KANSAS GEOLOGICAL SURVEY
OPEN-FILE REPORT 83-32

GROUND-WATER FLOW MANAGEMENT MODELING

by

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1983

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ABSTRACT

Two major techniques are described for integrating distributed parameter ground-water models with economic models: linking and embedding. With linking, the economic model and hydrologic model are run at different stages. The economic model is run with "known" hydrologic factors such as drawdown; the hydrologic model is then run with the output of the economic model as demands and the cycles are repeated. Embedding is accomplished in two manners; namely, difference equations and response functions. With difference equations, the ground-water hydrogeology is modeled by a set of linear equations derived from the partial differential equations of flow through porous media. This set of equations is embedded directly into the economic model as a constraint set. With the response functions, a linear relationship between drawdown and stress, such as pumping, can be determined through a series of analytical or numerical derivations for only those points where well development is anticipated. It is found that for highly non-linear hydrologic systems, the linking technique is preferable. For linear flow systems and certainly non-linear flow systems where good linear approximations exist or where a transformation induces a linear system, the response function approach is preferred.

Key Words: Ground-water, management, ground-water models

GROUND-WATER FLOW MANAGEMENT MODELING

INTRODUCTION

Ground-water flow management is concerned with the planning and management of water resources in the subsurface environment subject to possible economic and hydrologic interactions between these resources and surface waters. The management of ground-water resources is a problem in resource allocation in the broadest sense. The managers must develop policies that discern from a broad spectrum of feasible management alternatives and conflicting environmental, institutional and economic constraints. Ground waters have traditionally been developed to satisfy fully, or in part, municipal, industrial or agricultural demands, which in many cases will be competing demands for a scarce resource.

Regardless of the economic sector that is vying for the ground waters, there is a traditional set of management alternatives to be examined. These are the decisions as to quantity of pumpage for existing wells and the location, number and quantity of pumpage for new wells. The term location may refer to both spatial and temporal aspects if timing and staging of well fields are critical decisions.

The ground-water flow models have been developed as tools for managers to evaluate the hydrologic response of a ground-water system to existing or proposed planning and managing policies on a regional basis. These models consist of:

1. A two-dimensional partial differential equation of flow composed of state variables that characterize the condition of the system (e.g., hydraulic head, equation (1), policy or decision variables that provide control over

the state variables (e.g., pumpage) and spatially distributed parameters that define the flow and storage properties of the system (e.g., transmissivity and storage coefficients),

2. Boundary condition equations that specify conditions at the boundaries such as recharge (e.g., subsurface inflow), and discharge (e.g., base-flow to surface waters, subsurface outflow or evapotranspiration), and

3. Initial condition equations that portray the state of the system prior to the present state of development (usually taken as steady state).

The flow equation, boundary or initial condition equation may be nonlinear functions of the state and decision variables. The parameters, which reflect the hydrogeology of the ground-water system, may vary spatially. The boundaries of the system may be irregular in shape. These complexities prevent the determination of closed analytical solutions, and numerical techniques must be relied on in practice.

These models will predict the behavior of the flow system under prescribed stresses when a flow equation is selected, boundary and initial conditions have been determined, values of the parameters have been estimated, and a solution algorithm is available. The accuracy of prediction is dependent on how well each one of these tasks has been accomplished. It should be noted that the performance of these tasks is in the realm of the ground-water hydrologist, but the assessment of how much accuracy is required for the prediction belongs to the manager.

Solutions provided by these models do not provide optimal management policies directly but must be coupled with some type of economic model with a suitable objective function and constraints, because stresses in the traditional flow models are proscribed and not determined. The ground-water model component is a classic economic technological function which relates production (withdrawal of water) to changes in raw material (hydraulic head, drawdown or changes in concentration).

The economic model will consist of an objective function which may depend implicitly or explicitly on economics, be a function of a multitude of purposes, or be a scalar function of trade-offs among a multitude of noncommensurable objectives; and constraint sets which may represent physical, legal, economic or social limitations. The coupling of ground-water flow models and the economic model will constitute a management model for the ground-water flow system. The solution to the management model may be provided by classical optimization techniques or by mathematical programming.

DISTRIBUTED PARAMETER GROUND-WATER MODELS

Four types of ground-water flow models will be considered in this paper: confined-transient, confined-steady state, unconfined-transient, and unconfined-steady state. For each of the four models, the assumption is made that the natural recharge and discharge to and from the aquifers is not disturbed by pumping. They are all regional, distributed parameter, two-dimensional models. When the amount of water an aquifer releases from a storage is a function of the compressibility of water and aquifer media and not of the quantity that can be drained from void spaces, the aquifer is said to be confined. An unconfined aquifer flow model is required if storage releases are dependent only on void drainage or if the aquifer has a free surface.

a. Confined-transient

Confined aquifers are bounded by confining layers or bedrock. The confining layers may or may not have their own storage capabilities or be partially permeable. The transient, two-dimensional flow model for a confined aquifer is given by the equation

$$\frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial h}{\partial y}\right) - S\frac{\partial h}{\partial t} = q \tag{1}$$

The parameters T and S are transmissivity and storage coefficient, respectively. The source term q of equation (1) is given by the equation

$$q = \sum_{j=1}^{N_W} Q(j,t)\delta(\hat{x}-\hat{x}_j) \tag{2}$$

where Q(j,t) is the pumping at time t from the jth well, N_W is the number of wells, and $\delta(x-x_j)$ is a Dirac delta function indicating that wells are to be treated as line sources. Define drawdown (Figure 1), as

$$s = H_O - h \tag{3}$$

where H_O is the initial head condition. Then equation (1) becomes

$$\frac{\partial}{\partial x}\left(T\frac{\partial s}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial s}{\partial y}\right) = S\frac{\partial s}{\partial t} - \sum_{j=1}^{N_W}\left(Q(j,t)\delta(\hat{x}-\hat{x}_j)\right) \tag{4}$$

The most common boundary conditions used with equation (4) are constant head along a portion Γ of the boundary

$$s = 0 \tag{5}$$

or unperturbed flow along a portion Γ' of the boundary (Figure 2).

$$\frac{\partial s}{\partial n} = 0 \tag{6}$$

where n is normal to Γ'. The Γ and Γ' may not be the same portion of the boundary. Equation (6) is the mathematical representation of the assumption that pumpage from the aquifer does not effect the natural recharge and discharge patterns to and from the aquifer, i.e.,

$$\frac{\partial s}{\partial n} = \frac{\partial Ho}{\partial n} - \frac{\partial h}{\partial n} = 0 \tag{7}$$

GROUND-WATER MODEL VARIABLES

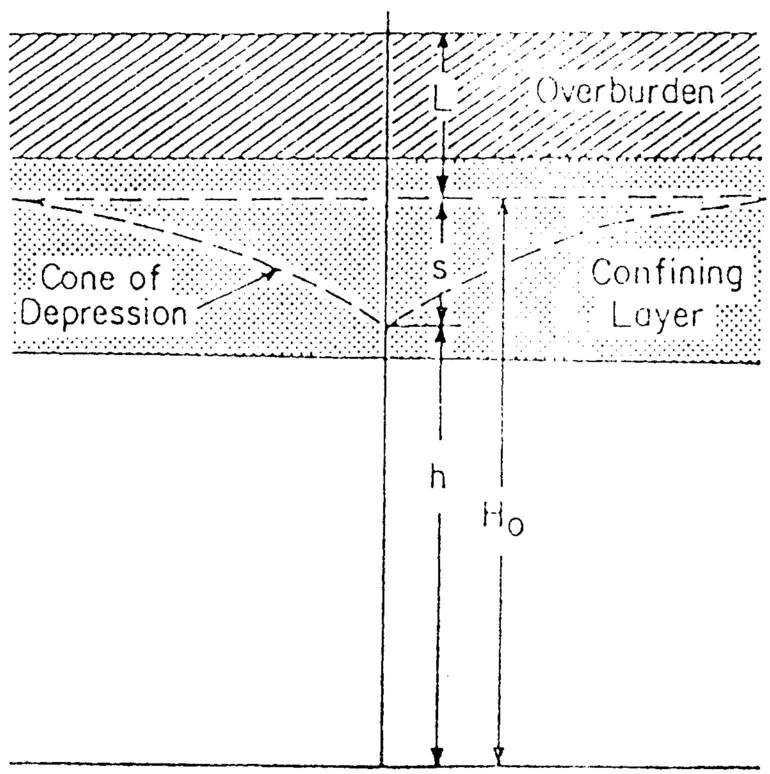
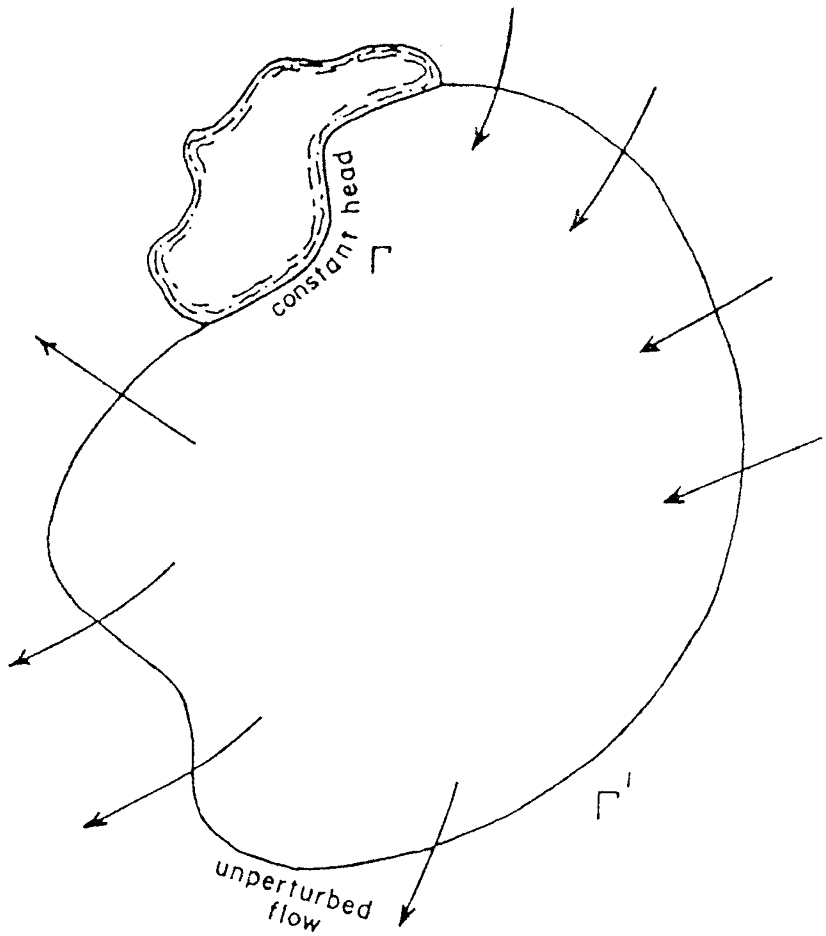


Figure 1. Ground-water Model Variables



Model Boundaries

Figure 2.

For unperturbed boundaries, equation (4) with boundary conditions given by equations (5) and (6) implies that all water supplied to wells must come from aquifer storage. In reality, the pumping from wells changes the recharge or discharge rates; it will induce leakage from an aquifer through any confining zones present or it will cause interaction with surface water by drawing waters that would normally be runoff or storage. Whenever pumpage effects disrupt the boundary conditions such that discharge is decreased or recharge is increased, water is said to be captured. If capture is known to occur, it must be explicitly modeled. Most capture terms are usually modeled as linear functions of drawdown. The most commonly modeled capture is capture through a confining layer from another aquifer. Capture phenomena will not be considered in this paper.

Initial conditions are commonly

$$s = 0 \quad \text{at } t = 0 \tag{8}$$

which implies the system is initially in steady state.

b. Confined-steady state

In many cases, the time period of management decision is longer than stress response processes within the aquifer. In such a situation, the system may arrive at a new steady state configuration. Since it matters little in terms of the decision process whether the new configuration was achieved instantaneously, or whether it was achieved after some real time passage within the management time period, steady state mathematical models can be used in place of equation (4) for the confined aquifer system,

$$\frac{\partial}{\partial x}\left(T\frac{\partial s}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial S}{\partial y}\right) = -\sum_{j=1}^{N_w} Q(j,t)\delta(\hat{x}-\hat{x}_j) \tag{9}$$

subject to the same boundary conditions as given by equations (5) and (6). It should be noted that unless there is a true constant head boundary or unless the water captured by a reduction in discharge or by an increase in recharge is sufficient to offset the pumping from wells, the aquifer will not achieve a steady state configuration and equation (9) will not be applicable as a ground-water flow model.

c. Unconfined-transient

In an unconfined aquifer, the head distribution has a free surface, and waters are released from storage when the movement of this surface creates drainage. The two-dimensional flow model for an unconfined aquifer is given by the Boussinesq equation

$$\frac{\partial}{\partial x}\left[kh\frac{\partial h}{\partial x}\right] + \frac{\partial}{\partial y}\left[kh\frac{\partial h}{\partial y}\right] = S_y\frac{\partial h}{\partial t} + \sum_{j=1}^{N_w} Q(j,t)\delta(\hat{x}-\hat{x}_j) \tag{10}$$

which has mathematical form similar to equation (1) if

$T = Kh$

and

$S_y = S$

The K is the two-dimensional hydraulic conductivity averaged over the vertical thickness and S_y is specific yield, a function gravity drainage. The other variables have their previous definitions. Boundary and initial conditions remain unchanged. Equation (10) is nonlinear in the variable h.

A linear approximation of equation (10) can be produced as follows: Let

$$\frac{S_y}{h} \simeq \frac{S_y}{\overline{H}} \qquad (11)$$

where

$$\overline{H} = \frac{1}{A} \iint H_o \, dA \qquad (12)$$

and A is the areal extent of the aquifer and make the change of variables

$$s^* = \frac{1}{\overline{H}} (sH_o - \frac{s^2}{2}) \qquad (13)$$

in equation (10), giving

$$\frac{\partial}{\partial x} (T' \frac{\partial s^*}{\partial x}) + \frac{\partial}{\partial y} (T' \frac{\partial s^*}{\partial y}) = S_y \frac{\partial s^*}{\partial t} - \sum_{j=1}^{N_w} Q(j,t)\delta(\hat{x}-\hat{x}_j) \qquad (14)$$

where

$$T' = \overline{H}k \qquad (15)$$

Equation (14) is linear and has an error of about the order $S/\overline{H}$. Note that equation (14) has the same mathematical form as equation (4).

Equation (12) implies that the initial condition s=0 at t=0 transforms to

$$s^*=0 \quad \text{at } t=0 \qquad (16)$$

and that the constant head boundary along Γ transforms to

$$s^* = 0 \quad \text{along } \Gamma \qquad (17)$$

The unperturbed boundary condition along Γ' transforms as follows: let Q be the flow per unit length through the boundary Γ', then along Γ'

$$Q = kh \frac{\partial h}{\partial n} = kH_0 \frac{\partial Ho}{\partial n} \tag{18}$$

because the boundary conditions are unperturbed. Thus along Γ',

$$h \frac{\partial h}{\partial n} = H_0 \frac{\partial Ho}{\partial n} \tag{19}$$

and substituting equations (3) and (13) gives

$$\frac{\partial s^*}{\partial n} = 0 \tag{20}$$

d. Unconfined steady state

An unconfined, steady state model is given by the equation

$$\frac{\partial}{\partial x}\left(T' \frac{\partial s^*}{\partial x}\right) + \frac{\partial}{\partial y}\left(T' \frac{\partial s^*}{\partial y}\right) = - \sum_{j=1}^{N_w} Q(j,t)\delta(\hat{x}-\hat{x}_j) \tag{21}$$

and is subject to boundary conditions given by equation (17) and/or (20). The s^* is defined by equation (13) without time dependence and T' is defined by equation (15). Equation (21) is not an approximation to the steady state formulation of equation (10),

$$\frac{\partial}{\partial x}\left(kh \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(kh \frac{\partial h}{\partial y}\right) = \sum_{j=1}^{N_w} Q(j,t)\delta(\hat{x}-\hat{x}_j) \tag{22}$$

but is exact. Again, care must be taken in applying equation (22) to field problems.

The set of ground-water models used in this paper are summarized in Table 1. For all the models, effects of wells are represented mathematically as point sources. It is commonly assumed that the wells fully penetrate the saturated thickness of the aquifer. Field data on well pumpages are tabulated as accumulated quantities over sets of time periods, usually of equal duration. Thus, a time series of pumping data will consist of rectangular pulses,

$$Q(k,n) = \sum_{i=1}^{n} q(k,i)\ [u(t-(i-1)\eta-u(t-i\eta)] \tag{23}$$

where η is the duration of the time period, $q(j,t)$ is the discharge from the jth well during the ith time period, n is such that $t = n\eta$, and $u(t-i\eta)$ is a unit step function (figure 3)

$$u(t-i\eta) = \begin{cases} 0 & t < i\eta \\ 1 & t \geq i\eta \end{cases} \tag{24}$$

If the flow system is modeled by a steady state model, the drawdowns return to an equilibrium during the pumping period. Thus, although a time series of different pumpages will create a time series of different drawdowns, the drawdown time series contains no memory of previous drawdown states, that is, drawdown effects for a time period are a function of pumping only from within that time period.

Bachmat et. al. (1978) reviewed the status of ground-water models as tools for ground-water resource management and included combined economic and physical models in the survey.

TABLE 1. REGIONAL GROUND-WATER FLOW MODELS

Conditions	Partial Differential Equation	Boundary and Initial Conditions
Confined-Transient	$\frac{\partial}{\partial x}\left(T\frac{\partial s}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial s}{\partial x}\right) - S\frac{\partial s}{\partial t} = -\sum_{j=1}^{N_w} Q(j,t)\delta(\hat{x}-\hat{x}_j)$	$s = 0$ at $t = 0$ $s(\Gamma) = 0$ or $\frac{\partial s}{\partial n}(\Gamma') = 0$
Confined-Steady State	$\frac{\partial}{\partial x}\left(T\frac{\partial s}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial s}{\partial x}\right) = -\sum_{j=1}^{N_w} Q(j)\delta(\hat{x}-\hat{x}_j)$	$s(\Gamma) = 0$ or $\frac{\partial s}{\partial n}(\Gamma') = 0$
Unconfined-Transient 1/	$\frac{\partial}{\partial x}\left(T'\frac{\partial s^*}{\partial x}\right) + \frac{\partial}{\partial y}\left(T'\frac{\partial s^*}{\partial y}\right) - S_y\frac{\partial s^*}{\partial t} = -\sum_{j=1}^{N_w} Q(j,t)\delta(\hat{x}-\hat{x}_j)$ $s^* = \frac{1}{\overline{H}}\left(s^*H_o - \frac{s^{*2}}{2}\right); \quad T' = K\overline{H}$	$s^* = 0$ at $t = 0$ $s^*(\Gamma) = 0$ or $\frac{\partial s^*(\Gamma')}{\partial n} = 0$
Unconfined-Steady State	$\frac{\partial}{\partial x}\left(T'\frac{\partial s^*}{\partial x}\right) + \frac{\partial}{\partial y}\left(T'\frac{\partial s^*}{\partial y}\right) = -\sum_{j=1}^{N_w} Q(j,t)\delta(\hat{x}-\hat{x}_j)$ $s^* = \frac{1}{\overline{H}}\left(s^*H_o - \frac{s^{*2}}{2}\right)$	$s^*(\Gamma) = 0$ or $\frac{\partial s^*(\Gamma')}{\partial n} = 0$

1/ Linear approximation

$$\frac{S_y}{h} = \frac{S_y}{\overline{H}}$$

PULSE PUMPING DEMAND

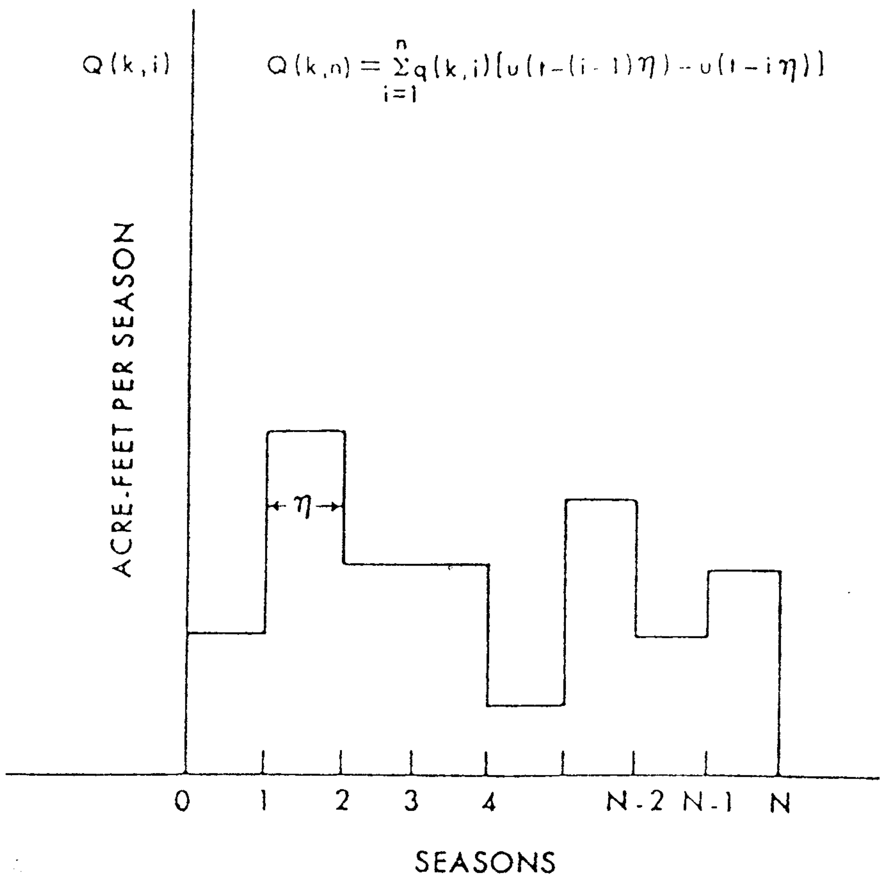


Figure 3

ECONOMIC MODELS

An economic model consists of an objective function which is a mathematical expression of a criteria to measure preference, and sets of constraints which act to limit the preferred actions. The objective function may be expressed in explicit economic terms such as costs and net benefits or in implicit economic terms such as total pumpage or drawdowns. In either case the objective function is used to search for a set of optimal actions subject to the limitations of the constraints. Thus, one may look for least cost, maximum net benefit, maximum pumpage or minimum drawdown.

Regardless as to whether the economics are implicitly or explicitly expressed in the objective function, two general types of management models have evolved. One is used to operate an existing or established system in the most optimal or efficient manner. For example, only operating costs or benefits are included for an explicit economics model and only pumpage or drawdown at existing wells are included for an implicit economics model. The other is used to expand the capacity of an existing system in an optimal or efficient manner. Thus, capital and fixed costs are included in explicit economics models, and pumpage and drawdown for proposed wells are included in the implicit economic models. The capacity expansion, implicit economic models are quite useful in establishing safe yields.

A number of example management models are presented. They are representative of the types of models applied in the field and will be used to demonstrate the techniques of combining physical and economic models. It should be noted that constraints relating pumping to drawdown have not been formulated mathematically in this section. This will be done in the following section.

a. Least cost pumping of ground water from an existing well field

An example of an explicit economic model for an established well field is a model which determines the pumpage from a set of N_w existing wells such that the pumping cost is minimized, subject to the constraints that the demand for water each time period is met (Figure 4).

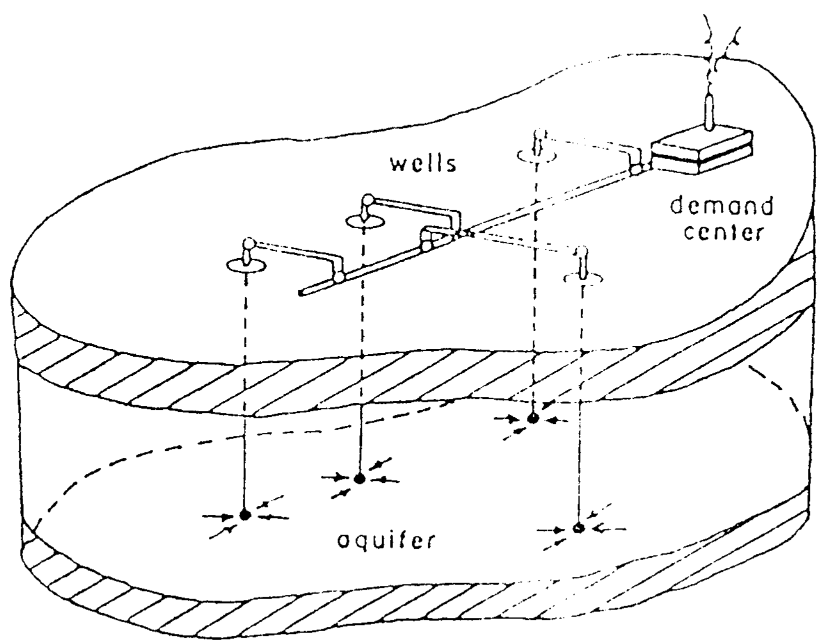


Figure 4

Let Q(k,n) be the quantity of water pumped from the kth well during the nth time period, s(k,n) be the drawdown at the kth well at the end of the nth time period, L(k) the initial lift at the kth well, c(k) the cost of pumping a volume of water per foot of lift and r the interest rate, then

$$Z = \sum_{k=1}^{N_W} \sum_{n=1}^{N_E} \frac{c(k)}{(1+r)^n} Q(k,n)[s(k,n) - L(k)] \quad (25)$$

is an objective function that may be used to determine the least cost operation of a set of N_W existing wells over N_E time periods. The term Q(k,n) [s(kn,) -L(k)] represents the energy consumed pumping Q(k,n) and thus equation (25) minimizes energy costs. The constraints are:

1. the demands D(n) for water during each time period is met

$$\sum_{k=1}^{N_W} Q(k,n) \geq D(n), \qquad n=1,\ldots,N_E \quad (26)$$

2. the quantity of water pumped from the kth well during any of the time periods may not exceed some design capacity

$$Q(k,n) \leq Q_{max}(k), \qquad n=1,\ldots,N_E;\ k=1,\ldots,N_W \quad (27)$$

3. The drawdown for the kth well may not exceed the design for any time period.

$$s(k,n) \leq s_{max}(k) \qquad n=1,\ldots,N_E;\ k=1,\ldots,N_W \quad (28)$$

4. and finally, there exists a constraint set which relates the effects of pumpage with drawdown.

If the latter constraint set establishes a linear relation between pumpage and drawdown, there these constraints, along with those represented by equations (25) through (28), constitute a quadratic programming problem.

There are software packages available for solving large quadratic programming problems. The solution of the quadratic program gives the quantity of water to be pumped from each well and the operating costs over the NE time periods.

b. Capacity expansion for a well field

An example of capacity expansion with explicit economics is when a manager of a ground-water system would like to determine the least cost spatial and temporal development of new wells along with the least cost operation of existing wells. Let $c_F(k)$ be the fixed cost of completing the kth well; $\delta(k,n)$ be a zero-one variable such that δ is one if the kth well is drilled in the nth time period and zero otherwise, then

$$Z = \{ \sum_{k=1}^{N_W} \sum_{n=1}^{N_E} \frac{c_f(k)}{(1+r)^n} \delta(k,n) + \sum_{k=1}^{N_W} \sum_{n=1}^{N_W} \frac{c_f(k)}{(1+r)^n} Q(k,n)[s(k,n)-L(k)]\} \qquad (29)$$

is an objective function that is used to determine the least cost expansion for a well field. The N_W now refers to both potential well site and actual established wells.

The constraints are

1. The demand, D(n), for water during each time period is met

$$\sum_{\varepsilon=1}^{N_W} Q(k,n) \geq D(n), \qquad n=1,\ldots,N_E \tag{30}$$

2. The well must be completed before it is pumped,

$$\varepsilon(k,n)Q_{max}(k) - Q(k,n) \geq 0 \tag{31}$$

where,

$$\varepsilon(k,n) = \sum_{i=1}^{n} \delta(k,i) \tag{32}$$

and $\varepsilon(k,i)$ is a zero-one variable. Note that this constraint also insures that the kth well does not exceed its design pumping capacity.

The rest of the constraint sets are the same as for operating an existing field.

The explicit economics capacity expansion model constitutes a mixed integer, quadratic program when drawdowns are linearly related to pumpage.

c. Safe yield for a ground-water system

A typical problem faced by a water manager is to determine what is the maximum quantity of water that may be removed from an aquifer without producing some undesirable drawdown effect. If there is capture available, either from decreasing discharge, increasing recharge or reducing evapotranspiration, then the ground-water system may go to a new steady state provided that the maximum quantity pumped does not exceed the capture. If pumpage exceeds capture or no capture is available, the aquifer is mined and the ground-water system remains in a transient state. In either the steady state or the transient state, the maximum quantity of water pumped is called the safe yield.

If $Q(k,n)$ is the quantity of water pumped from the kth well during the nth time period, then

$$Z = \{ \sum_{k=1}^{N_W} \sum_{n=1}^{N_E} Q(k,n)\} \tag{33}$$

is an objective function to determine the maximum quantity of water to be extracted from a set of N_W wells over N_E time periods

The constraints are

1. A well may not pump more than its design capacity

$$\varepsilon(k,n)Q_{max}(k) - Q(k,n) \geq 0 \tag{34}$$

where $\varepsilon(k,n)$ is a zero-one variable

2. A well must pump at least a minimal quantity to be economical

$$\varepsilon(k,n)Q_{min}(k) - Q(k,n) \geq 0 \tag{35}$$

Note that if the kth well is pumped during the nth time period

$$Q_{min}(k) \leq Q(k,n) \leq Q_{max}(k) \tag{36}$$

3. The drawdown (or a function of drawdown) shall not exceed a specified value.

$$s(k,n) \leq \delta_{max}(k,n) \tag{37}$$

The constraints relating drawdown to pumpage remain the same as the previous examples.

d. Pollution Control in Contaminated Ground-Water Basins

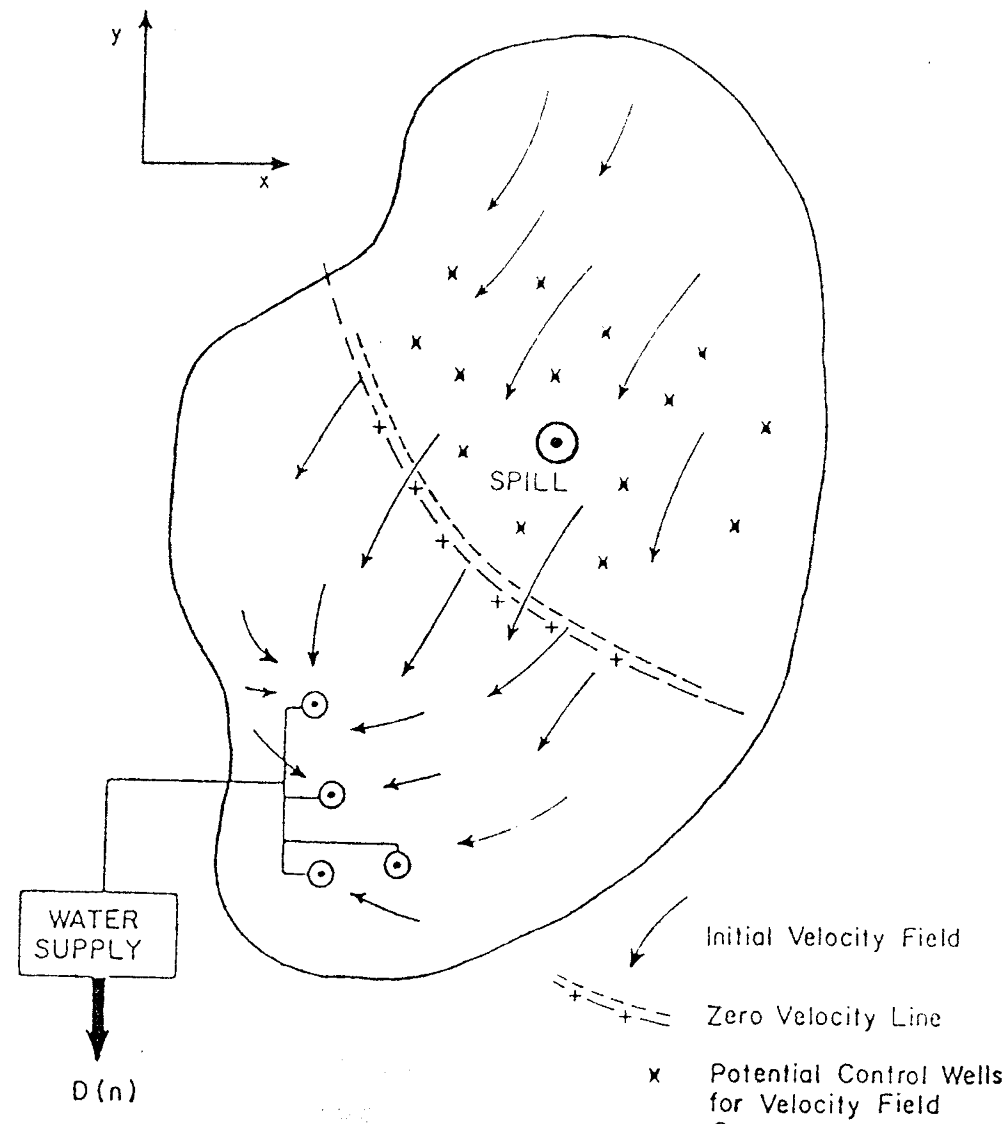
A ground-water manager wishes to develop an optimal strategy for deploying interception wells to control ground-water contamination, while operating a system of wells to supply water to meet demands. The objective is to determine the minimum number of interception wells, their location and their pumpage such that the water supply demands are met by pumping of existing, non-contaminated wells and that a desired velocity field is created at a set of observation points (Figure 5).

Let $Q_I(k,n)$ be the quantity of water pumped from the kth interception well in the nth time periods. The objective function is

$$\min \left\{ \sum_{n=1}^{N_E} \sum_{k=1}^{N_{Iw}} Q_I(k,n) \right\} \tag{38}$$

where N_{Iw} is the number of potential interception wells.

Figure 5



Interception Well Model

The constraints are

1) The pumping demand must be met

$$\sum_{j=1}^{N_{pw}} Q(j,n) \geq D(n) \qquad n=1,\ldots,N_E \tag{39}$$

where N_{pw} is the number of pumping wells

2. The velocity field at the observation points has the desired value. Since velocity is a vector it will have an x and y component.

$$VI_x(m,n) + VP_x(m,n) + VO_x(m,n) = VDIS_x(m), \qquad m=1,\ldots,N_O$$
$$VI_y(m,n) + VP_y(m,n) + VO_y(m,n) = VDIS_y(m), \qquad n=1,\ldots,N_E \tag{40}$$

where $VI_x(m,n)$ and $VI_y(m,n)$ are the x and y component of velocity at the mth observation point during the nth time period due to pumping velocity from the interception wells, $VP_x(m,n)$ and $VP_y(m,n)$ are the x and y components of velocity at the mth observtion point during the nth time period due to pumping from the water supply wells, $VO_x(m,n)$ and $VO_y(m,n)$ are the x and y components at the mth observation point during the nth time period due to past pumping practices and the initial velocity field, and $VDIS_x(m\ )$ and $VDIS_y(m)$ are the desired velocity components at the mth observation point.

3. A function relationship between velocity components and pumping. (Figure 5).

The above objective function and constraint sets constitute a linear program. There are many software packages available for solving large scale linear programs.

INTEGRATING PHYSICAL AND ECONOMIC MODELS

The functional relations between drawdown or velocity components and pumpage have not been specified in any of the previous management models. It is through this set of constraints that the coupling of the physical system and the economic system occurs. Two general methods have evolved to integrate economic models with the physical, distributed parameter, ground-water models. These methods are called embedding and linking.

Embedding

Embedding is accomplished in two manners; namely difference embedding and response functions.

With difference embedding, the ground-water system is modeled by a set of linear equations which represents a finite difference or finite element approximation to the partial differential equation describing ground-water flow in the aquifer. This set of equations is embedded directly into the linear programming management model as a constraint set.

It should be noted that there is an advantage of using a finite element model in that the number of nodes needed is less than required in a finite difference model of the same prediction accuracy.

For example, if the ground-water system is modeled by the partial differential equation (1), then a finite difference approximation is given by

$$\frac{T(i-1/2,j)}{\Delta y^2}[s(i-i,j,n) - s(i,j,n)] - \frac{T(i+1/2,j)}{\Delta y^2}[s(i+i,j,n) - s(i,j,n)]$$

continued

$$+ \frac{T(i,j-1/2)}{\Delta x^2} [s(i,j-i,n) - s(i,j,n)] - \frac{T(i,j+1/2)}{\Delta x^2} [s(i,j+1,n) - s(i,j,n)] -$$

$$\frac{S(i,j)}{n} s(i,j,n) = \frac{S(i,j)}{n} s(i,j,n-1) + Q(i,j,n) \tag{41}$$

where

$$T(i-1/2,j) = 2 \frac{T(i-1,j)T(i,j)}{T(i-i,j)+T(i,j)}$$

$$T(i+1/2,j) = 2 \frac{T(i+1,j)T(i,j)}{T(i+i,j)+T(i,j)}$$

$$T(i,j-1/2) = 2 \frac{T(i,j-1)T(i,j)}{T(i,j-i)+T(i,j)} \tag{42}$$

$$T(i,j+1/2) = 2 \frac{T(i,j+1)T(i,j)}{T(i,j+1)+T(i,j)}$$

a rectangular difference grid has been used. The index n represents the nth time step and i and j represent row and column number, respectively, of the rectangular grid imposed to produce the discrete form. The values of Δx and Δy are the representative distances between node point; n is the value of the time step. The kth well pumps $Q(i_k,j_k,n)$ during the nth time period, where i_k,j_k are the row and column number for the well. Equation (40) becomes a constraint set to provide the relation between pumping with drawdown needed in the management models.

Longenbough (1970), Aguado and Remson (1974), Aguado et. al. (1974, 1977), Remson et. al. (1974), and Alley et. al. (1976) have used this technique extensively to examine management plans for various types of groundwater problems, such as dewatering an excavation, and maximizing heads at desired locations. The approach proved to be feasible for a wide variety of hydrologic situations (Aguado and Remson, 1974), such as confined and unconfined configurations, steady-state and transient flow, and one and two-dimensional systems. The principle advantage of difference embedding is that every node point of the difference grid can be a potential site for wells. There are two major disadvantages to the difference embedding. The first is that the number of constraints can become inhibitively large. For example, if a 2000 node ground-water model is embedded in a management model and if there are ten time steps in the design period, 20000 constraints are added to the management model. The second disadvantage is that this set of constraints has a mathematical form that is difficult for mathematical programming techniques to solve. Very large errors are induced and the programming methods may never establish an optimal solution. $\sum Op \neq Op \sum$

Response Function

An alternative method of embedding is by response functions. Response functions represent an efficient approach to the problem of coupling a distributed parameter ground-water model with an economic and management model. The approach involves determining a set of functions that relates pumping over time at wells to drawdowns or velocity components at those wells (Maddock, 1972 and Morel-Seytoux, 1975).

The response function techniques are applicable to linear, partial differential equations and thus can be applied to all four drawdown models listed in Table 1. Because the partial differential equations with their initial (if transient models) and boundary conditions are linear, there exists a Green's function such that

$$s'(\hat{x},t) = \int_{\hat{x}} \int_{o}^{t} F(x',\tau) G(\hat{x},\hat{x}',t-\tau)\, dx' d\tau \tag{43}$$

where

$$F(\hat{x}',\tau\ ) = \sum_{J=1}^{N_W} Q(\hat{x}_j,\tau)\delta(\hat{x}-\hat{x}_j) \tag{44}$$

and s'(x,t) is equal to either s(x,t) or s*(x,t) depending on which equation in Table 1 is used. For a design period of NE equal duration time periods of length η, and NW well sites, the drawdown or function of drawdown at the kth well at the end of the nth time period can be rewritten as s'(k,n) and is given by the equation

$$s'(k,n) = \sum_{j=1}^{N_W} \sum_{i=1}^{n} Q(j,i) \int_{(i-i)\eta}^{i\eta} G(\hat{x}_k,\hat{x}_j,\ n\eta-\tau)\, d\tau \tag{45}$$

Define

$$\beta(k,j,n-i+1) = \int_{(i-i)\eta}^{i\eta} G(\hat{x}_k,\hat{x}_j,\ n\eta-\tau)\, d\tau \tag{46}$$

and equation (44) becomes

$$s'(k,n) = \sum_{J=1}^{N_W} \sum_{i=1}^{n} Q(j,i)\beta(k,j,n-i+1) \tag{47}$$

If k is an arbitrary point X = (x,y) and s' is the drawdown s then

$$s(\hat{x},n) = \sum_{J=1}^{N_W} \sum_{i=1}^{n} \beta(\hat{x},j,n-i+1)Q(j,i) \tag{48}$$

recalling that

$$s = H_O - h \tag{49}$$

gives

$$h(\hat{x},n) = H_O(\hat{x}) - \sum_{J=1}^{N_W} \sum_{i=1}^{n} \beta(\hat{x},j,n-i+1)Q(j,i) \tag{50}$$

taking the derivative with respect to x or y, multiplying by the hydraulic conductivity K(isotropic) and dividing by the porosity, ϕ gives

$$\frac{k}{\theta}\frac{\partial h}{\partial x} = \frac{k}{\theta}\frac{\partial H_O}{\partial x} - \sum_{J=1}^{N_W} \sum_{i=1}^{n} \frac{k}{\theta}\frac{\partial \beta}{\partial x}(\hat{x},j,n-i+1)Q(j,i) \tag{51}$$

$$\frac{k}{\theta}\frac{\partial h}{\partial y} = \frac{k}{\theta}\frac{\partial H_O}{\partial y} - \sum_{J=1}^{N_W} \sum_{i=1}^{n} \frac{k}{\theta}\frac{\partial \beta}{\partial y}(\hat{x},j,n-i+1)Q(j,i)$$

but by the definition of velocity

$$V_x = \frac{k}{\theta}\frac{\partial h}{\partial x}$$
$$V_y = \frac{k}{\theta}\frac{\partial h}{\partial y} \qquad (52)$$

hence at the mth observation point in the nth time period

$$V_x(m,n) = V_{ox}(m) - \sum_{J=1}^{N_w}\sum_{i=1}^{n} \beta_{vx}(m,j,n-i+1)Q(j,i)$$
$$V_y(m,n) = V_{oy}(m) - \sum_{i=1}^{N_w}\sum_{i-1}^{n} \beta_{vy}(m,j,n-i+1)Q(j,i) \qquad (53)$$

where $v_{ox}(m)$ and $v_{oy}(m)$ are the x and y components of the initial velocity field at the mth observation points before development.

If the steady state groundwater flow model is applicable, the drawdown-pumping relation is given by

$$s'(k) = \sum_{J=1}^{N_w} \beta(k,j)Q(j) \qquad (54)$$

and the velocity field components are given by

$$V_x(m) = V_{ox}(m) - \sum_{J=1}^{N_w} \beta_{vx}(m,j)Q(j)$$
$$V_y(m) = V_{oy}(m) - \sum_{i=1}^{N_w} \beta_{vy}(m,j)Q(j) \qquad (55)$$

the β's, βvx's and βvy's are response functions. They are constants and independent of pumping and drawdown. For example $\beta(k,j,n-i+1)$ measures the increment of drawdown of the kth well at the nth time period due to unit pumping at the jth well during the ith time period. The $\beta vx(m,j,n-i+1)$ and $\beta vy(m,j,n-i+1)$ are the x and y components of velocity respectively at the mth observation point at the nth time period due to unit pumping at the jth well during the ith time period ($i<n$). The response functions are related to the well hydraulics, being functions of the distances between wells, the transmissivity and storage qualities of the aquifer, the boundary and initial conditions, and the partial differential equation chosen to emulate the flow phenomena. In practice, the response functions are determined by a simulation model (Maddock, 1975) because the irregularly-shaped boundaries and nonhomogeneous parameters make analytical determination impossible. Once the response functions are computed by simulation and saved, they do not have to be re-calculated. The algebraic form (Maddock, 1972) of the response function allows the behavior of the ground-water system to explicitly be included in a management model.

The use of response functions is best illustrated in the work of Maddock (1972, 1973, and 1974), Haimes and Dreizen (1977), Dreizen and Haimes (1977), Maddock and Haimes (1975), Morel-Seytoux (1975), Morel Seytoux and Daly (1975), and Larson et al. (1976).

The principle advantage to using response functions is that only potential well sites need be considered in the management model and the extraneous equations of the difference-embedding technique for sites of non-interest are not present. The disadvantages are that the ground-water system must be modeled by a linear set of equations, and if the number of potential well sites is large, the storage requirements for β's may become prohibitively large.

Linking

The linking approach requires that the economic model and physical model be run at different stages. The stage may be different time period or different iteration within a time period. The physical model is usually represented by a simulation model and the economic model is represented by a mathematical program. In most cases, the mathematical program determines pumpage while the simulation model emulates the response of the ground-water system to the pumpage. The simulation model provides a detailed incorporation of aquifer properties such as free surface conditions. The economic model provides detailed pricing and cost information along with the pumpage. Linking will require modification of the mathematical formulation of the management model. For example, consider the previous economic model which found the least cost pumpage for a system with existing wells. The linking approach requires that there are two stages within each time period. In the mathematical programming stage (economic model), stage 1, the drawdowns are assumed known. For the nth time period the objective function of the mathematical program is

$$\min\{Z(n) = \sum_{k=1}^{N_W} \frac{C(k)}{(1+r)^n} Q(k,n)[s(k,n)+L(k)]\} \quad (56)$$

Because the s(k,n)'s are known, the objective function is linear. The constraints are

$$\sum_{k=1}^{N_W} Q(k,n) \geq D(n) \qquad n=1, \ldots, N_E \qquad (57)$$

and

$$Q(k,n) \leq Q_{max}(k) \qquad k=1, \ldots, N_W; \; n=1, \ldots, N_E \qquad (58)$$

The mathematical program is a linear program and its solution provides the pumpage values, Q(k,n). The Q(k,n) calculated become input into the second stage, which is a ground-water simulation model (physical model). The output of the simulation model is a new set of drawdowns. These drawdowns are used in the objective function of the linear program for the next time period. This cycle is continued until the design horizon N_E has been reached.

Linked model approaches are represented in the work of Martin et al. (1969), Bredehoeft and Young (1970) and Young and Bredehoeft (1972). Martin used a two-stage linking model with the ground-water system modeled and solved by an analog model, and the economic model modeled by a linear program. The objective of the management model was to maximize agricultural profits given ground-water costs based on the quantity of water pumped and the lift. For a set of lifts, the linear programming management model determined the optimal crop configuration and the water requirements from wells. The pumpage was manually applied to the analog model, resulting in a new set of lifts for the following pumping period. The resulting head configuration was entered into the linear program to determine the new cropping pattern. This process was repeated until the design horizon was reached. Bredehoeft and Young (1970) replaced the electric analog model with a digital model, which allowed the crop configuration and head distributions over the design horizon to be determined in a single run of the computer. In their approach, the ground-water hydrology was modeled by a simulation model based on numerical approximation to a set of partial differential equations, and the management model by a linear program. Both models were contained in the same control program but were run at separate stages.

The reformulation of the least cost pumpage model to expedite the use of linking has produced a management model that will be suboptimal over the N_E cycles and will underestimate pumping costs (s(k,n)) in equation (55) in the drawdown at the beginning of the time period.

Previous studies using a linked approach to integrating economic and distributed parameter ground-water models (Bredehoeft and Young, 1970; Young and Bredehoeft, 1972; and Martin et al., 1969) demonstrate that the linking method can be used to solve nonlinear problems. These studies dealt with unconfined alluvial fill aquifers, in which transmissivity varied with saturated thickness, resulting in nonlinearities in the equations of flow. Thus, the simulation model used in Stage 2 can be based directly on equation (10), the Boussinesq model, rather than on the approximation, equation (14). Linking reduces the number of dependent variables in the objective function and constraint sets, and the number of constraints since each optimization takes place for a single time period. As a general rule, mathematical programming solution algorithms run-times increases exponentially with the number of constraints and variables. Thus, it will take less time to run N_E smaller mathematical programs than one that includes all the N_E time periods.

Although there are several advantages to the linking combinatorial technique, there is a major disadvantage of employing this technique due to the nature of simulation. Using a linked approach does not guarantee a global optimum, nor even a local optimum for that matter. If a large number of decision variables exists for the problem in question, a thorough examination of the response of the ground-water system may involve a large number of additional simulation runs. The cost of the additional number of simulation runs may discourage a regulatory agency from using a linked approach to integrating economic and distributed parameter ground-water models. Bredehoeft and Young (in 1970 and 1972) were able to identify an approximate optimal management plan, since the strategy resulting in maximum benefits was selected from a relatively small number of alternatives.

The search for an optimal management strategy which will involve many decision variables, such as pumpage, crop selection, well spacings and locations, may not be achieved without prohibitive costs of manpower and computer time.

The following conclusions and recommendations are made with respect to combining economic and physical models.

1. Linear ground-water models exist for confined or unconfined ground-water systems. A linear approximation is applied to the unconfined system.

For the confined, steady-state and transient ground-water system, the resulting model is linear in drawdown. For the unconfined, steady-state ground-water system, the model is linear in a function of drawdown, i.e.,

$$s^* = \frac{1}{H}\left(SH_o - \frac{S^2}{2}\right)$$

For the unconfined, transient ground-water system, the approximation is also linear in a function of drawdown. Mixed unconfined and confined ground-water systems were not examined in this paper.

2. For the four economic models considered, Model 1 is a quadratic program, Model 2 is a mixed integer-quadratic program, Model 3 is a mixed integer program, and Model 4 is a linear program, when the relation between drawdown and pumping is linear. The mathematical form of these models is representative of those applied in the field.

3. The principle advantages and disadvantages for methods of integrating physical and economic models, linking difference-embedding and response functions, are summarized in Table 3. The following recommendations are made of the applicability of the three methods.

a. If the relation between drawdown and pumping can only be modeled by a non-linear relation, then the linking technique should be used.

b. If very large numbers of potential well sites (i.e. $NW \geq 500$) are to be considered, it may be preferable to accept the problems with the inversion of the simplex algorithm and use the difference-embedding technique.

c. If difference-embedding is used, a finite element approximation is preferable to a finite difference approximation because the number of constraint equations is less with finite elements.

d. For a linear relation between pumping and drawdown, and for NW $<$ 500, the response function technique is preferable.

ACKNOWLEDGEMENT

The work upon which this paper is based was supported in part by funds provided by OWRT Project No. A-095-ARIZ, as authorized under the Water Resource Research Act of 1964, and funds from the Kansas Geological Survey.

COMBINED ECONOMICAL AND PHYSICAL MODELS

TABLE 3

ADVANTAGES	DISADVANTAGES
Linking	
1. Can be used with a non-linear ground-water model	1. Suboptimization
2. Small constraint set - single time period optimization	2. Addition simulations runs necessary to define optimal region
3. Global optimum for steady-state explicit economic models	3. Drawdown limitation constraints can not be introduced
	4. Ground-water costs under-estimated in Models 1 and 2
Difference-embedding	
1. Any grid node may be a potential well site	1. Very large constraint sets
2. Finite difference or finite element models may be used in the embedding process	2. Mathematical form of drawdown-pumping constraint set produces large error in simplex methods
3. Global optimums for linear ground-water models	3. Suboptimize non-linear models
	4. Restricted to linear ground-water models
Response Function	
1. Greatly reduced constraint sets	1. Restricted to linear ground-water models
2. Global optimums	2. Large β storage requirements if the number of potential well sites is large

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