. Changing the parameters by delta allows the new head to be estimated using the sensitivity coefficients and the first order Taylor series expansion. The sensitivity coefficients are a measure of how much the head changes when a parameter is changed by a small amount, and therefore, are very useful to examine in some detail. However, it is very difficult to compare the relative effects of certain parameters by looking at these coefficients because the numerical values of the parameters vary widely. A more diagnostic sensitivity coefficient is defined in this slide. Normalized sensitivities are obtained by multiplying by the parameter. These are indicated by a prime. Parameter variations are now defined as the delta value over the parameter, and are expressed as a decimal fraction of the original parameter. The normalized sensitivity coefficients can be plotted on the same graph and will give a good indication of the relative importance of various parameters.

One other type of sensitivity coefficient will also be useful. Each of the previously defined normalized sensitivity coefficients can be written explicitly showing how H_0, the initial displacement, and h, the relative head enter. This slide clearly shows that the normalized sensitivity coefficients are directly proportional to H_0. Thus, the height of the initial displacement should be as large as possible to achieve greater sensitivity. If H_0 is factored out of each we are left with the normalized sensitivities to relative head; denoted by small u prime in this slide. These sensitivity functions are only functions of alpha and beta and are generic for any set of parameters. For example, the next slide shows one curve for u'_T and u'_S for alpha=.001; all possible values of T are represented by these two curves as long as alpha remains constant. Note that this plot clearly shows that the system is much more sensitive to T than S. The sensitivities to T and S peak at discrete values of beta, and have a certain characteristic width. In general, measurements should be taken such that a good number are made near maximum sensitivity and such that the shape of the sensitivity curve through time is well represented. Plotting sensitivities allows the quantity and timing of measurements to be appropriately selected. The next slide shows sensitivity curves of T and S for three different values of alpha. The sensitivity to T increases and that to S decreases with decreasing alpha. Therefore, as the casing radius increases or S decreases, slug-test responses are less sensitive to S.

The next slide shows the normalized sensitivity of relative head to H_0 which is actually just the relative head. In some slug tests an offset is observed between the measured and fitted initial displacement H_0. Therefore, it may be useful to use H_0 as a fitted parameter, and utilize the sensitivity to H_0 shown here. Notice that the largest sensitivity is at small times. In other words, early measurements are more useful in determining H_0, not a surprizing result.

PARAMETER ESTIMATION AND ERROR ANALYSIS

A common method of performing parameter estimation involves minimizing a squared error functional as shown in this slide. An attempt is made to minimize the error between the observed head and the calculated head by changing the parameters a small amount delta. A computer automated least squares technique using sensitivity analysis has been implemented to calculate the deltas and iterate until convergence is achieved. The superscript m is an iteration index.

The sensitivity design matrix [A] defined in this slide is a sum over time of products for any two sensitivity coefficients. If we are fitting all three parameters H_0, T, and S then the sensitivity design matrix is 3 x 3. The least squares solution for the delta parameter changes can be expressed in terms of the inverse of [A]. In general, the solution is well behaved if the diagonal elements are large and nearly equal and the off-diagonal elements are small. This will be the case if the sensitivity coefficients are large and do not have similar shapes over the measurement times. One way to measure the similarity of the sensitivity coefficients is to define the sensitivity correlation matrix as shown in this slide; it will have ones on the diagonal and the off-diagonal terms will vary between $\pm$ 1. If any of the off-diagonal terms are exactly one, the inverse of [A] does not exist and the inverse problem can not be solved for aquifer parameters. From a practical standpoint, anytime the off-diagonal elements of the correlation matrix get above .9 the [A] matrix becomes ill-conditioned rather rapidly and the inverse solution becomes more unreliable.

As long as an inverse of [A] can be found, the reliability of the parameter estimates can be assessed by looking at the parameter covariance matrix defined in this slide. The form shown here results from some simplifying assumptions about the errors in head such as additive, zero mean, noncorrelated and constant variance. With these assumptions the estimated standard errors of the parameters are given by the square roots of the diagonal elements of the parameter covariance matrix.

These ideas can be applied to the generic sensitivity functions u'_T and u'_S shown earlier which were functions only of alpha and beta. The next slide shows the correlation between u'_T and u'_S as a function of alpha. The sensitivity coefficients have been sampled twenty times per log cycle for beta ranging from 10^{-3} to 10^{-5}. Notice that the correlation between S and T increases as alpha decreases and goes to about .99 when alpha is 10^{-5}.

However, even if alpha is four orders of magnitude greater at 10^{-1} the correlation is still about .91. This is the reason why it is usually difficult to determine both S and T reliably from a slug test. The next slide shows the estimated standard error for S and T as a function of alpha for the same sampling and assuming the standard error in head is about one percent of H_0. Notice that the error in T remains below 6% while the error in S climbs to about 66% when alpha is 10^{-5}. This reflects the fact shown earlier that the sensitivity coefficient maximum for T is much greater than that for S.

TYPICAL APPLICATION

The next series of slides are produced using some typical parameters that we have found recently for slug tests in the Kansas River alluvium and will be used to illustrate some of the points made earlier. This slide shows the normalized sensitivity coefficients for H_0, T and S when the hydraulic conductivity is 25 ft./ day, the storage coefficient is 10^{-4}, and the radii r_c and r_s are both one inch: This means alpha and S are the same. The initial displacement of water H_0 is 300 inches or about 25 feet. The screened interval on the well is 30 inches. The maximum normalized sensitivity to H_0 occurs at small times and is about 300. However, the maximum values for the normalized sensitivity to T and S occur at about 25 seconds and are -100 for U_T and about -10 for U'_S This clearly shows the system is less sensitive to S. Notice that if data had only been taken during the first 20 seconds the curves have very similar shapes and the correlation between parameters would be large. The majority of information about this slug test will be obtained in the first 100 seconds of data; after that, all sensitivities are low and decreasing. For good definition of H_0, a good number of measurements at early time are needed. However, for the best definition of T and S a good sampling of measurements should occur near the sensitivity maxima around 25 seconds.

The next slide shows the effect of the initial displacement H_0 assuming a standard error in head of .25 inches. We saw earlier that the normalized sensitivities were directly proportional to H_0, so the estimated standard error should be inversely proportional to H_0. We see that as the initial head is varied from 50 to 300 inches the estimated standard error of all parameters decreases by nearly an order of magnitude. This figure was produced by assuming a thousand equally spaced measurements over 250 seconds.

The next slide shows the effect of the number of equally spaced measurements over the first 250 seconds. We have plotted points for 10, 50, 100 and 1000 measurements. There is a significant decrease in the estimated standard error for all parameters as the number of measurements is increased. Looking at this kind of data allows one to plan the number of measurements necessary for the desired experimental accuracy.

Placement of measurements in time is also important as we have noted earlier, because sampling at maximum sensitivity and minimizing correlation are both critical. This slide shows the predicted estimated standard error for the parameters when 50 measurements one second apart are used at different time periods. The moving center point of this measurement interval is plotted versus estimated standard error here. The lowest estimated standard errors occur when measurements are made over the first 50 seconds which includes sampling of all the sensitivity maxima. In general, as the 50 measurements are moved to larger times the estimated standard error increases due to decreasing sensitivity at larger times. However, the curve for S shows a kink which is caused by changing correlation with T.

CONCLUSIONS

In conclusion, sensitivity analysis is useful for designing better slug tests. T and S usually can not both be reliably estimated due to high correlation. Estimated standard errors of the parameters are inversely proportional to H_0. Therefore, large initial heads should be used. Generally, an increased number of measurements improves parameter estimation. Measurement times should be selected to minimize sensitivity correlation and to sample at points of maximum sensitivity.

Cooper - Bredehoeft - Papadopulos Analytical Solution for Slug Tests

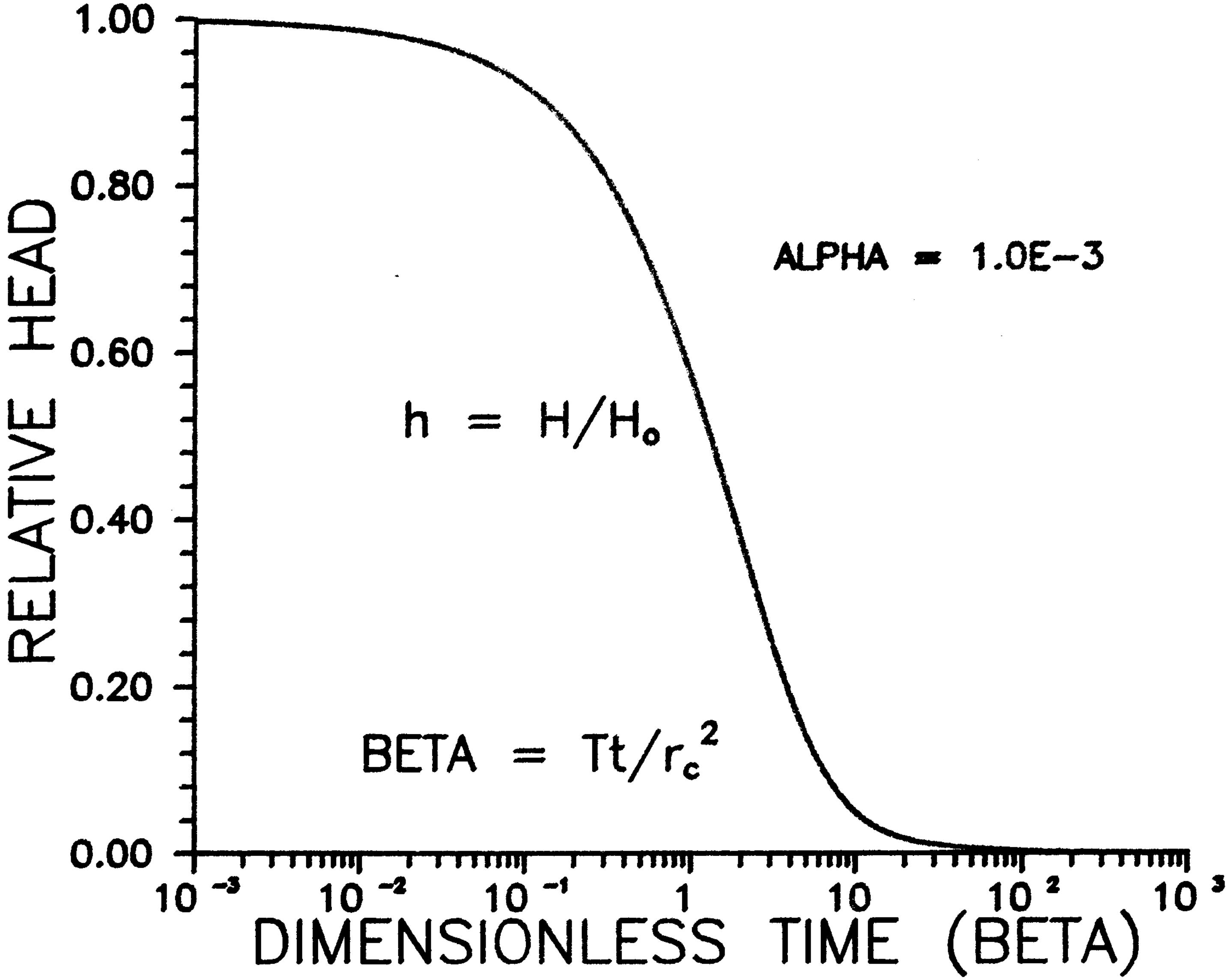
$$H = \frac{8H_o\alpha}{\pi^2} \int_0^\infty \frac{e^{-\beta u^2/\alpha}}{u\,\Delta(u)}\,du$$

$$\Delta(u) = \left[\, u J_0(u) - 2\alpha J_1(u) \right]^2 + \left[\, u Y_0(u) - 2\alpha Y_1(u) \right]^2$$

$$\beta = \frac{Tt}{r_c^2}$$ J, Y are Bessel Functions

$$\alpha = \frac{r_s^2}{r_c^2} S$$ r_s and r_c are screen and casing radii

$$\frac{H}{H_0} = h \equiv$$ relative head


RELATIVE HEAD
1.00
0.80
0.60
0.40
0.20
0.00
ALPHA = 1.0E-3
h = H/H$_o$
BETA = Tt/r$_c$^{2}
10^{-3} 10^{-2} 10^{-1} 1 10 10^{2} 10^{3}
DIMENSIONLESS TIME (BETA)

TAYLOR EXPANSION FOR SLUG TEST FUNCTION

$$\mathbf{H}^* \cong \mathbf{H}^m + \mathbf{U}_T^m \Delta T^m + \mathbf{U}_S^m \Delta S^m + \mathbf{U}_{H_0}^m \Delta H_0^m$$

$\mathbf{H}^*$: vector of heads based on true parameters T^*, S^*, H_0^*

$\mathbf{H}^m$: vector of heads based on current estimates T^m, S^m, H_0^m

$\mathbf{U}_T^m = \frac{\partial \mathbf{H}^m}{\partial T^m}$, $\mathbf{U}_S^m = \frac{\partial \mathbf{H}^m}{\partial S^m}$, $\mathbf{U}_{H_0}^m = \frac{\partial \mathbf{H}^m}{\partial H_0^m}$: sensitivities to T^m, S^m, H_0^m

ΔT^m, ΔS^m, ΔH_0^m : unknown perturbations in transmissivity, storage coefficient, and initial head

NORMALIZED SENSITIVITIES

CAN WRITE: $$\mathbf{H}^e - \mathbf{H} = T\,\mathbf{U}_T\left(\frac{\Delta T}{T}\right) + S\,\mathbf{U}_S\left(\frac{\Delta S}{S}\right) + H_0\,\mathbf{U}_{H_0}\left(\frac{\Delta H_0}{H_0}\right)$$

OR: $$\mathbf{H}^e - \mathbf{H} = \mathbf{U}'_T \Delta T' + \mathbf{U}'_S \Delta S' + \mathbf{U}'_{H_0} \Delta H'_0$$

$$\mathbf{U}'_T = T\frac{\partial \mathbf{H}}{\partial T}\ ,\ \mathbf{U}'_S = S\frac{\partial \mathbf{H}}{\partial S}\ ,\ \mathbf{U}'_{H_0} = H_0\frac{\partial \mathbf{H}}{\partial H_0}\ :$$

normalized sensitvities to T, S, H_0

$$\Delta T' = \frac{\Delta T}{T} \qquad \Delta S' = \frac{\Delta S}{S} \qquad \Delta H'_0 = \frac{\Delta H_0}{H_0}$$

NORMALIZED SENSITIVITIES
TO RELATIVE HEAD

$$U'_T = T\frac{\partial H}{\partial T} = H_0 T\frac{\partial h}{\partial T} = H_0 u'_T$$

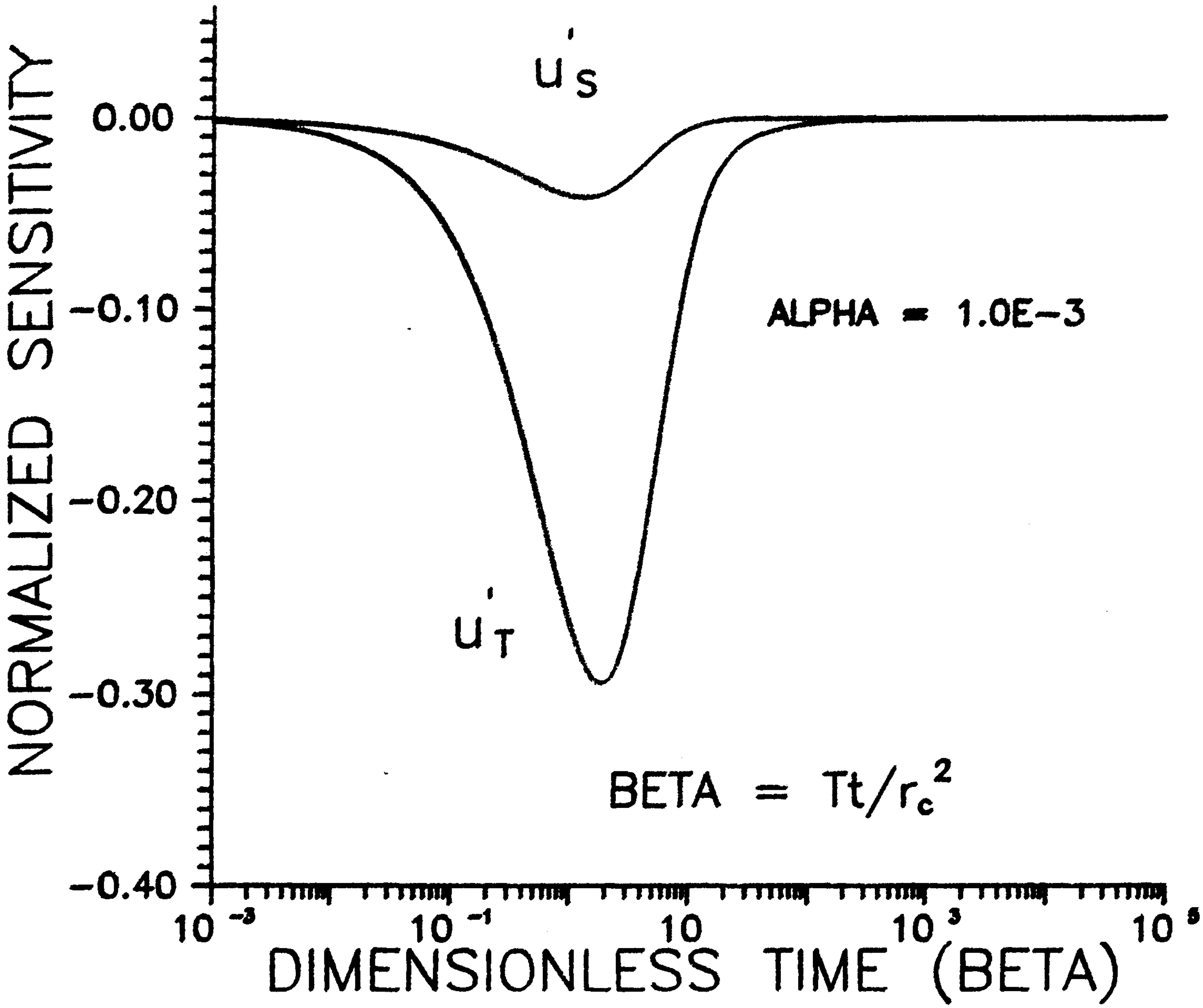
$$u'_T = T\frac{\partial h}{\partial T} = \beta\frac{\partial h}{\partial \beta}$$

$$U'_S = S\frac{\partial H}{\partial S} = H_0\, S\frac{\partial h}{\partial S} = H_0\, u'_S$$

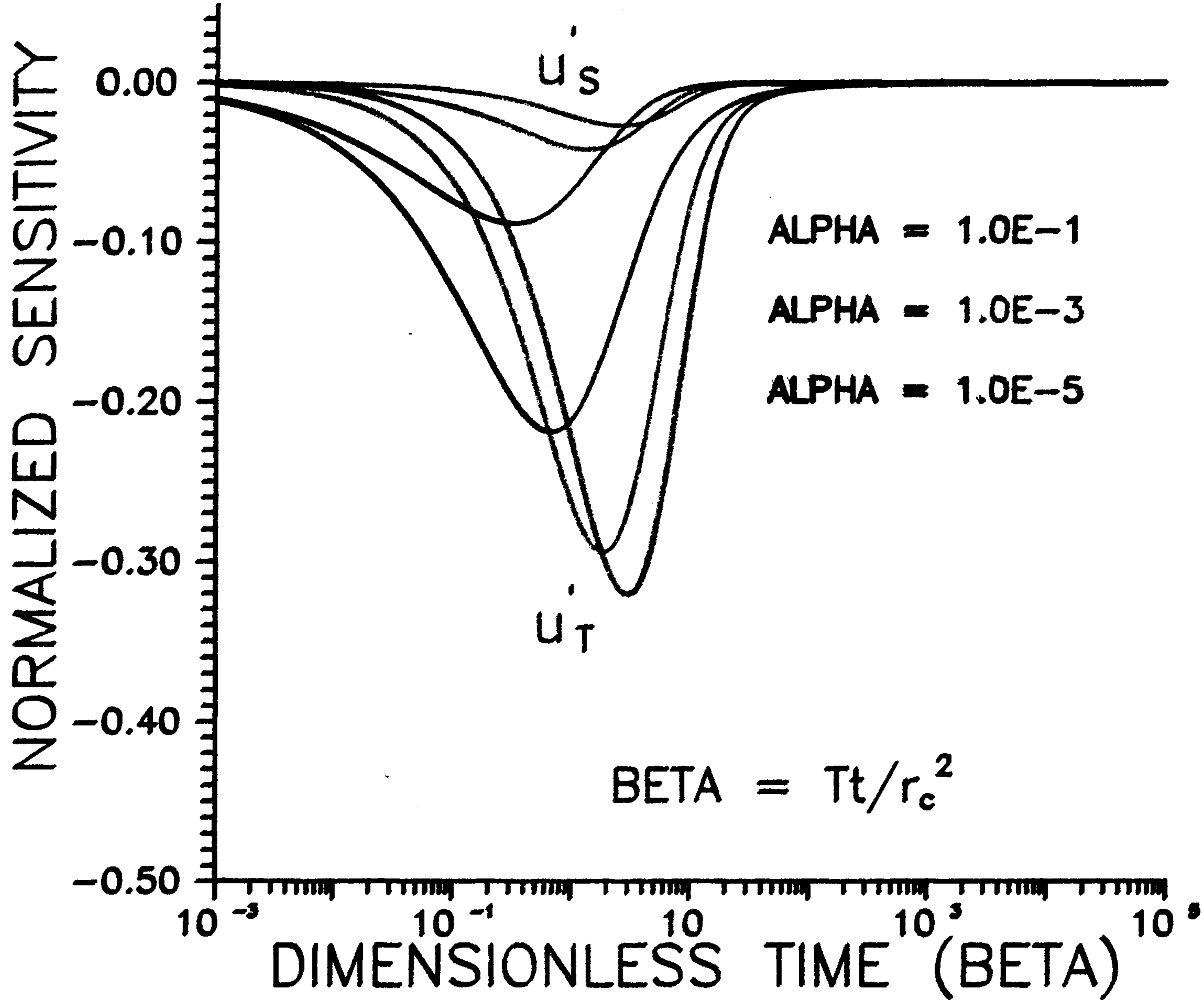
$$u'_S = S\frac{\partial h}{\partial S} = \alpha\frac{\partial h}{\partial \alpha}$$

$$U'_{H_0} = H_0\frac{\partial H}{\partial H_0} = H_0\, h = H = H_0\, u'_{H_0}$$

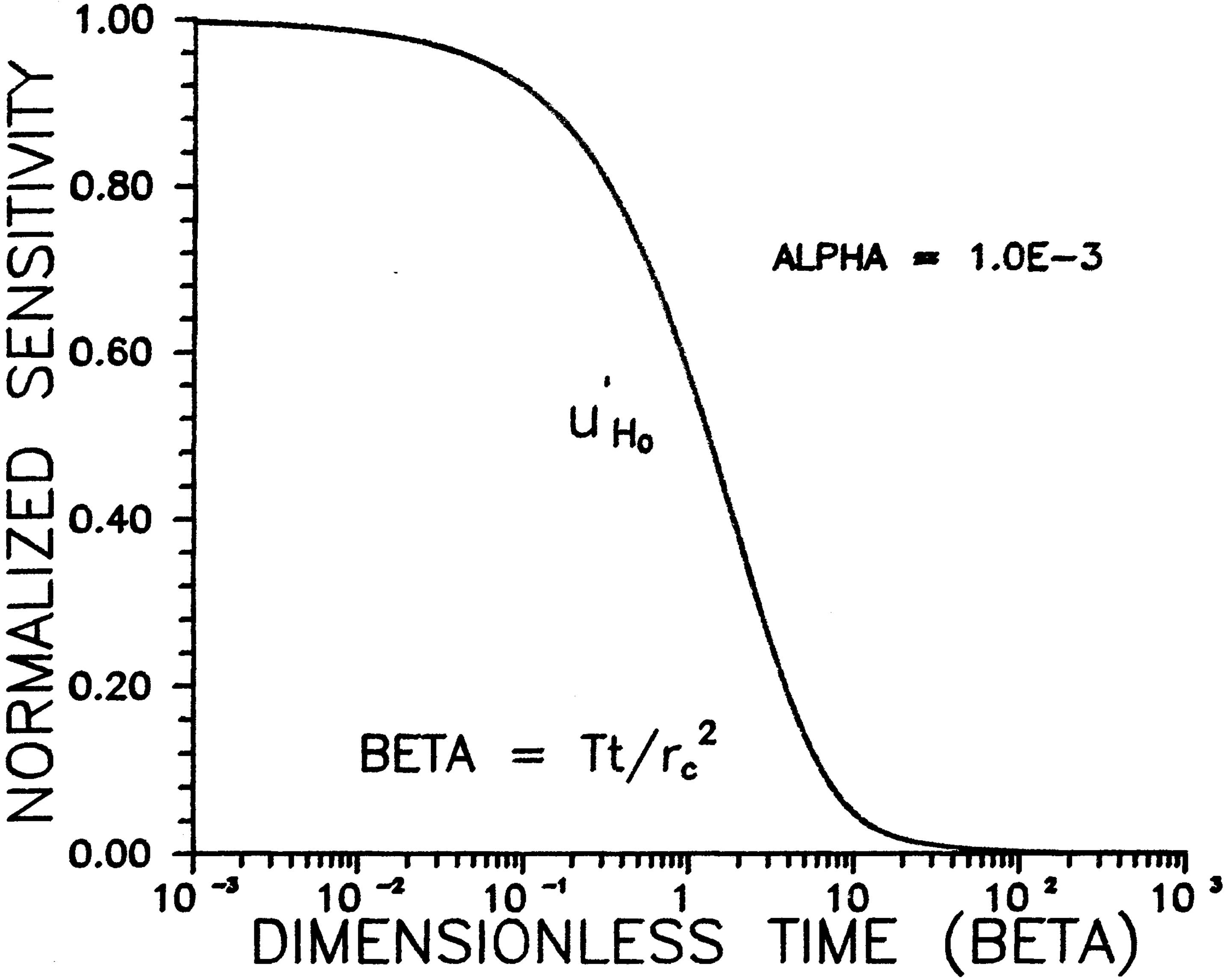
$$u'_{H_0} = h$$



u'S
u'T
0.00
−0.10
−0.20
−0.30
−0.40
NORMALIZED SENSITIVITY
ALPHA = 1.0E−3
BETA = Tt/rc2
10 −3
10 −1
10
10 3
10 5
DIMENSIONLESS TIME (BETA)



NORMALIZED SENSITIVITY
0.00
-0.10
-0.20
-0.30
-0.40
-0.50
u's
u'T
ALPHA = 1.0E-1
ALPHA = 1.0E-3
ALPHA = 1.0E-5
BETA = Tt/rc2
10 -3
10 -1
10
10 3
10 5
DIMENSIONLESS TIME (BETA)


NORMALIZED SENSITIVITY
1.00
0.80
0.60
0.40
0.20
0.00
ALPHA = 1.0E-3
U'_{H_0}
BETA = Tt/r_c^2
10^{-3}
10^{-2}
10^{-1}
1
10
10^{2}
10^{3}
DIMENSIONLESS TIME (BETA)

PARAMETER ESTIMATION USING LINEARIZATION

OBJECTIVE: Minimize $E = \sum_i \left[H_i^e - H_i \right]^2$

H_i^e = observed head at index point i

H_i = calculated head at index point i

ASSUME: $\mathbf{H}^e = \mathbf{H}^* + \varepsilon$

ε = error vector

THUS: $\mathbf{H}^e - \mathbf{H}^m = \mathbf{U}_T^m \Delta T^m + \mathbf{U}_S^m \Delta S^m + \mathbf{U}_{H_0}^m \Delta H_0^m + \varepsilon$

SOLVE: New parameter estimates

$T^{m+1} = T^m + \Delta T^m \qquad S^{m+1} = S^m + \Delta S^m \qquad H_0^{m+1} = H_0^m + \Delta H_0^m$

SENSITIVITY DESIGN MATRIX

$$a_{ij} = \left[A\right]_{ij} = \sum_{k=1}^{n} U_i\,(t_k)\,U_j\,(t_k)$$

$$i, j = H_0, T, S$$

SENSITIVITY CORRELATION MATRIX

$$c_{ij} = \left[C\right]_{ij} = \frac{a_{ij}}{\sqrt{a_{ii}a_{jj}}}$$

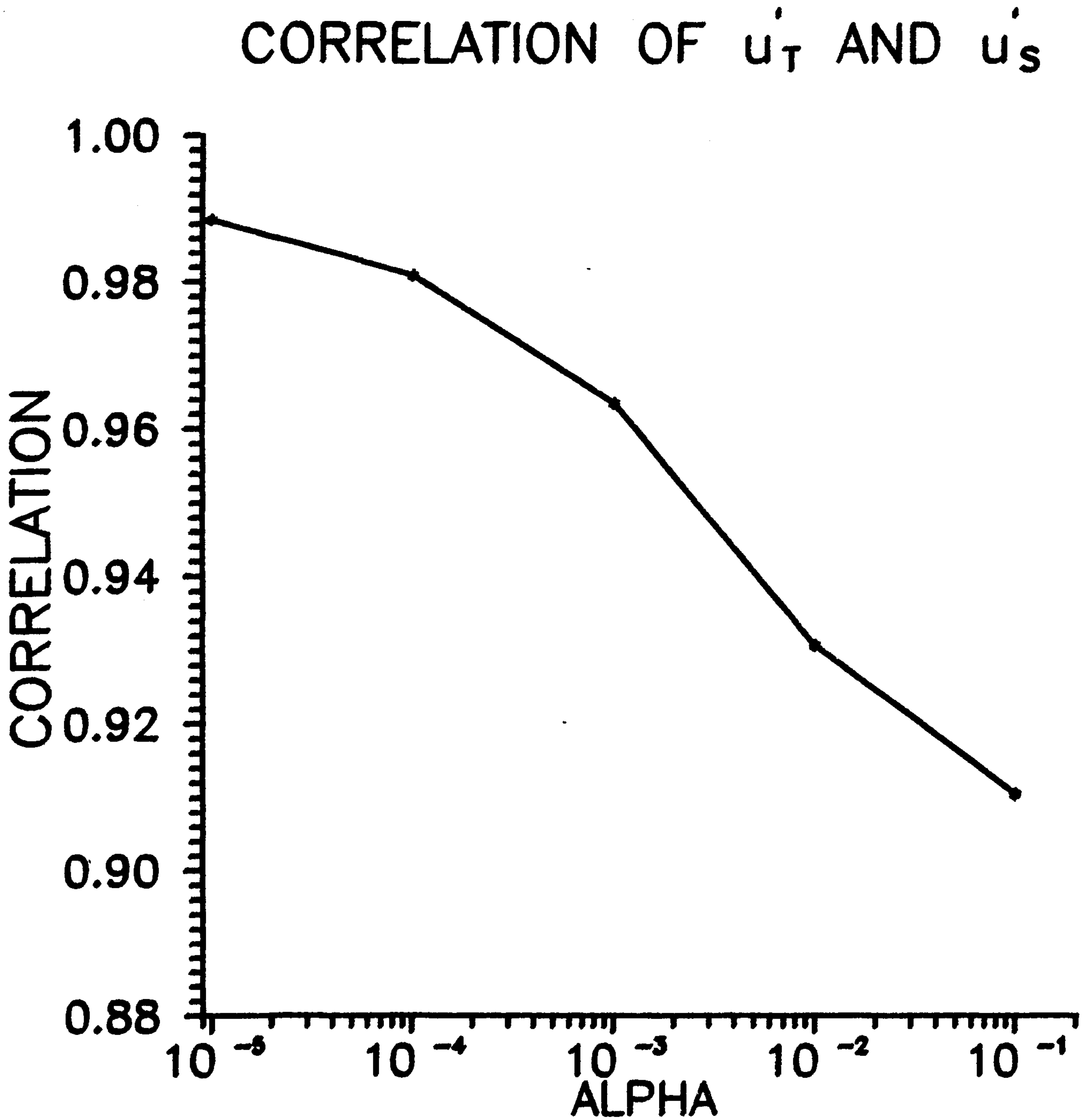
PARAMETER COVARIANCE MATRIX

$$\text{COV}(P) = \left[B\right] = \left[A\right]^{-1} \sigma^2$$

σ = head variance

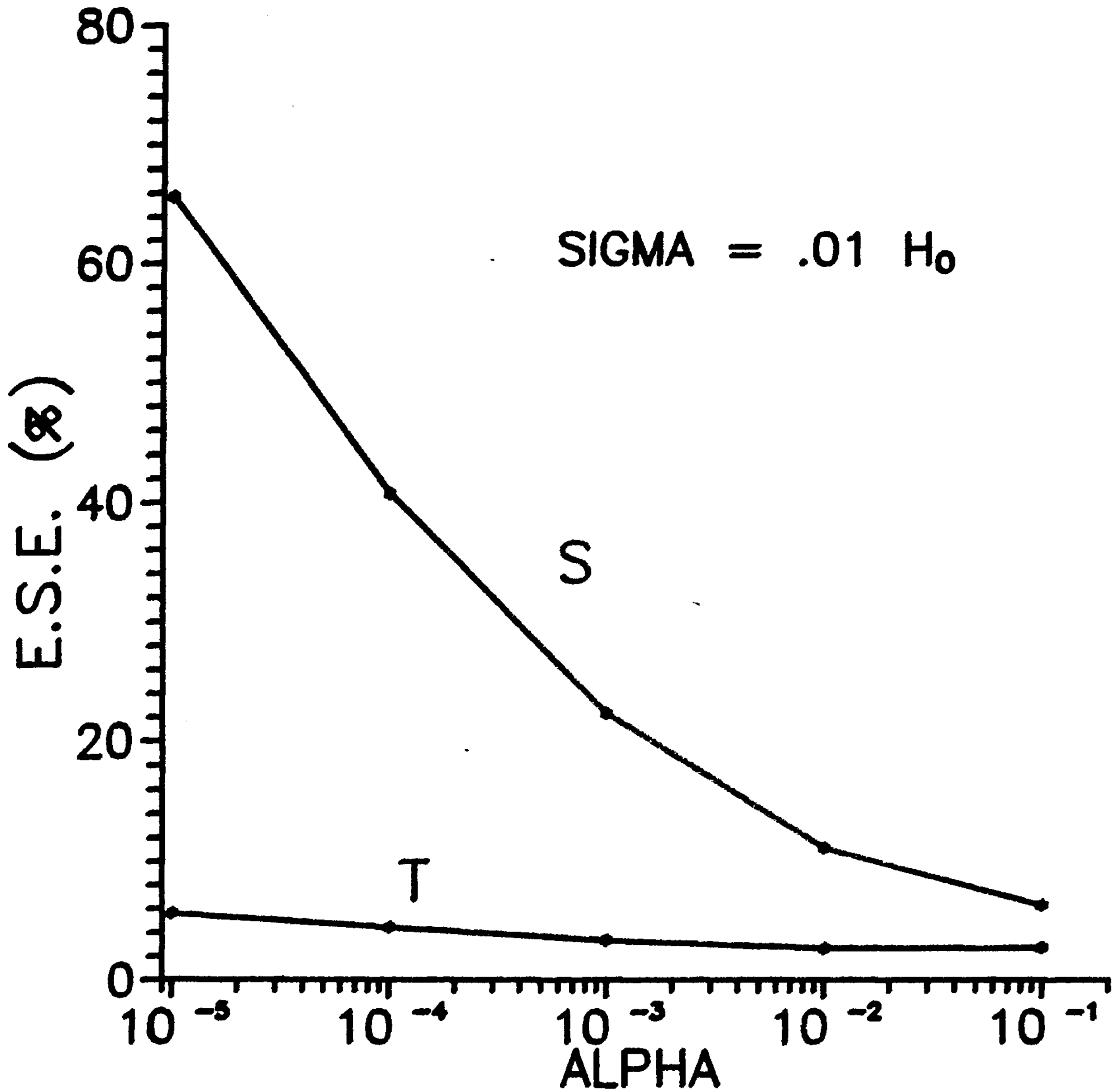
Estimated Standard Error
of Parameter i

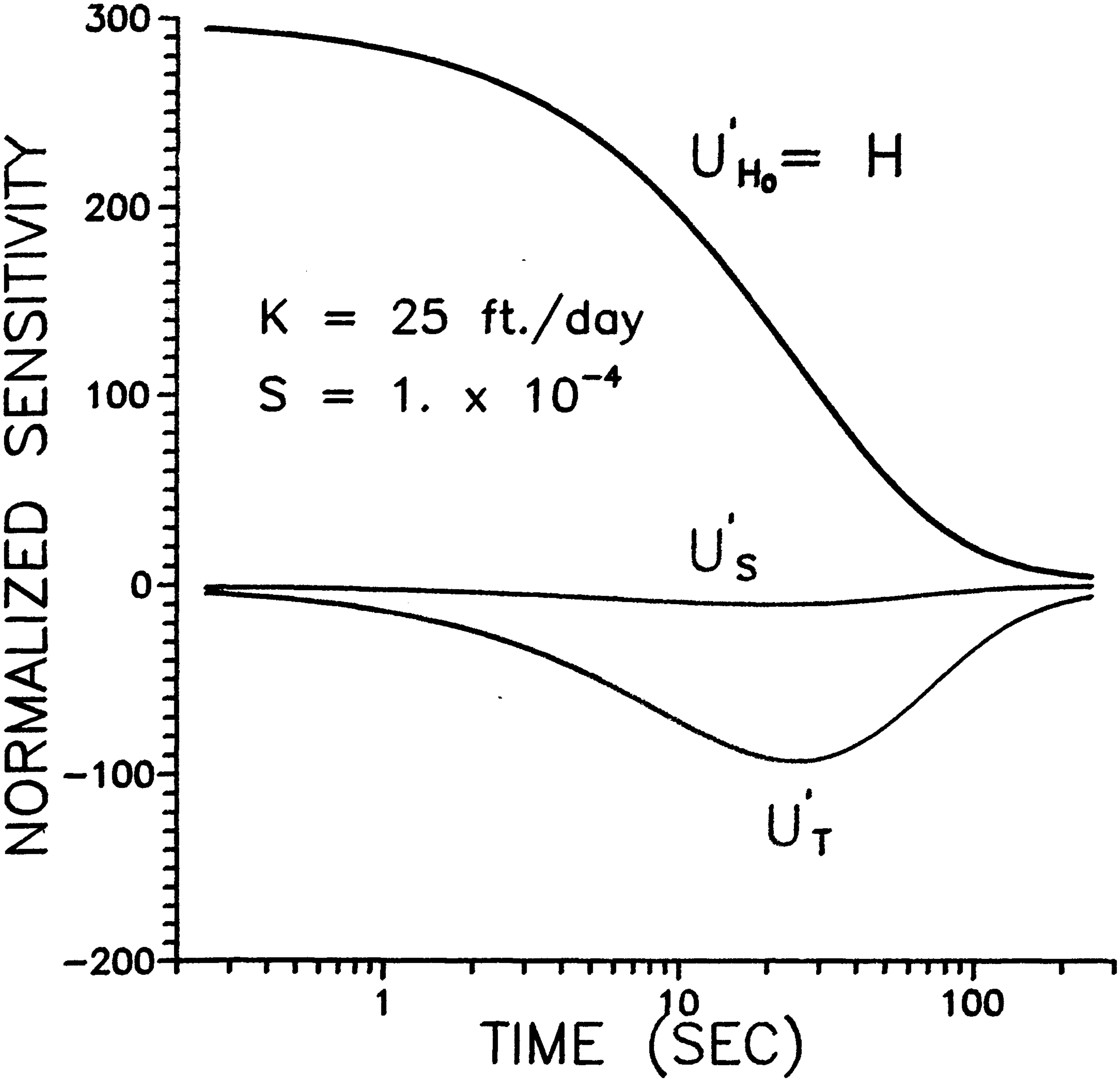
$$\sqrt{b_{ii}}$$



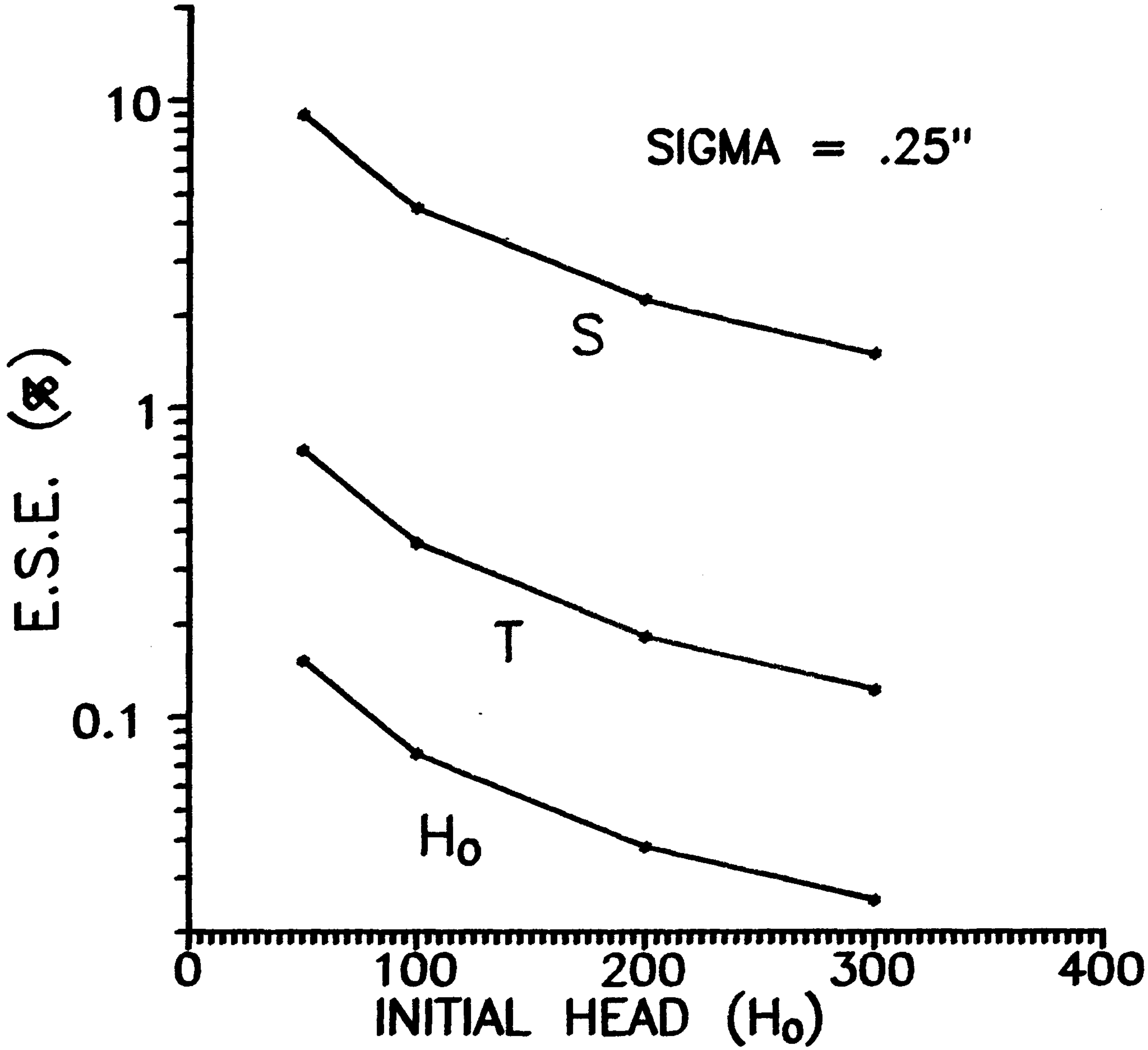
CORRELATION OF u'_T AND u'_S
1.00
0.98
0.96
0.94
0.92
0.90
0.88
CORRELATION
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
ALPHA

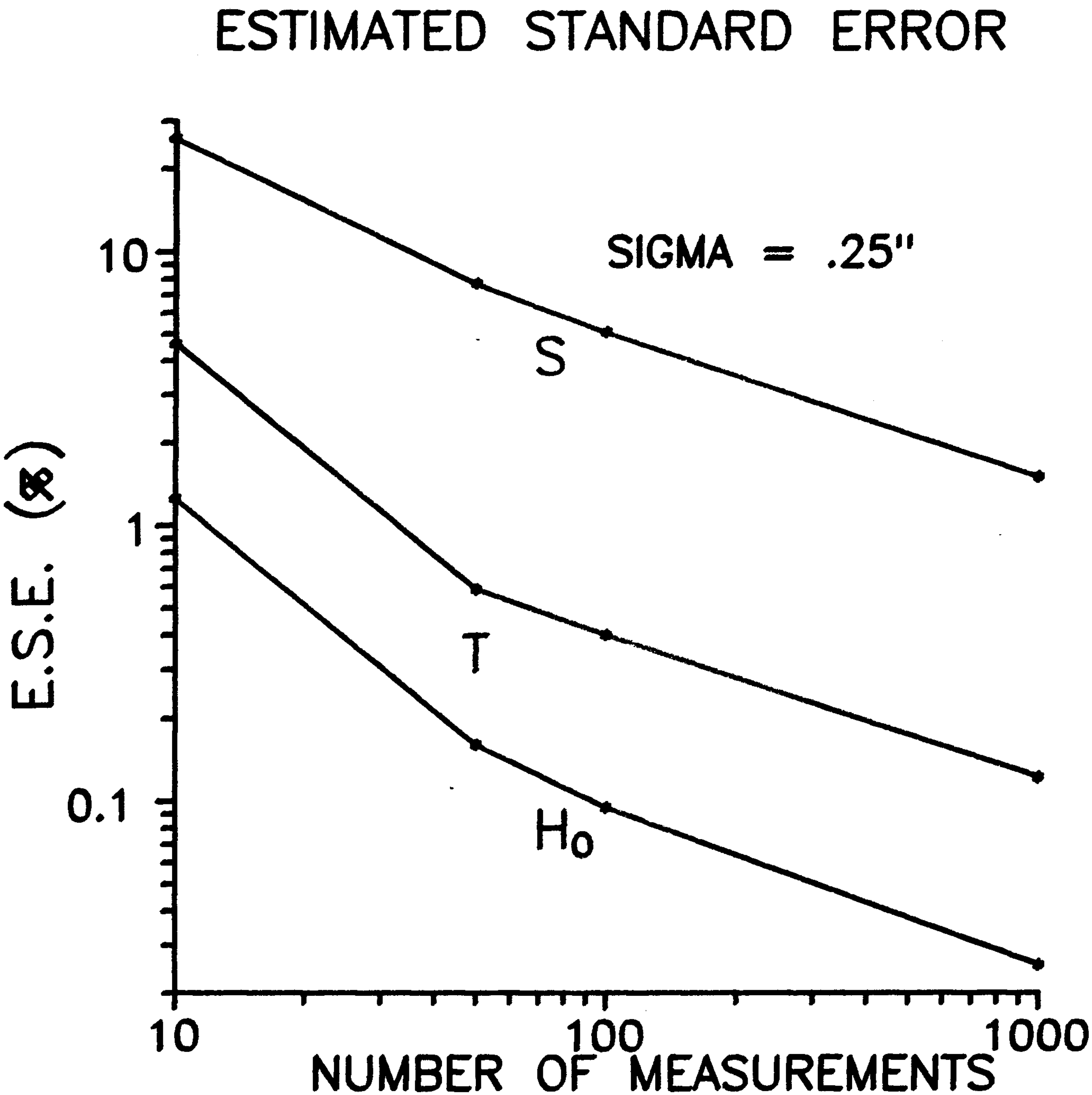
ESTIMATED STANDARD ERROR



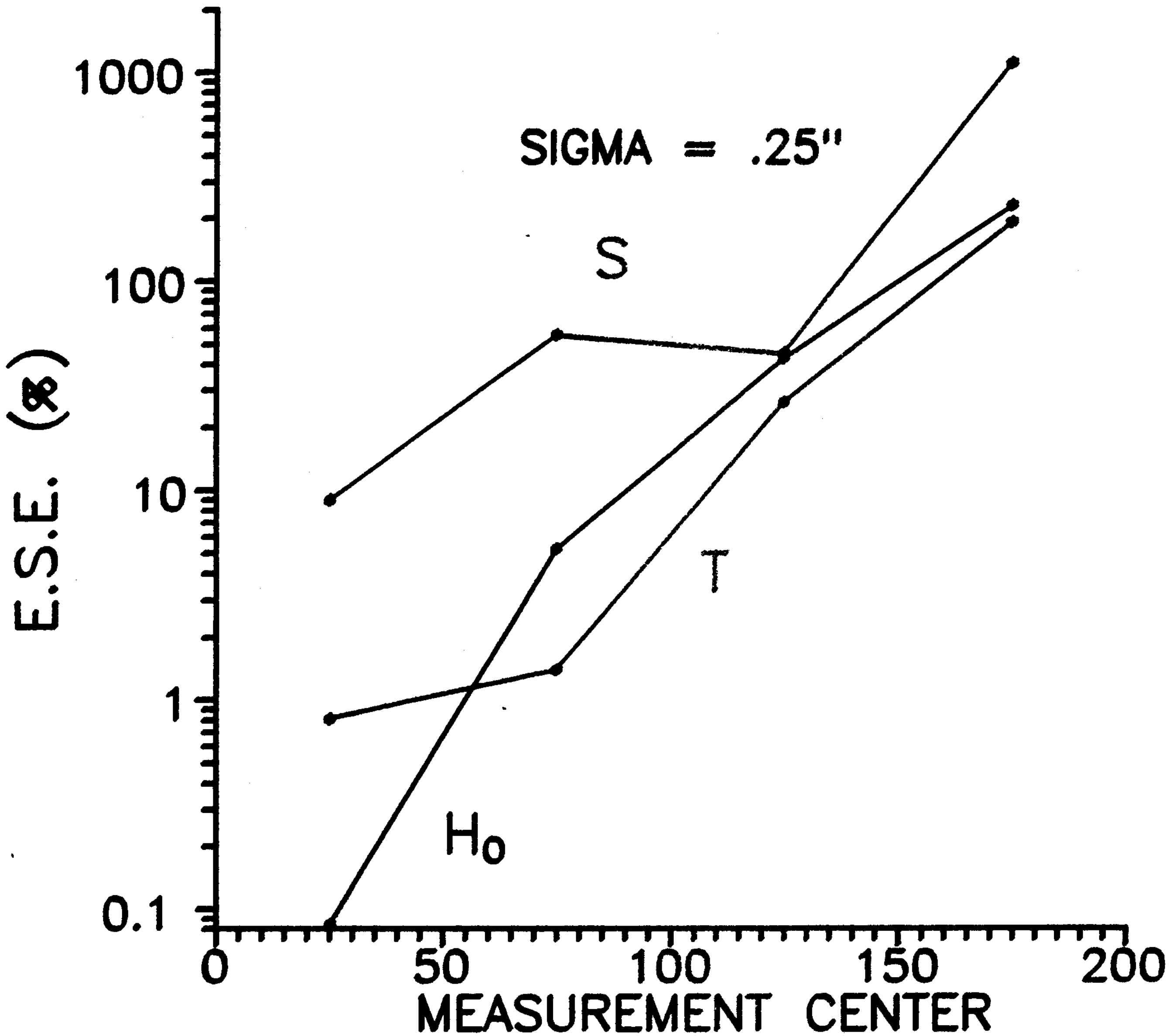

NORMALIZED SENSITIVITY
300
200
100
0
−100
−200
1
10
100
TIME (SEC)
U'Ho = H
K = 25 ft./day
S = 1. x 10^-4
U'S
U'T

ESTIMATED STANDARD ERROR




ESTIMATED STANDARD ERROR
SIGMA = .25"
S
T
H_0
E.S.E. (%)
10
1
0.1
10
100
1000
NUMBER OF MEASUREMENTS

ESTIMATED STANDARD ERROR



CONCLUSIONS

1. Sensitivity analysis is useful for designing better slug tests.

2. T and S usually can not both be reliably estimated due to high correlation.

3. Estimated standard errors of the parameters are inversely proportional to H_0. Therefore large initial heads should be used.

4. Generally, an increased number of measurements improves parameter estimation.

5. Measurement times should be selected to minimize sensitivity correlation and to sample at points of maximum sensitivity.