APPROXIMATE MODEL OF SALT-WATER UPCONING IN AQUIFERS

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INTRODUCTION

The motion of the salt-water/fresh-water transition zone due to pumpage in coastal and inland aquifers is of importance for water resources development. Unstable salt-water upwelling (intrusion of salt water into wells) has caused contamination of many wells.

Perhaps the most general description of this problem is given by a variable-density hydrodynamic dispersion model (Bear, 1979); however, the difficulties associated with the numerical solution (mainly due to the numerical dispersion) make this approach impractical in many cases.

One way to avoid these problems is to utilize the sharp interface approximation. This approximation has been used quite extensively by two types of models: 1) a three-dimensional flow model (Liu et al., 1981; Kemblowski, 1984, and 1985a), and 2) a horizontal flow model which uses the Dupuit-Forchheimer approximation (Shamir and Dagan, 1971; Mercer et al., 1980; McElwee, 1985; Kemblowski, 1985b). Kemblowski (1985b) analyzes the impact of this approximation and concludes that, as long as the size of the sinks or sources (drains, rivers, etc) is of the same order as that of the fresh-water zone, this approximation is quite acceptable.

However, in some cases the sharp interface approach cannot be used. Due to transient boundary conditions (pumpage, river stage fluctuations), a transition zone between the fresh-water and salt-water zones may develop. A simplified description of the transition zone that uses the concept of the boundary layer was presented by Rubin and Pinder

(1977) and Rubin (1983). In this paper, we use this concept to develop a solution for salt-water upconing and to analyze the mechanism of the development of the transition zone.

DEVELOPMENT OF THE MODEL

The development of this solution assumes several approximations; therefore, it is important to emphasize that all of them have to be verified if a practical problem is to be solved. It is assumed that the Dupuit-Forchheimer approximation is valid in the fresh-water and salt-water zones, thus the Darcian velocities in both zones are horizontal. The velocity distribution function (shape function) and the concentration distribution function within the transition zone are also assumed to be known and are similar to finite element basis functions. The velocity distribution is given by

$$V_T(\eta) = U \cdot F(\eta) + V \cdot G(\eta) \tag{1}$$

where U and V are fresh-water and salt-water Darcian velocities, $V_T(\eta)$ is the groundwater velocity in the transition zone, η is a dimensionless vertical coordinate given by

$$\eta = \frac{Z - Z_b}{\delta} \,. \tag{2}$$

Z is the vertical coordinate, Z_b is the elevation of the bottom of the transition zone, and δ is the transition-zone thickness.

The concentration distribution may be described in a similar manner by

$$C(\eta) = C_F K(\eta) + C_S L(\eta) \tag{3}$$

C_F is the fresh-water concentration and C_S is the salt-water concentration. Assuming that the fresh-water concentration is equal to zero, equation 3 may be simplified to

$$C(\eta) = C_s L(\eta) \tag{4}$$

In the following development, we assume that the aquifer is unconfined. The analysis may be easily modified for confined conditions. Equation 4 may be used to derive a relationship between the fresh-water and salt-water potentials (ϕ^f and ϕ^s), which are defined by

$$\phi^f = z + p/\gamma_f \tag{5}$$

and

$$\phi^s = z + p/\gamma_s \tag{6}$$

where p is the pressure and γ_f and γ_s are fresh-water and salt-water specific weights.

Assuming that the flow in the transition zone is horizontal, we obtain the following relationship for the pressure difference between the top and the bottom of the transition zone:

$$\Delta p = p_t - p_b = - \int_{Z_b}^{Z_t} \gamma dZ \qquad (7)$$

Z_t is the elevation of the top of the transition zone. Using the following relation between the specific weight and the concentration (the equation of state)

$$\gamma = \gamma_f(1 + \alpha C), \qquad (8)$$

we may evaluate the pressure difference as follows:

$$\Delta p = - \int_{Z_b}^{Z_t} \gamma_f(1 + \alpha C) dz = - \gamma_f(\delta + \alpha C_s \delta \int_o^1 L(\eta) d\eta)$$

$$= - \gamma_f(\delta + \alpha\delta C_s \bar{L}) \qquad (9)$$

where $\bar{L} = \int_o^1 L(\eta) d\eta$.

This pressure difference may be also estimated from equations 5 and 6.

$$\Delta p = \gamma_f \phi^f - \gamma_s \phi^s - \gamma_f Z_t + \gamma_s Z_b \qquad (10)$$

Combining equations 8, 9, and 10 leads to

$$\gamma_f(1+\alpha C_s)\phi^s - \gamma_f \phi^f + \gamma_f Z_t - \gamma_f(1+\alpha C_s) Z_b = \gamma_f(\delta + \alpha\delta C_s \bar{L}) \qquad (11a)$$

or

$$(1+\alpha C_s)\phi^s - \phi^f - \alpha C_s Z_b = \alpha\delta C_s \bar{L} \qquad (11b)$$

Using equation 11b, the salt-water potential may be expressed as follows:

$$\phi^s = \frac{1}{1+a} (\phi^f + aZ_b + a\bar{L}\delta) \qquad (12)$$

where $a = \alpha C_s$.

FLOW CONTINUITY FOR FRESH-WATER AND TRANSITION ZONES

We are considering unconfined groundwater flow in the fresh-water and transition zones. The continuity equation for both zones as a unit may be written as follows:

$$\nabla(q) = - n \frac{\partial \phi^f}{\partial t} + n \frac{\partial Z_b}{\partial t} + N, \qquad (13)$$

where:

q = horizontal flow rate in the fresh-water and transition zones per unit width $(\frac{L^2}{T})$

n = effective porosity (dimensionless)

ϕ^f = fresh-water potential (groundwater level for unconfined conditions) (L)

N = recharge (positive) or discharge (negative) $(\frac{L}{T})$

Vector q may be expressed by

$$q = U(\phi^f - Z_t) + \int_{Z_b}^{Z_t} V_T \, dZ \qquad (14)$$

After substituting equation 1 into equation 14, we obtain

$$q = U(\phi^f - Z_t) + \delta U \int_o^1 F(\eta)d\eta + \delta V \int_o^1 G(\eta)d\eta. \tag{15}$$

When we define $\bar{F}$ and $\bar{G}$ as

$$\bar{F} = \int_o^1 F(\eta)d\eta \text{ and } \bar{G} = \int_o^1 G(\eta)d\eta, \tag{16}$$

equation 13 may be written as follows:

$$\nabla\left[(\phi^f - Z_t + \delta\bar{F})U + \delta\bar{G}V\right] = n\left(\frac{\partial Z_b}{\partial t} - \frac{\partial \phi^f}{\partial t}\right) + N \tag{17}$$

where:

$$U = -k_f \nabla\phi^f \tag{18}$$

$$V = -k_s \nabla\phi^s. \tag{19}$$

k_f and k_s are hydraulic conductivities in the fresh- and salt-water zones, respectively.

Substituting equations 12, 18, and 19 into equation 17, we obtain

$$\nabla\left[k_f(\phi^f - Z_t + \delta\bar{F})\nabla\phi^f + \frac{k_s\delta\bar{G}}{1+a}(\nabla\phi^f + a\nabla Z_b + a\bar{L}\nabla\delta)\right]$$

$$= n\left(\frac{\partial\phi^f}{\partial t} - \frac{\partial Z_b}{\partial t}\right) - N. \tag{20}$$

Defining

$$A_1 = k_f(\phi^f - Z_t + \delta\bar{F}) + \frac{k_s\delta\bar{G}}{1+a}, \tag{21}$$

$$B_1 = \frac{k_s \delta \bar{G} a}{1+a} \tag{22a}$$

and

$$C_1 = B_1 \bar{L} \tag{22b}$$

equation 20 may be written as follows:

$$\nabla(A_1 \nabla \phi^f + B_1 \nabla Z_b + C_1 \nabla \delta) = n(\frac{\partial \phi^f}{\partial t} - \frac{\partial Z_b}{\partial t}) - N. \tag{23}$$

MASS TRANSPORT IN THE TRANSITION ZONE

The solute transport in the transition zone is assumed to be of the advection type, with the exception of dispersive saltwater flux at the bottom of the transition zone. Under such assumptions the horizontal mass transport equation may be written as follows:

$$\nabla Q = - \frac{\partial M}{\partial t} - nD_T \frac{\partial C}{\partial Z}\bigg|_{Z_b} + P \tag{24}$$

where

Q = solute mass flux in the transition zone,

M = mass of the solute in the transition zone,

D_T = transverse dispersion coefficient out the bottom of the transition zone and

P = a source or sink in the transition zone.

The transverse dispersion coefficient is further defined as

$$D_T = \frac{\alpha_T}{n} \mid U \mid + D_f \tag{25}$$

α_T is the transverse dispersivity and D_f is the molecular diffusion constant.

The solute mass flux in equation 24 may be calculated as follows:

$$Q = \int_{Z_b}^{Z_t} V_T C dz = C_s \delta \left[U \int_o^1 F(\eta) L(\eta) d\eta + V \int_o^1 G(\eta) L(\eta) d\eta \right] . \qquad (26)$$

Denoting

$$\overline{FL} = \int_o^1 F(\eta) L(\eta) d\eta, \text{ and } \overline{GL} = \int_o^1 G(\eta) L(\eta) d\eta, \qquad (27)$$

we obtain

$$Q = C_s (\delta U \overline{FL} + \delta V \overline{GL}) \qquad (28)$$

The second term in equation 24 may be expressed as follows:

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \left(\int_{Z_b}^{Z_t} nC dz \right) = \frac{\partial}{\partial t} \left(\delta n C_s \int_o^1 L(\eta) d\eta \right)$$

$$= n C_s \bar{L} \frac{\partial \delta}{\partial t} \qquad (29)$$

The dispersive flow through the bottom of the transition zone may be written as

$$n D_T \left. \frac{\partial C}{\partial Z} \right|_{Z_b} = n D_T \left[\frac{\partial C}{\partial \eta} \frac{\partial \eta}{\partial Z} \right]_{\eta=0} = \frac{n D_T C_s}{\delta} \left. \frac{dL(\eta)}{d\eta} \right|_{\eta=0} \qquad (30)$$

Denoting

$$L'(0) = \left.\frac{dL(\eta)}{d\eta}\right|_{\eta=0}, \qquad (31)$$

we obtain

$$nD_T \left.\frac{\partial C}{\partial Z}\right|_{Z_b} = nD_T C_s L'(0)/\delta. \qquad (32)$$

In equation 24 P represents the solute mass flux due to a source or sink that is operative in the transition zone. Suppose Q_p is the strength of the source or sink that is evenly distributed vertically through the fresh and transition zones considered as a single unit. We will only consider a sink for P. In that case, the water pumped will have the appropriate concentration for the transition zone. The total solute mass flux is obtained by integrating over the transition zone.

$$P = \frac{Q_p}{(\phi^f - Z_b)} \int_{Z_b}^{Z_t} C dZ = \frac{Q_p \delta C_s}{(\phi^f - Z_b)} \int_0^1 L(\eta) d\eta$$

$$P = \frac{C_s Q_p \delta \overline{L}}{(\phi^f - Z_b)}. \qquad (33)$$

Substituting equations 28, 29, 32, 33, 18, 19, and 12 into equation 24 we obtain

$$\nabla\left[\delta(k_f \overline{FL} + \frac{k_s \overline{GL}}{1+a})\nabla\phi^f + \delta \frac{k_s \overline{GLa}}{1+a} \nabla Z_b + \delta \frac{k_s \overline{GLaL}}{1+a} \nabla\delta\right]$$

$$= n\overline{L} \frac{\partial \delta}{\partial t} + nD_T L'(0)/\delta - \frac{Q_p \delta \overline{L}}{(\phi^f - Z_b)} \qquad (34)$$

Equation 34 can be simplified by defining

$$A_2 = \delta\left(k_f\overline{FL} + \frac{k_s\overline{GL}}{1+a}\right) \tag{35}$$

$$B_2 = \delta\left(\frac{k_s\overline{GLa}}{1+a}\right) \tag{36}$$

$$C_2 = B_2\overline{L} \tag{37}$$

The resulting form for equation 34 is

$$\nabla(A_2\nabla\phi^f + B_2\nabla Z_b + C_2\nabla\delta) = n\overline{L}\,\frac{\partial\delta}{\partial t} + D_T L'(0)/\delta - \frac{Q_p\delta\overline{L}}{(\phi^f - Z_b)} \tag{38}$$

which is very similar to equation 23.

FLOW CONTINUITY IN THE SALT-WATER ZONE

Salt-water flow continuity may be written as follows:

$$\nabla(q_s) = -n\,\frac{\partial Z_b}{\partial t} + Q_s \tag{39}$$

where

$$q_s = (Z_b - Z_B)\ V \tag{40}$$

Z_B = the elevation of the bottom of the aquifer.

Q_s = areal "recharge" of salt water in the salt-water zone (from underlying formations), may be expressed as a function of the salt-water potential (leaky aquifer conditions).

Substituting equations 12, 19, and 40 into equation 39, we obtain

$$\nabla(k_s(Z_b-Z_B)/(1+a)\ (\nabla\phi^f + a\nabla Z_b + a\bar{L}\nabla\delta)$$

$$= n\frac{\partial Z_b}{\partial t} - Q_s \qquad (41)$$

Introducing the coefficients

$$A_3 = \frac{k_s(Z_b - Z_B)}{1+a} \qquad (42)$$

$$B_3 = A_3 a \qquad (43)$$

$$C_3 = B_3\bar{L} \qquad (44)$$

allows equation 41 to be written as

$$\nabla(A_3\nabla\phi^f + B_3\nabla Z_b + C_3\nabla\delta) = n\frac{\partial Z_b}{\partial t} - Q_s. \qquad (45)$$

Equations 23, 38, and 45 complete the mathematical description of the approximate model of salt-water upconing.

SHAPE FUNCTIONS OF VELOCITY AND CONCENTRATION

The velocity distribution within the transition zone is approximated by equation 1. The boundary conditions, which shape functions $F(\eta)$ and $G(\eta)$ are to satisfy, are

$$F(0) = 0 \qquad F(1) = 1 \tag{46}$$

$$\frac{dF}{d\eta}(0) = 0 \qquad \frac{dF}{d\eta}(1) = 0 \tag{47}$$

$$G(0) = 1 \qquad G(1) = 0 \tag{48}$$

$$\frac{dG}{d\eta}(0) = 0 \qquad \frac{dG}{d\eta}(1) = 0. \tag{49}$$

The conditions given by equations 47 and 49 are not necessary; however, they assure the smoothness of the velocity distribution.

The lowest order polynomials that satisfy equations 46 through 49 are given by

$$F(\eta) = -2\eta^3 + 3\eta^2 \tag{50}$$

$$G(\eta) = 2\eta^3 -3\eta^2 + 1 \ . \tag{51}$$

The values of the integrals in equation 16 may now be calculated:

$$\overline{F} = \int_o^1 (-2\eta^3 + 3\eta^2) = 1/2 \tag{52}$$

$$\overline{G} = \int_o^1 (2\eta^3 - 3\eta^2 + 1) \ d\eta = 1/2 \tag{53}$$

The shape function of the concentration distribution ($L(\eta)$ in equation 4) has the following boundary conditions:

$$L(0) = 1 \qquad\qquad L(1) = 0 \tag{54}$$

$$\frac{dL}{d\eta}(1) = 0 \; . \tag{55}$$

The lowest order polynomial that satisfies this equation is given by

$$L(\eta) = \eta^2 - 2\eta + 1 \tag{56}$$

The derivative of this function at the bottom of the transition zone (which is related to the concentration gradient and therefore to the dispersion through the bottom of the transition zone, given by equation 32) can now be evaluated.

$$L'(0) = \left.\frac{dL(\eta)}{d\eta}\right|_{\eta=0} = -2 \tag{57}$$

The values of the integrals in equations 9 and 27 may now be calculated

$$\overline{L} = \int_o^1 (\eta^2 - 2\eta + 1)\; d\eta = 1/3 \tag{58}$$

$$\overline{FL} = \int_o^1 F(\eta)L(\eta)\; d\eta = 1/15 \tag{59}$$

$$\overline{GL} = \int_o^1 L(\eta)G(\eta)\; d\eta = 4/15 \tag{60}$$

NUMERICAL SOLUTION

Equations 23, 38, and 45 form a set of three simultaneous equations which must be solved for ϕ^f, Z_b, and δ. For a complete solution appropriate boundary conditions and initial conditions must also be given. The left-hand side of all three equations consist of terms of the following form:

$$\frac{\partial}{\partial X}\left(W \frac{\partial U}{\partial X}\right) \tag{61}$$

In order to implement a numerical solution, a grid system with grid spacing ΔX is introduced in the X direction. Since the three equations are vertically averaged and since we are only considering a cross-sectional model, no spatial derivatives other than X appear in ∇.

A centered finite difference approximation to equation 61, located at ~~mode~~ node i, is given by

$$\left[\frac{\partial}{\partial X}\left(W \frac{\partial U}{\partial X}\right)\right]_i = \left[W_{i+1/2} U_{i+1} - \left(W_{i+1/2} + W_{i-1/2}\right)U_i + W_{i-1/2} U_{i-1}\right]/\Delta X^2 \tag{62}$$

The half-integer values of W are usually evaluated at node points by the approximation

$$W_{i+1/2} = \frac{1}{2}\left(W_{i+1} + W_i\right) \tag{63}$$

Using this approximation, equation 62 becomes

$$\left[\frac{\partial}{\partial x}\left(W\frac{\partial U}{\partial x}\right)\right]_i = \left[\left(W_{i+1} + W_i\right)U_{i+1} - \left(W_{i+1} + 2W_i + W_{i-1}\right)U_i + \left(W_i + W_{i-1}\right)U_{i-1}\right]/2\Delta x^2 \qquad (64)$$

In equation 64, as applied to equations 23, 38, and 45, W may be A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, or C_3 and U may be ϕ^f, Z_b, or δ.

In addition to spatial derivatives, the right-hand sides of equations 23, 38, and 45 involve various time derivatives. These time derivatives may be approximated several ways. We consider only two ways. Let time be discretized into invervals Δt and let n, when used as a superscript, denote the nth time step. If n is the current time step for which values are known (either from the initial conditions or a previous time step solution) then n+1 represents the unknown values at a new time step which are to be solved for. The time derivative may be evaluated as

$$\left[\frac{\partial U}{\partial t}\right]_i^n = \frac{U_i^{n+1} - U_i^n}{\Delta t} \qquad (65)$$

or

$$\left[\frac{\partial U}{\partial t}\right]_i^{n+1} = \frac{U_i^{n+1} - U_i^n}{\Delta t} \,. \qquad (66)$$

Equation 65 implies the whole equation is evaluated at time n, and the end result is an explicit procedure with its attendant restriction on time step magnitude. Equation 66 indicates the equation is evaluated at time n+1 and gives an implicit procedure. In equations 65 and 66, U may be ϕ^f, Z_b or δ.

Using an implicit procedure (equation 66) on equation 23 gives

$$
\begin{aligned}
&\left(\frac{A_{1i+1}+A_{1i}}{2\Delta x^2}\right)^{n+1}\phi^{f\,n+1}_{i+1} - \left(\frac{A_{1i+1}+2A_{1i}+A_{1i-1}}{2\Delta x^2}+\frac{n}{\Delta t}\right)^{n+1}\phi^{f\,n+1}_{i} \\
&\quad + \left(\frac{A_{1i}+A_{1i-1}}{2\Delta x^2}\right)^{n+1}\phi^{f\,n+1}_{i-1} + \left(\frac{B_{1i+1}+B_{1i}}{2\Delta x^2}\right)^{n+1} Z^{n+1}_{bi+1} \\
&+ \left(\frac{n}{\Delta t} - \frac{B_{1i+1}+2B_{1i}+B_{1i-1}}{2\Delta x^2}\right)^{n+1} Z^{n+1}_{bi} + \left(\frac{B_{1i}+B_{1i-1}}{2\Delta x^2}\right)^{n+1} Z^{n+1}_{bi-1} \\
&\quad + \left(\frac{C_{1i+1}+C_{1i}}{2\Delta x^2}\right)^{n+1}\delta^{n+1}_{i+1} - \left(\frac{C_{1i+1}+2C_{1i}+C_{1i-1}}{2\Delta x^2}\right)^{n+1}\delta^{n+1}_{i} \\
&\quad + \left(\frac{C_{1i}+C_{1i-1}}{2\Delta x^2}\right)^{n+1}\delta^{n+1}_{i-1} = \frac{n}{\Delta t}\left(Z^n_{bi} - \phi^{f\,n}_{i}\right) - N^{n+1}_i . \qquad (67)
\end{aligned}
$$

In equation 67 the new time level values of the dependent variables have been collected on the left, while the known previous time level values and the recharge (which should be known) have been kept on the right-hand side. If we consider A, B, and C known, then equation 67 has nine unknowns and can be written for each active node. For convenience, equation 67 can be written in more compact form.

$$
\begin{aligned}
&\left[a^{(1)}_i\phi^f_{i-1} + a^{(2)}_i Z_{bi-1} + a^{(3)}_i\delta_{i-1} + b^{(1)}_i\phi^f_i + b^{(2)}_i Z_{bi}\right. \\
&\left. + b^{(3)}_i\delta_i + c^{(1)}_i\phi^f_{i+1} + c^{(2)}_i Z_{bi+1} + c^{(3)}_i\delta_{i+1}\right]^{n+1} = d^{(1)}_i . \qquad (68)
\end{aligned}
$$

The new coefficients in equation 68 (a, b, c) can be easily identified by comparison with equation 67 and then referring to the definition of A_1, B_1, and C_1 (equations 21 and 22) and using the values of $\bar{F}$ and $\bar{G}$ given by equations 52 and 53.

$$a_i^{(1)} = \left\{k_f\left[(\phi_i^f + \phi_{i-1}^f) - (Z_{bi} + Z_{bi-1}) - (\delta_i + \delta_{i-1})/2\right]\right.$$

$$\left. + \frac{k_s}{2(1+a)}\left(\delta_i + \delta_{i-1}\right)\right\}/2\Delta X^2 \qquad (69)$$

$$a_i^{(2)} = \frac{k_s a}{4\Delta X^2(1+a)}\left(\delta_i + \delta_{i-1}\right) \qquad (70)$$

$$a_i^{(3)} = a_i^{(2)}/3 \qquad (71)$$

$$c_i^{(1)} = a_{i+1}^{(1)} \qquad (72)$$

$$c_i^{(2)} = a_{i+1}^{(2)} \qquad (73)$$

$$c_i^{(3)} = c_i^{(2)}/3 \qquad (74)$$

$$b_i^{(1)} = -\frac{n}{\Delta t} - a_i^{(1)} - c_i^{(1)} \qquad (75)$$

$$b_i^{(2)} = \frac{n}{\Delta t} - a_i^{(2)} - c_i^{(2)} \qquad (76)$$

$$b_i^{(3)} = -a_i^{(3)} - c_i^{(3)} \qquad (77)$$

$$d_i^{(1)} = \frac{n}{\Delta t}\left(Z_{bi} - \phi_i^f\right)^n \qquad (78)$$

The same implicit procedure used to obtain equations 67 through 78 can be applied to the remaining two equations, 38 and 45. This results in two more equations with nine unknowns similar to equation 68.

$$\Big[a_i^{(4)}\phi_{i-1}^f + a_i^{(5)}Z_{bi-1} + a_i^{(6)}\delta_{i-1} + b_i^{(4)}\phi_i^f + b_i^{(5)}Z_{bi} + b_i^{(6)}\delta_i + c_i^{(4)}\phi_{i+1}^f + c_i^{(5)}Z_{bi+1} + c_i^{(6)}\delta_{i+1}\Big]^{n+1} = d_i^{(2)} \tag{79}$$

$$a_i^{(7)}\phi_{i-1}^f + a_i^{(8)}Z_{bi-1} + a_i^{(9)}\delta_{i-1} + b_i^{(7)}\phi_i^f + b_i^{(8)}Z_{bi} + b_i^{(9)}\delta_i + c_i^{(7)}\phi_{i+1}^f + c_i^{(8)}Z_{bi+1} + c_i^{(9)}\delta_{i+1}\Big]^{n+1} = d_i^{(3)}. \tag{80}$$

The coefficients appearing in equations 79 and 80 are obtained in a manner similar to that used for equations 69 through 78.

$$a_i^{(4)} = \left[k_f + \frac{4k_s}{1+a}\right]\left(\delta_i + \delta_{i-1}\right)/30\Delta X^2 \tag{81}$$

$$a_i^{(5)} = \frac{2k_s a}{15\Delta X^2(1+a)}\left(\delta_i + \delta_{i-1}\right) \tag{82}$$

$$a_i^{(6)} = a_i^{(5)}/3 \tag{83}$$

$$c_i^{(4)} = a_{i+1}^{(4)} \tag{84}$$

$$c_i^{(5)} = a_{i+1}^{(5)} \tag{85}$$

$$c_i^{(6)} = c_i^{(5)}/3 \tag{86}$$

$$b_i^{(4)} = -a_i^{(4)} - c_i^{(4)} \tag{87}$$

$$b_i^{(5)} = -a_i^{(5)} - c_i^{(5)} \tag{88}$$

$$b_i^{(6)} = \left(b_i^{(5)} - \frac{n}{\Delta t}\right)/3 \tag{89}$$

$$d_i^{(2)} = -\frac{n}{3\Delta t}\,\delta_i^n - \frac{2D_{Ti}^{n+1}}{\delta_i^{n+1}} - \frac{Q_{pi}^{n+1}\;\delta_i^{n+1}}{3\left(\phi_i^{fn+1} - Z_{bi}^{n+1}\right)} \tag{90}$$

$$a_i^{(7)} = \frac{k_s}{2\Delta X^2(1+a)}\left(Z_{bi} + Z_{bi-1} - Z_{Bi} - Z_{Bi-1}\right) \tag{91}$$

$$a_i^{(8)} = (a)\;a_i^{(7)} \tag{92}$$

$$a_i^{(9)} = a_i^{(8)}/3 \tag{93}$$

$$c_i^{(7)} = a_{i+1}^{(7)} \tag{94}$$

$$c_i^{(8)} = a_{i+1}^{(8)} \tag{95}$$

$$c_i^{(9)} = c_i^{(8)}/3 \tag{96}$$

$$b_i^{(7)} = -a_i^{(7)} - c_i^{(7)} \tag{97}$$

$$b_i^{(8)} = (a)\;b_i^{(7)} - \frac{n}{\Delta t} \tag{98}$$

$$b_i^{(9)} = (a)\;b_i^{(7)}/3 \tag{99}$$

$$d_i^{(3)} = -\frac{n}{\Delta t}\,Z_{bi}^n - Q_{si}^{n+1} \tag{100}$$

Equations 68, 79, and 80 form a tri-tridiagonal system of equations. Von Rosenberg (1969) has shown how to solve this set of simultaneous equations directly. He gives an efficient algorithm which takes advantage of the sparseness of the coefficient matrix. If we have N active nodes, it is seen that equations 68, 79, and 80 form a 3N x 3N set of matrix equations to be solved for the 3N unknowns. ϕ^f, Z_b and δ must be found for each of the N active nodes. However, it is clear from equations 68, 79, and 80 that only a maximum of 9 unknowns appear in each equation. Therefore, in the 3N x 3N matrix, only a central band of 9 diagonals is nonzero. In fact, the three equations (68, 79, and 80) only have 6 unknowns for the first and last active nodes. This is so because some type of boundary condition must be applied at the ends of the model and this eliminates 3 of the unknowns. We have written a program to implement von Rosenberg's tri-tridiagonal solution on equations 68, 79, and 80.

Unfortunately, a single solution of equations 68, 79, and 80 for a given time step does not give the correct answer, since the coefficients defined by equations 69-78 and 81-100 are themselves functions of ϕ_f, Z_b, and δ at the new time level n+1. In other words, we have a nonlinear set of equations in 68, 79, and 80. Therefore, an iterative scheme must be used at each time step until sufficient accuracy has been obtained. At the beginning of each time step the preceeding time step values of ϕ^f, Z_b, and δ are used to evaluate the coefficients and start a new iteration. Iteration ceases when there is little change between iterations in ϕ^f, Z_b, and δ. This convergence criteria is specified by the user.

Alternatively, an explicit procedure for solving equations 23, 38, and 45 can be derived by using equation 65 for the time derivatives. If this is done, the evaluation of each equation is at time n. The only place where the new time level values for ϕ^f, Z_b, and δ appear is in the time derivative. Therefore, we have only one unknown in each equation. In this case, the equations decouple and they can be solved sequentially for δ, Z_b, and ϕ^f with simple algebra, no simultaneous equation solution or matrix solution is required. However, as mentioned earlier, the time step must be kept quite small with this technique to prevent stability problems.

The explicit technique can be implemented easily using the a, b, and c coefficients defined by equations 69-78 and 81-100 with only minor changes. First, the $n/\Delta t$ part in equations 75, 76, 89, and 98 must be dropped. Second, the time evaluations in equations 90 and 100 must all be at the old time level n. With these changes, we can define a new set of coefficients (ac, bc, ce, and de) for the explicit procedure.

$$ae_i^{(m)} = a_i^{(m)} \qquad m = 1,2,3 \quad 9 \tag{101}$$

$$ce_i^{(m)} = c_i^{(m)} \qquad m = 1,2,3 \quad 9 \tag{102}$$

$$be_i^{(m)} = -a_i^{(m)} - c_i^{(m)} \qquad m = 1,2,3 \quad 9 \tag{103}$$

$$de_i^{(1)} = d_i^{(1)} \tag{104}$$

$$de_i^{(2)} = -\frac{n}{3\Delta t}\,\delta_i^n - \frac{2D_{Ti}^n}{\delta_i^n} - \frac{Q_{pi}^n\,\delta_i^n}{3(\phi_i^f - Z_{bi})^n} \tag{105}$$

$$de_i^{(3)} = -\frac{n}{\Delta t} Z_{bi}^n - Q_{si}^n \tag{106}$$

The explicit equations can then be written in the following form:

$$\phi_i^{fn+1} = Z_{bi}^{n+1} + \frac{\Delta t}{n} \Big[ae_i^{(1)} \phi_{i-1}^f + ae_i^{(2)} Z_{bi-1} + ae_i^{(3)} \delta_{i-1} + be_i^{(1)} \phi_i^f + be_i^{(2)} Z_{bi} + be_i^{(3)} \delta_i + ce_i^{(1)} \phi_{i+1}^f + ce_i^{(2)} Z_{bi+1} + ce_i^{(3)} \delta_{i+1} \Big]^n - \frac{\Delta t}{n} de_i^{(1)} \tag{107}$$

$$\delta_i^{n+1} = \frac{3\Delta t}{n} \Big[ae_i^{(4)} \phi_{i-1}^f + ae_i^{(5)} Z_{bi-1} + ae_i^{(6)} \delta_{i-1} + be_i^{(4)} \phi_i^f + be_i^{(5)} Z_{bi} + be_i^{(6)} \delta_i + ce_i^{(4)} \phi_{i+1}^f + ce_i^{(5)} Z_{bi+1} + ce_i^{(6)} \delta_{i+1} \Big]^n - \frac{3\Delta t}{n} de_i^{(2)} \tag{108}$$

$$Z_{bi}^{n+1} = \frac{\Delta t}{n} \Big[ae_i^{(7)} \phi_{i-1}^f + ae_i^{(8)} Z_{bi-1} + ae_i^{(9)} \delta_{i-1} + be_i^{(7)} \phi_i^f + be_i^{(8)} Z_{bi} + be_i^{(9)} \delta_i + ce_i^{(7)} \phi_{i+1}^f + ce_i^{(8)} Z_{bi+1} + ce_i^{(9)} \delta_{i+1} \Big]^n - \frac{\Delta t}{n} de_i^{(3)} \tag{109}$$

It is clear that equations 108 and 109 may be solved in any order since they only depend on old time level values. However, equation 107 can only be solved after equation 109 has been solved since Z_{bi}^{n+1} is needed on the right-hand side of 107.

RESULTS AND CONCLUSIONS

We have written implicit and explicit computer programs to implement the equations given in the previous section. In order to check the validity of the computer programs, several tests were run. In particular, analytical results can be obtained for simple cases. The simplest test was to see if initial horizontal surfaces for ϕ^f, Z_b and δ were maintained with no imposed stresses. Then the effects of constant areal recharge of freshwater, constant areal recharge of saltwater, and constant areal hydrodynamic dispersion were tested independently. The program results checked with known analytical behavior.

The area of interest in this study is the alluvial river valley of the Smoky Hill River between Salina and Solomon. McElwee (1985) has worked with this area using the sharp interface approximation. In applying the transition zone model currently under investigation, we will use the same physical parameters he used. These are only gross averages for the area and cannot be considered site specific.

$$k_f = 300 \text{ ft/day}$$

$$n = .15$$

$$N = .00197 \text{ ft/day}$$

$$a = .2$$

$$\gamma_f = 1.0$$

$$\gamma_s = 1.2$$

$$Q_s = .0000974 \text{ ft/day}$$

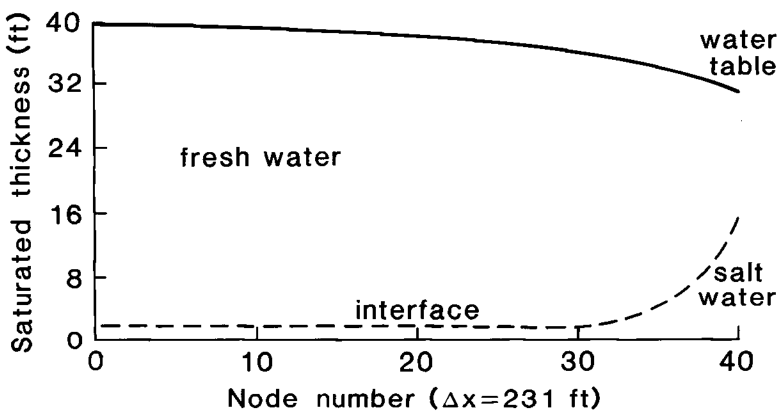
$$\Delta x = 231 \text{ ft}$$

The model was set up with 79 active nodes. The river is at node 40 with a base flow from the groundwater system of $Q_p = .154$ ft/day. Barrier boundaries (zero slope) are assumed for all three quantities ϕ^f, Z_b, and δ at nodes 0 and 80. The maximum saturated thickness in this area is about 40 feet. Using this data McElwee (1985) found that unstable upconing was occurring under the river. The profile generated by this sharp interface approximation is shown in Figure 1. The interface was assumed to intercept the bottom of the river when it was 10 feet below the water table at node 40.

As further checks on the validity of the new computer programs, a few other tests were run. The transition zone and saltwater zone can be disabled in the code so only the freshwater zone is modeled. We ran a few situations to verify that the freshwater zone was being modeled correctly. By setting $\delta = 0$, we have the sharp interface approximation. We attempted to reproduce the sharp interface result shown in Figure 1. The interface produced by the current programs was a little wider near the river than Figure 1, but was of acceptable accuracy. Lastly, the implicit and explicit programs were written separately using different numerical techniques. Therefore, a good comparison of results between the two would indicate both are working correctly. This was the case for several test runs. These transition zone programs are still in the development stage and need further testing along with the addition of more useful features.

One of the objectives of this study was to see if the transition zone model developed here would significantly affect the previous sharp interface results. The new parameter to be introduced is the dispersivity α_T. This will control the amount of hydrodynamic dispersion that is occurring. We have run the transition zone model for a range of α_T letting it approach a steady state profile in each case. The profiles for two values of α_T are shown in Figures 2 and 3. It is seen that in both cases we still have unstable upconing under the river. Additional runs indicate that for all reasonable values of α_T we still get unstable upconing. This was a little surprising. We had originally supposed that for large enough α_T a stable upconing situation would occur. It appears that the saturated thickness in this area is just too thin for that to happen.

The conclusion seems to be that although dispersion has an effect on the groundwater concentration profile, it is not the major mechanism feeding saltwater to the river. The convective effect causes the unstable upconing and feeds the major part of the saltwater to the river. Field data collected for this project suggest that the concentrations fall off away from the river faster than shown in Figure 3. Therefore, one might expect an upper limit on α_T in this area to be about 1.0. If one is primarily intersted in the concentration profile in the groundwater system, it is clear that the transition zone must be considered either by the present approximate model or a more rigorous model. However, it appears that the unstable upconing is a strong enough effect that the discharge of saltwater to the river can be understood mainly in terms of the sharp interface approximation (McElwee, 1985).



K_f=300 ft/day

γ_s=1.2 gm/cm^3

Figure 1. Sharp interface approximation profile.

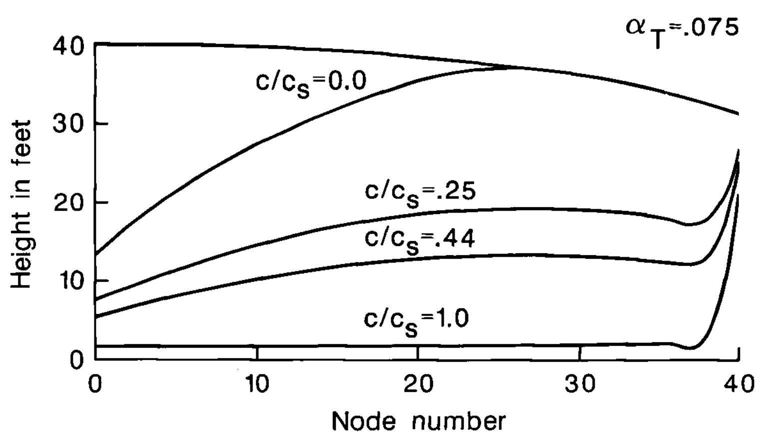


Figure 2. Salt concentration profile for α_T = .075

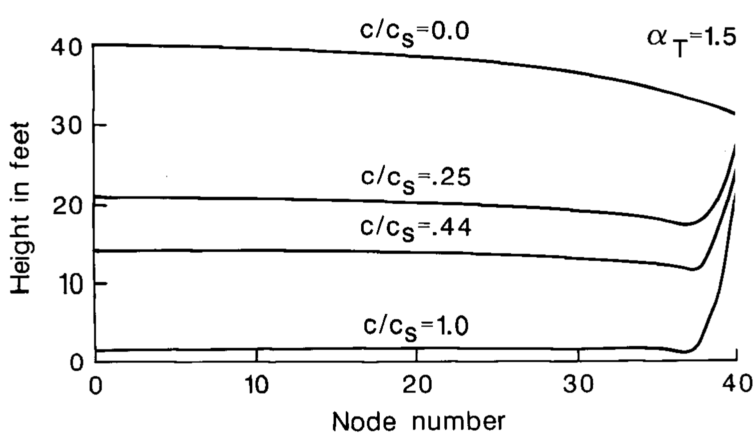


Figure 3. Salt concentration profile for α_T = 1.5

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