

Key  
CF  
82-13

THE 2-D INVERSE PROBLEM AS A SUITE OF 1-D PROBLEMS

Carl D. McElwee  
Kansas Geological Survey  
The University of Kansas  
Lawrence, Kansas 66044

Prepared for Presentation at  
Fall AGU Meeting  
San Francisco, California  
1982

## ABSTRACT

Aquifer parameters obtained from the inverse process may be nonunique or have large errors. The sources of error and difficulty may be classified into three major areas: uncertainty in knowledge of basic groundwater hydrology (including appropriate mathematical equations), uncertainty in data measurement, and nonuniqueness and insensitivity of the chosen model. This paper will deal primarily with the last area. A steady-state groundwater model containing  $n$  node points has  $n^2$  associated sensitivity coefficients if the transmissivity is allowed to vary at every node. To condense this, the transmissivity is usually assigned to constant values in certain zones. However, this greatly reduces the degree of transmissivity variation that can be considered. A more efficient approach is to contour the historical head values and draw an appropriate number of streamlines. The 2-D problem may then be broken into a number of independent 1-D problems along various streamtubes. The collection of 1-D problems is much easier to deal with analytically and numerically. Since the transmissivity on one streamline does not affect the head on any other, a great deal of insensitivity and instability has been removed. Analytical expressions for the sensitivity coefficients along the streamtube may be derived which are easy and inexpensive to evaluate. Sensitivity coefficients can delineate areas of low sensitivity, where several nodes may be summed to obtain a minimum sensitivity and make the inverse process more stable. Using sensitivity coefficients, one can look at the effect of error in head along the streamtube on the transmissivity distribution. In general, the error in head is more critical in areas of lower sensitivity. In summary, this is a very efficient inverse procedure which gives diagnostic output and allows a rather general variation in transmissivity.

## INTRODUCTION

The groundwater inverse problem involves solving for the aquifer parameters (usually transmissivity and storativity) from known historical values of hydraulic head. The general conclusion reached by most investigators is that the groundwater inverse process is nontrivial and fraught with difficulty and uncertainty. The aquifer parameters obtained from the inverse process may be nonunique, have large errors, or be completely unobtainable. The sources of error and difficulty may be classified into three major areas: uncertainty in knowledge of basic groundwater hydrology (including appropriate mathematical equations), uncertainty in data measurement, and nonuniqueness and insensitivity of the chosen model. This paper will deal primarily with the last area, but will have some impact on the second area.

A sensitivity coefficient (**slide #1**) can be defined as the limit of the ratio of the change in hydraulic head at one point to the change in transmissivity at the same or another point. A steady-state groundwater model containing  $n$  node points has  $n^2$  associated sensitivity coefficients if the transmissivity is allowed to vary at every node. This makes for a cumbersome problem if  $n$  is very large. To condense this, the transmissivity is usually assigned constant values in a certain small number of zones. However, this greatly reduces the degree of transmissivity variation that can be considered.

A more efficient approach (**slide #2**) is to contour the historical head values and draw an appropriate number of streamlines. The 2-D problem may then be broken up into a number of independent 1-D problems along various streamtubes. The collection of 1-D problems is much easier to deal with analytically and numerically. The number of sensitivity coefficients that

must be considered has been reduced substantially. Since the transmissivity on one streamtube does not affect the head on any other streamtube, those sensitivity coefficients are identically zero. Thus, each streamtube becomes an independent 1-D problem. Working along streamlines should remove a great deal of insensitivity and instability from the inverse process.

Contouring the hydraulic head and drawing in the streamlines presents its own set of problems. The data points are usually not as dense and as accurate as we would like. However, this is the basic data we have to work with and the accuracy of the inverse process hangs in the balance. Any available geohydrologic information should be incorporated into the contouring process. The exact position of the contours and streamlines will depend on the data accuracy, the interpolation strategy, and our knowledge of the flow system. However, if we cannot draw the head contours and streamlines with some degree of confidence, we probably do not have enough information to attempt any inverse procedure.

#### **HEAD ALONG STREAMTUBE**

An analytical solution for the head along a streamtube may be easily derived. Its numerical approximation is shown in this slide (**slide #3**).  $Q$  is the flux of water between two nodes separated by a space  $\Delta x$ .  $x$  is measured along the streamtube.  $L$  is the width of the streamtube,  $T$  is transmissivity, and  $h$  is the head.  $Q$  represents the total flux of water between two nodes and depends on such things as recharge, discharge, and boundary conditions.  $h_N$  is the head at one end of the streamtube. Of course, the accuracy of this numerical formula increases as the node spacing,  $\Delta x$ , and streamtube width,  $L$ , decrease. This formula is very easy and inexpensive to evaluate on a

computer. Notice that only the ratio  $Q/T$  appears in this formula. There is a basic nonuniqueness,  $Q$  and  $T$  cannot be determined independently.

### **SENSITIVITY COEFFICIENTS**

The sensitivity coefficients can be obtained very easily by differentiating the expression for the hydraulic head. It is convenient to work with two kinds of sensitivity coefficients. The first (**slide #4**) is the sensitivity with respect to transmissivity. Notice that these sensitivity coefficients are inversely proportional to the square of the transmissivity. Therefore, areas of low transmissivity have greater sensitivity. The second type (**slide #5**) of sensitivity coefficient involves the head at the end of the streamtube,  $h_N$ .  $h_N$  can be considered a boundary condition at one end of the streamtube. In the inverse procedure to be described shortly, we shall try to determine the transmissivity and  $h_N$  simultaneously to obtain the best fit to the head data. For this reason, the sensitivity coefficients of the second type must be considered. For convenience, in the following equations  $T$  and  $h_N$  shall be dropped as subscripts of  $U$ ; but, when  $j$  equals  $n$ , we know we are dealing with the second type of sensitivity coefficient.

### **LEAST SQUARES INVERSE PROCEDURE**

If an aquifer parameter, denoted by  $P$ , is changed by an amount  $\Delta P$ , the perturbed head,  $h^*$ , may be obtained to first order by using a Taylor series as shown in this slide (**slide #6**). The squared error between the observed head,  $h_e$ , and the new head after changing the parameters by  $\Delta P$  is shown in this slide (**slide #7**). A necessary condition for the error to be a minimum is to set the partial derivative with respect to  $\Delta P_n$  equal to zero. This results in a set of  $n$  simultaneous equations for the  $\Delta P$ 's. Two new quantities  $\alpha$  and  $f$ ,

which involve the sensitivity coefficients, can be defined for conciseness (slide #8). At this point, we have  $n$  equations for  $n$  parameter changes based on  $n$  values of observed head. In other words, this is the direct inverse procedure. Since we are using a first order Taylor's series, a few iterations are usually necessary to allow the  $\Delta P$ 's to approach sufficiently close to zero.

On the other hand, if we choose to use fewer than  $n$  aquifer parameters, we have the indirect inverse procedure (slide #9). We may choose to let certain adjacent nodes have the same value of  $T$ . This is accomplished very easily in our system of  $n$  simultaneous equations.  $\alpha$  is the coefficient matrix for our set of equations. Summing the rows and columns that correspond to nodes having the same value of  $T$  results in a new set of  $M$  simultaneous equations for  $M$   $\Delta P$ 's.  $\alpha'$  and  $f'$  are the appropriate new quantities after summation. This defines the indirect inverse procedure and a few iterations are usually required to allow  $\Delta P$  to approach zero.

The sensitivity coefficients can be used to delineate areas of low sensitivity. However, for a streamtube with only 10 nodes, there are 100 sensitivity coefficients. For this reason, it is usually more convenient to look at the diagonal elements of  $\alpha$ . From the definition of  $\alpha$ , it is seen that these diagonal elements are a measure of the total model sensitivity to the transmissivity at a certain node. If the diagonal elements are smaller in a certain region, several nodes may be summed to obtain a minimum sensitivity and make the inverse process more stable.

The sensitivity coefficients can also be used to evaluate the effect of an error in head at node  $i$  on the calculated transmissivity at node  $j$ , as shown in this slide (slide #10). In general, the error in head is more critical in areas of lower sensitivity.

## **HYPOTHETICAL MODEL**

We have tested these ideas on some hypothetical models. For very accurate data, everything seems to work very well. This slide (**slide #11**) shows the head and streamlines for a hypothetical model where  $h$  is specified on the upper boundary, the lower boundary is a barrier to flow, and  $Q$  is specified on the ends of the model. This plot was made using the model output for 28 node points. The contours were produced by Surface II, an automated contouring package developed at KGS. All the contour maps shown from here on are made by Surface II without direct human intervention to change any of the contours. In fact, it is seen that the head lines and streamlines are not everywhere perpendicular. Human intervention could substantially improve this map. However, we wanted to test the method with a data set known to contain errors. We used the 5 streamtubes shown on this slide to implement the inverse process. The position of the streamlines and the lines of constant head were obtained by digitizing the crossings. This was done on a digitizing table with computer printout of the x-y coordinates.

The actual transmissivity distribution is shown in this slide (**slide #12**). It is an increasing function, trending from the lower left-hand corner to the upper right-hand corner.

## **DIRECT INVERSE CALCULATIONS**

The transmissivity distribution obtained from the direct inverse process is shown in this slide (**slide #13**). There is noticeable error along the upper edge and the upper right-hand corner. This is also where the streamlines and head lines were not perpendicular. So we are seeing the result of error in our original data. The next slide (**slide #14**) shows the percent error for the

direct inverse process. Generally the error is less than 10% with some 15-20% errors on the upper edge of the model.

#### **INDIRECT INVERSE CALCULATIONS**

The next slide (**slide #15**) shows the results for the transmissivity distribution using the indirect inverse procedure. In this case, consecutive node points along the streamtubes have been placed in groups of three with appropriate condensation of the system of equations. In this case, the number of observed head values will be three times greater than the number of values of calculated transmissivity. The percent error is shown in the next slide (**slide #16**). It is seen that the error along the upper edge of the model has decreased by about 5%. However, the error in the lower left-hand corner has increased to about 20%. Examination of the data going into the contouring program shows that the error should only be about 15%. The contouring program may have trouble near corners and has introduced some extra error there.

In this particular model there is no recharge or discharge in the interior of the model. Thus, the flux along a streamtube is constant. As mentioned earlier, there is a basic nonuniqueness in groundwater problems so that only the ratio of flux to transmissivity can be determined. For this particular model, that means that the absolute value of the transmissivity can be varied by varying the assumed flux along the streamtube. In the real world, we rarely know the fluxes with great accuracy. Therefore, the fluxes may be adjusted within limits to establish a certain correlation of T values perpendicular to the streamlines. It is clear that this correlation must be established from other geohydrologic information because it is not contained in the head data. Suppose we know the transmissivity at one node point near the center of each streamtube from whatever source of information. This I

call the modified indirect inverse procedure. The results are shown on this slide (slide #17). As one might expect, this is the best result we have obtained for this model. The percent errors are shown in the next slide (slide #18). The errors are generally in the range of 5-10% over most of the model. There is still a contouring problem in the lower left-hand corner.

#### **SUMMARY AND CONCLUSIONS**

In summary, the inverse procedure that I have outlined decouples the 2-D problem into a number of 1-D problems. This should eliminate some of the insensitivity and instability since flow is only considered along streamlines. Analytical expressions for the head and sensitivity are easy to obtain and inexpensive to evaluate. The sensitivity coefficients or the diagonal elements of  $\alpha$  can be used to delineate areas of low sensitivity. The sensitivity coefficients can also be used to evaluate the effect of head error on the calculated transmissivity. In summary, this is a very efficient inverse procedure which gives diagnostic output and allows a rather general variation in transmissivity.

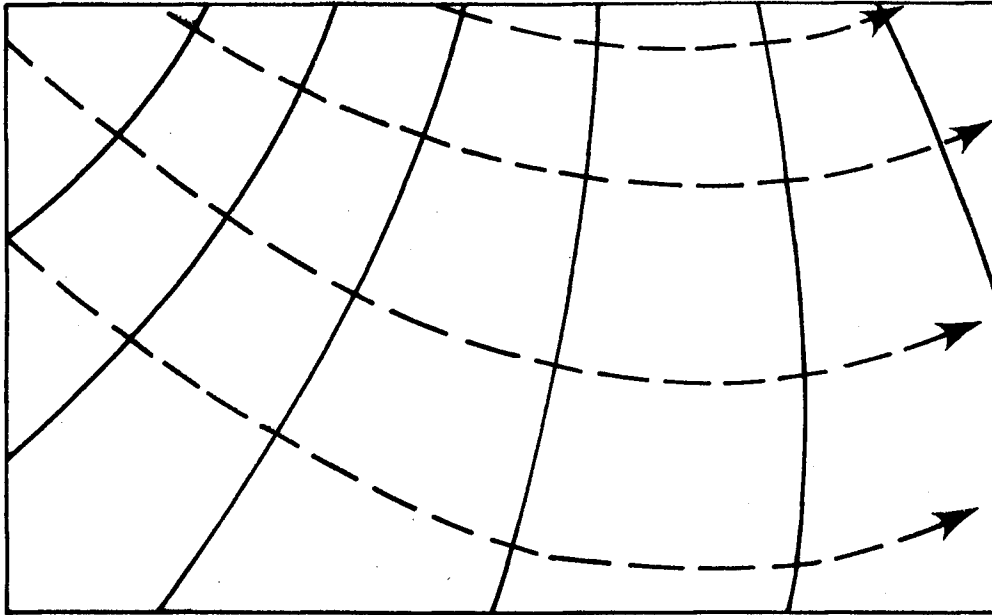
SENSITIVITY COEFFICIENTS

$$U_{Ti;j} = \frac{\Delta h_i}{\Delta T_{j+\frac{1}{2}}} \quad i, j = 1, 2, \dots, N$$

$T_{j+\frac{1}{2}} \equiv$  Transmissivity between nodes  
j and j+1

$h_i \equiv$  Head at node i

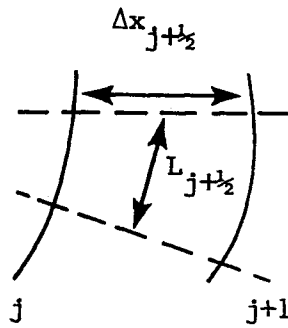
# SCHEMATIC OF HEAD MAP



— — — LINES OF CONSTANT HEAD  
- - - STREAMLINES

ANALYTICAL SOLUTION FOR HEAD

$$h_i = \sum_{j=i}^{N-1} \frac{Q_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}}{L_{j+\frac{1}{2}} T_{j+\frac{1}{2}}} + h_N$$



SENSITIVITY COEFFICIENTS

$$U_{Ti;j} = \begin{cases} -\frac{Q_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}}{L_{j+\frac{1}{2}} T_{j+\frac{1}{2}}^2} & j \geq i \\ 0 & \text{otherwise} \end{cases}$$

$1 \leq i \leq N$   
 $1 \leq j \leq N-1$

SENSITIVITY COEFFICIENTS

$$U_{h_N i; N} = \frac{\Delta h_i}{\Delta h_N} = 1 \quad 1 \leq i \leq N$$

Drop T and  $h_N$  as subscripts of U

PERTURBED HEAD

$$h_i^* \approx h_i + \sum_{j=1}^N U_{i;j} \Delta P_j$$

$$\Delta P_j = \begin{cases} \Delta T_{j+\frac{1}{2}} & 1 \leq j \leq N-1 \\ \Delta h_N & j = N \end{cases}$$

Slide #6

INVERSE PROCEDURE

$$\text{ERROR} = \sum_{i=1}^N (h_{e_i} - h_i - \sum_{j=1}^N U_{i;j} \Delta P_j)^2$$

$$\frac{\partial (\text{ERROR})}{\partial (\Delta P_j)} = 0 \text{ for minimization}$$

Slide #7

DIRECT INVERSE PROCEDURE

$$\alpha_{j,k} = \sum_{i=1}^N U_{i;j} U_{i;k}$$

$$f_j = \sum_{i=1}^N U_{i;j} (h_{e_i} - h_i)$$

$$\sum_{k=1}^N \alpha_{j,k} \Delta P_k = f_j$$

INDIRECT INVERSE PROCEDURE

$$\sum_{m=1}^M \alpha'_{\ell,m} \Delta P_m = f'_{\ell}$$

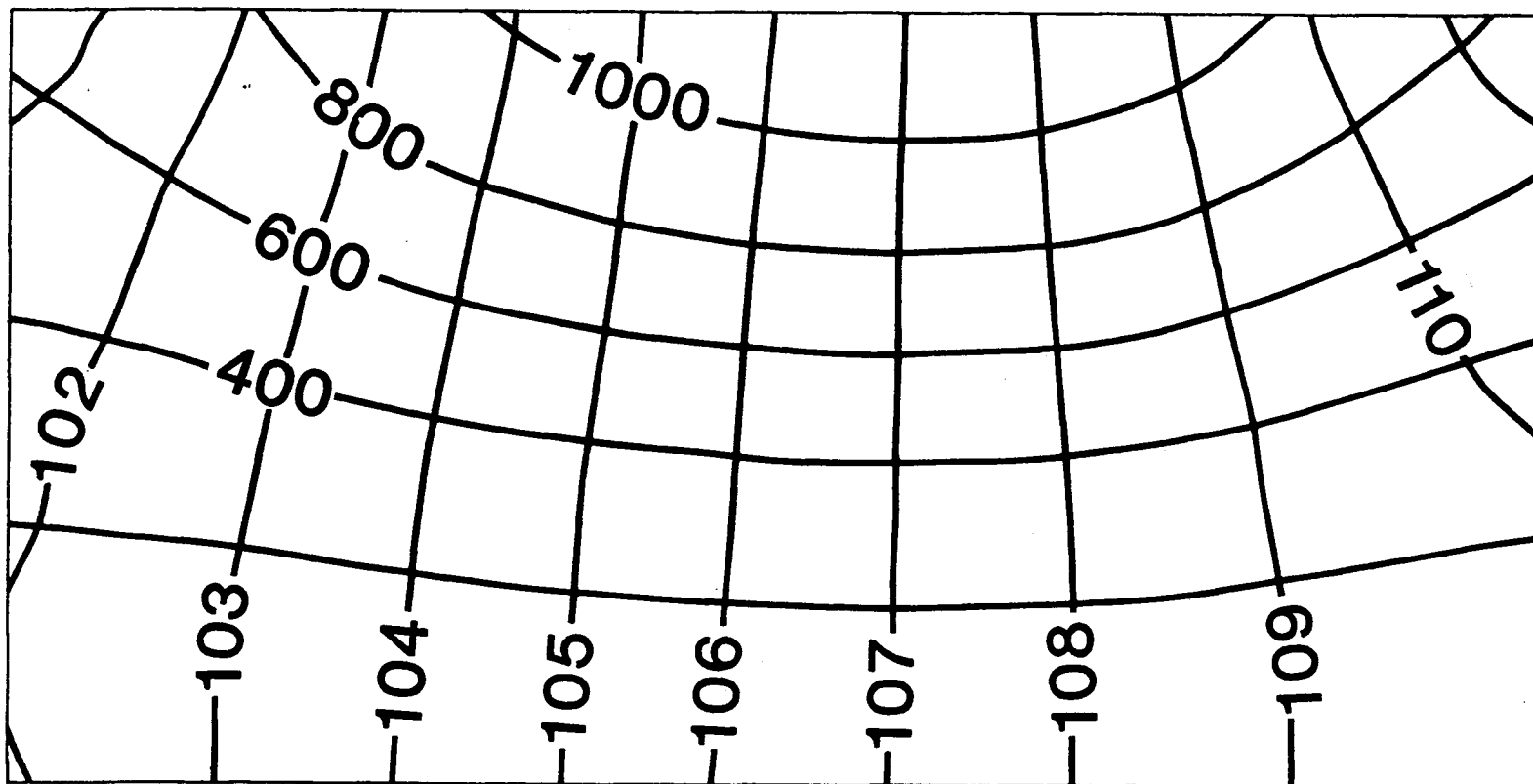
$$M < N$$

Slide #9

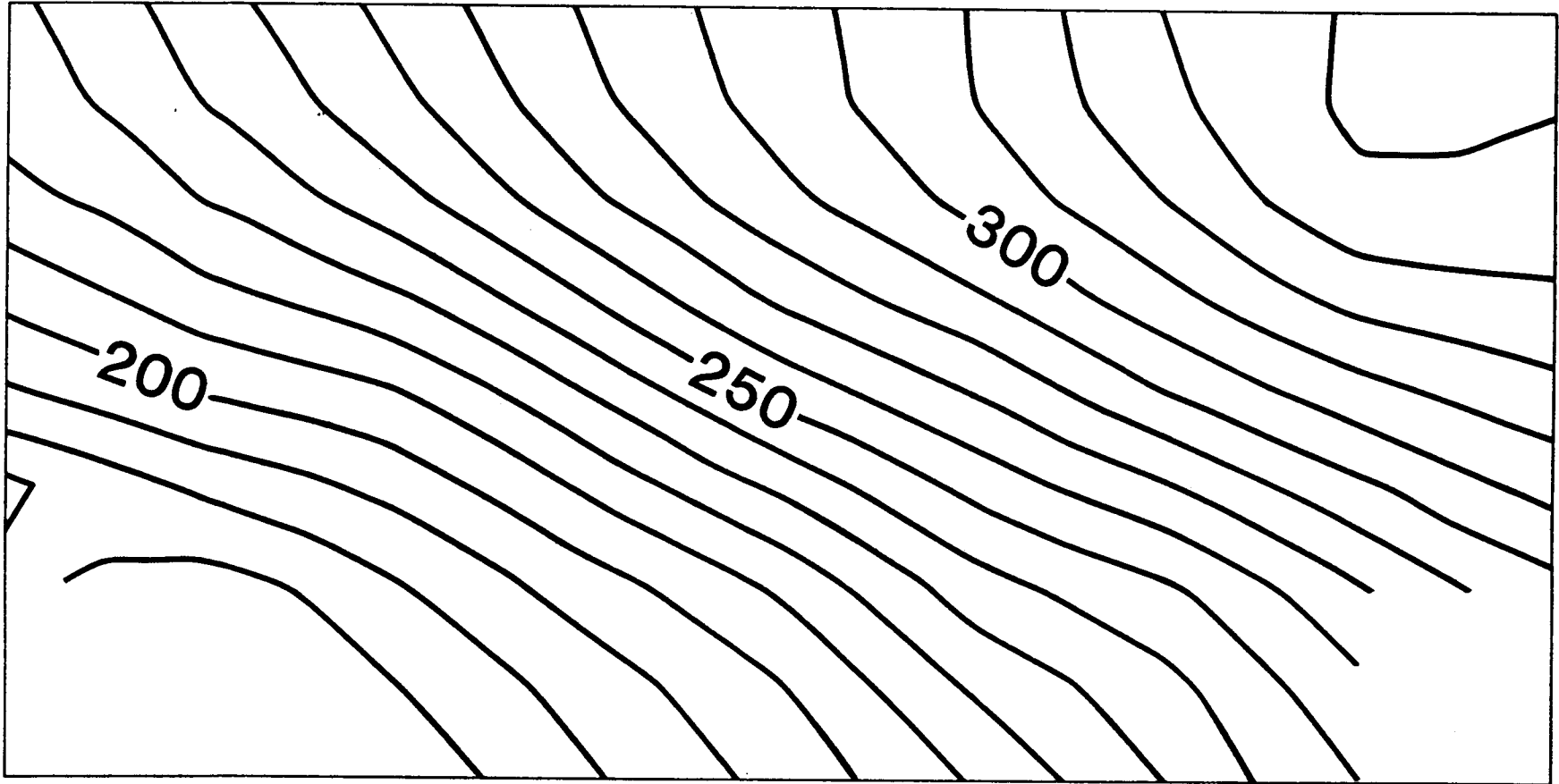
Relation of parameter error to head error

$$\Delta P_j = \frac{\Delta h_i}{U_{i;j}}$$

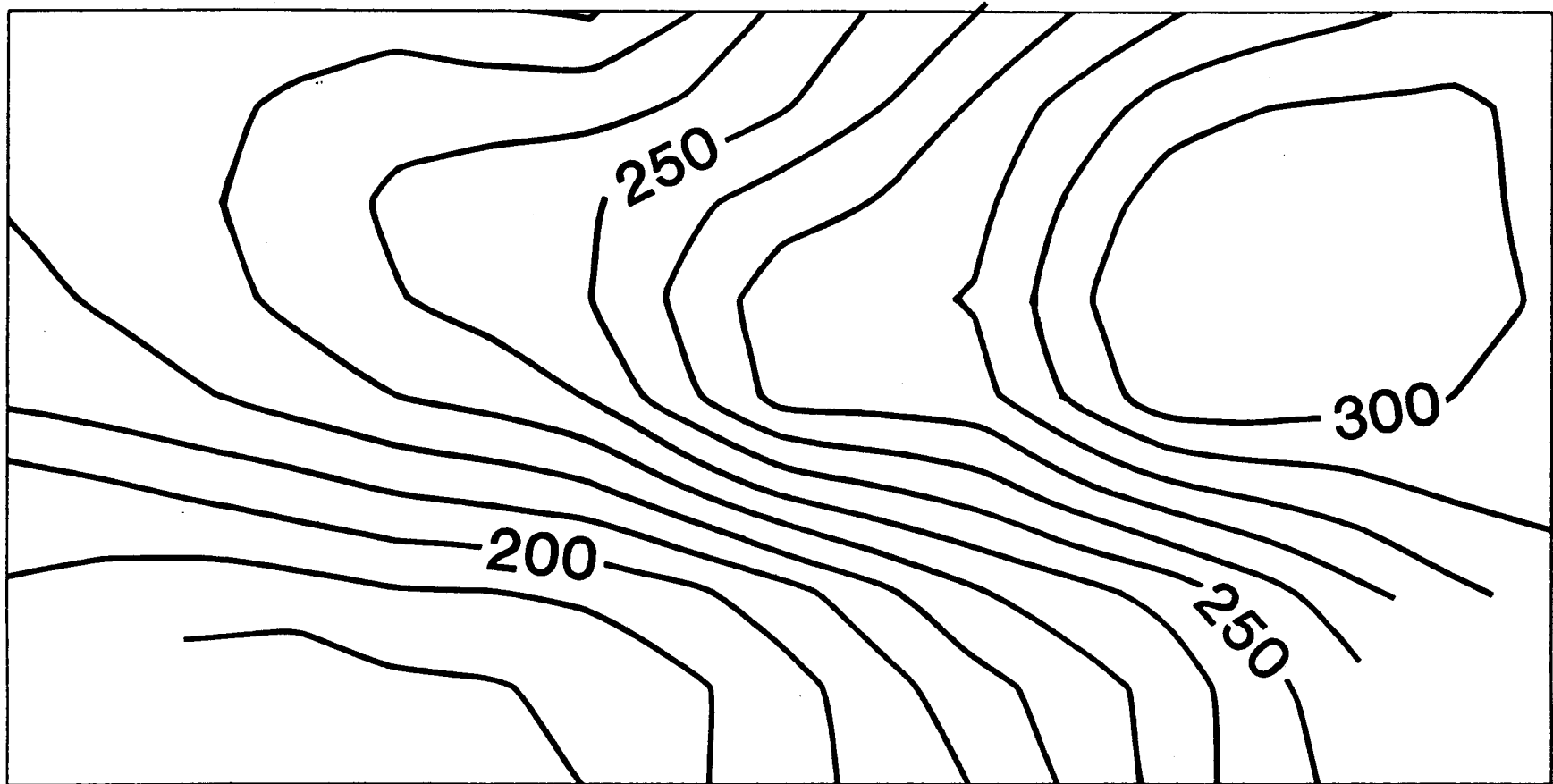
Slide #10



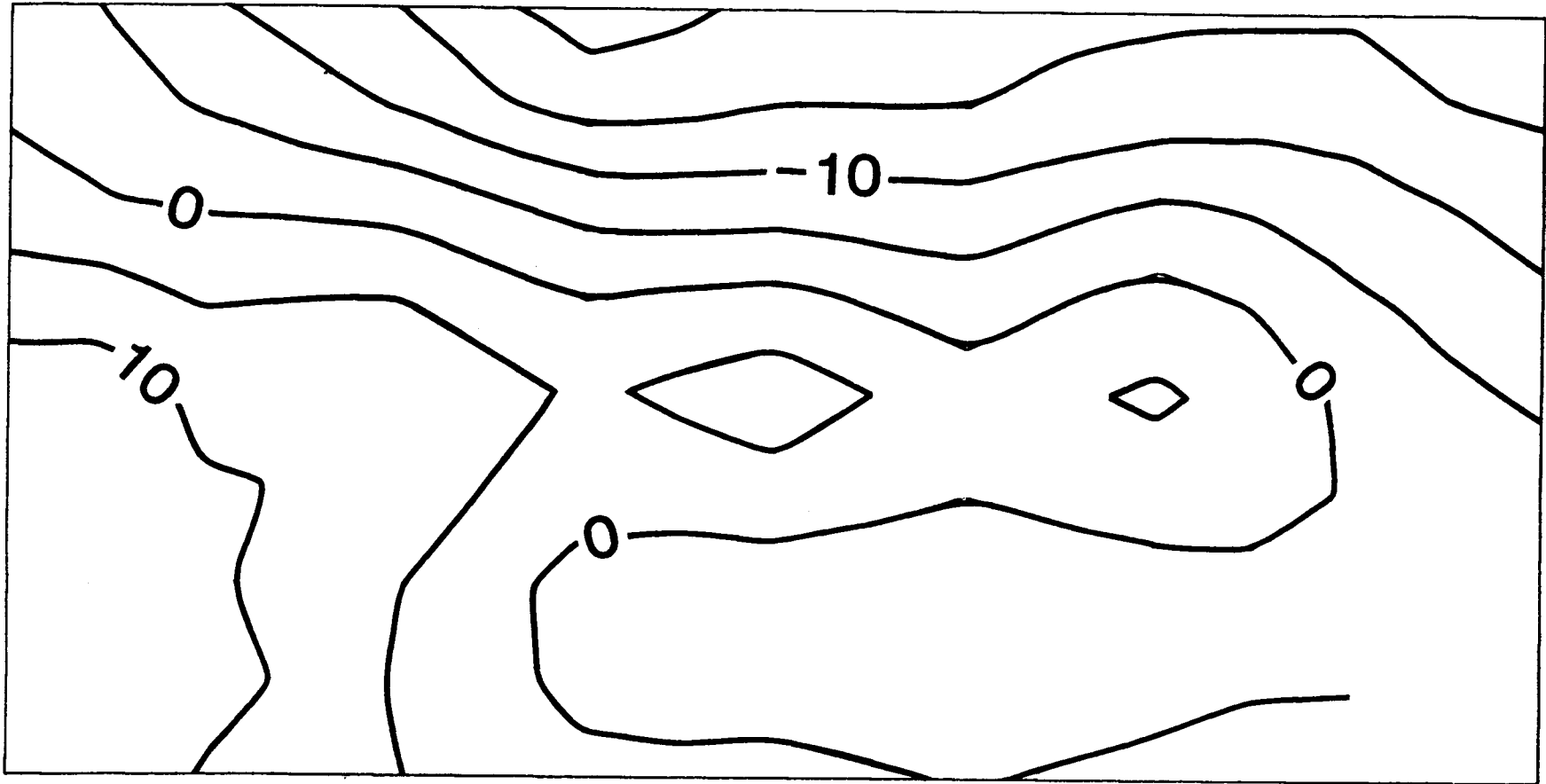
**Model Head And Streamlines**



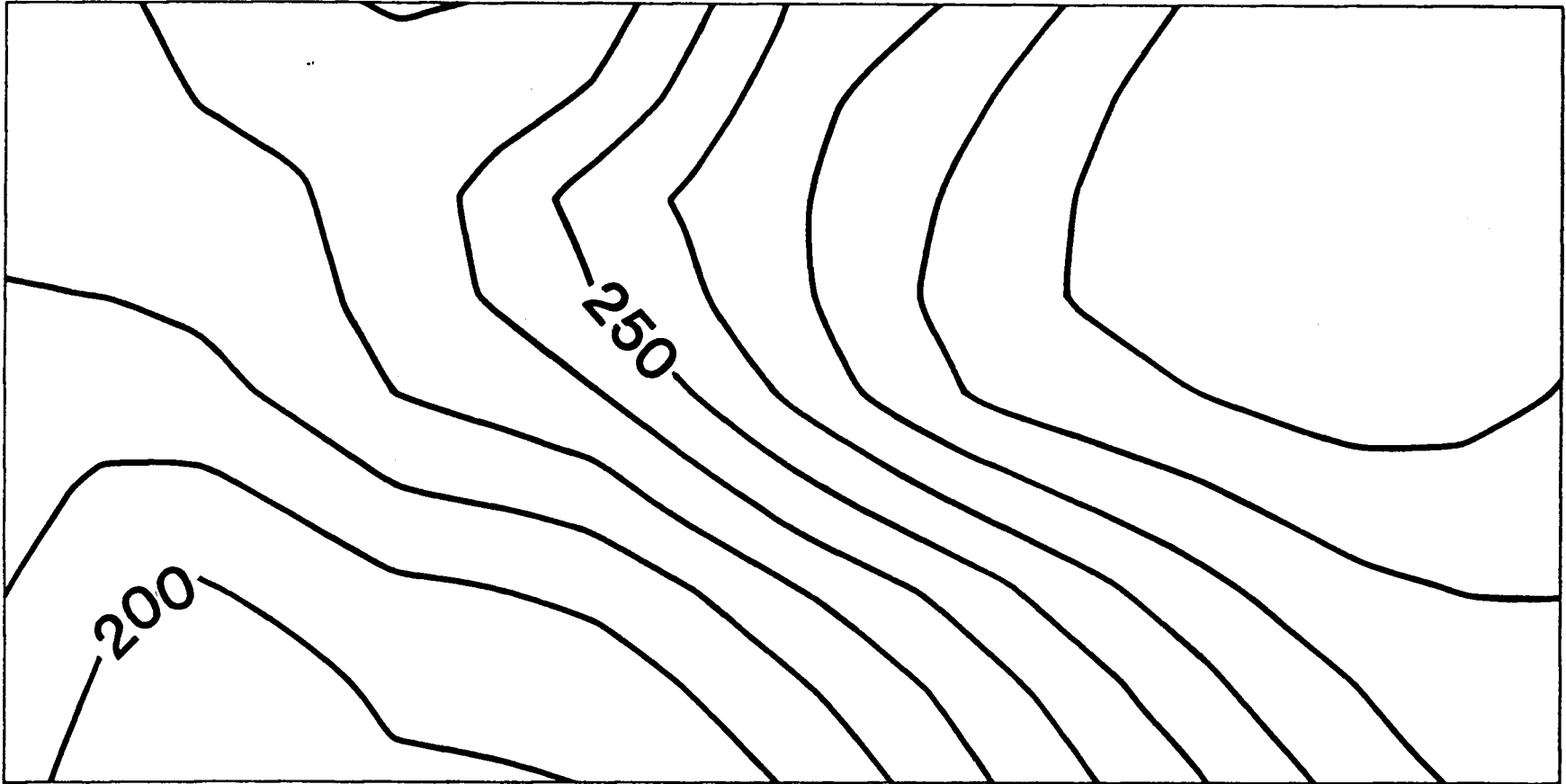
**Transmissivity Distribution**



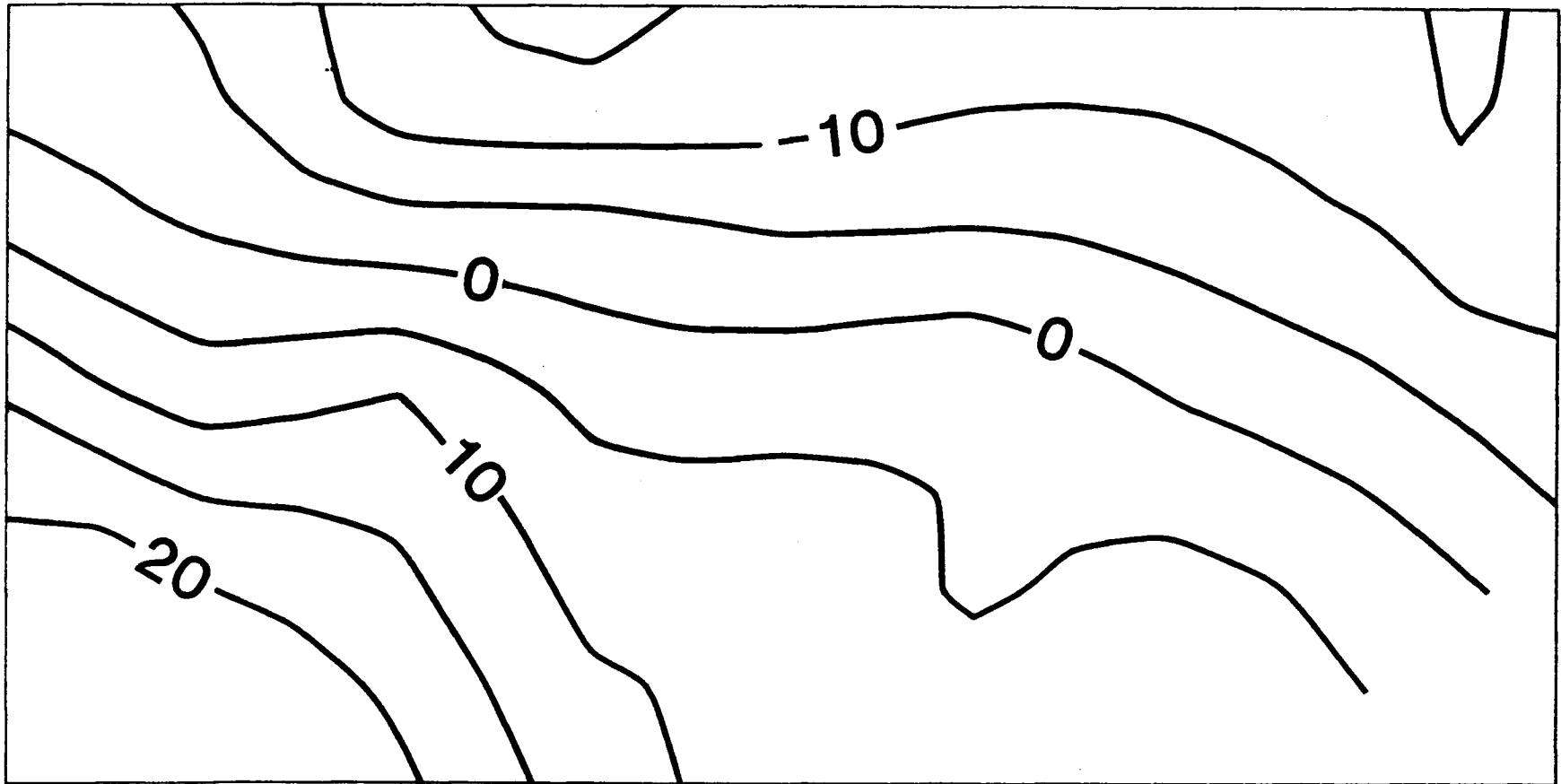
**Direct Transmissivity Distribution**



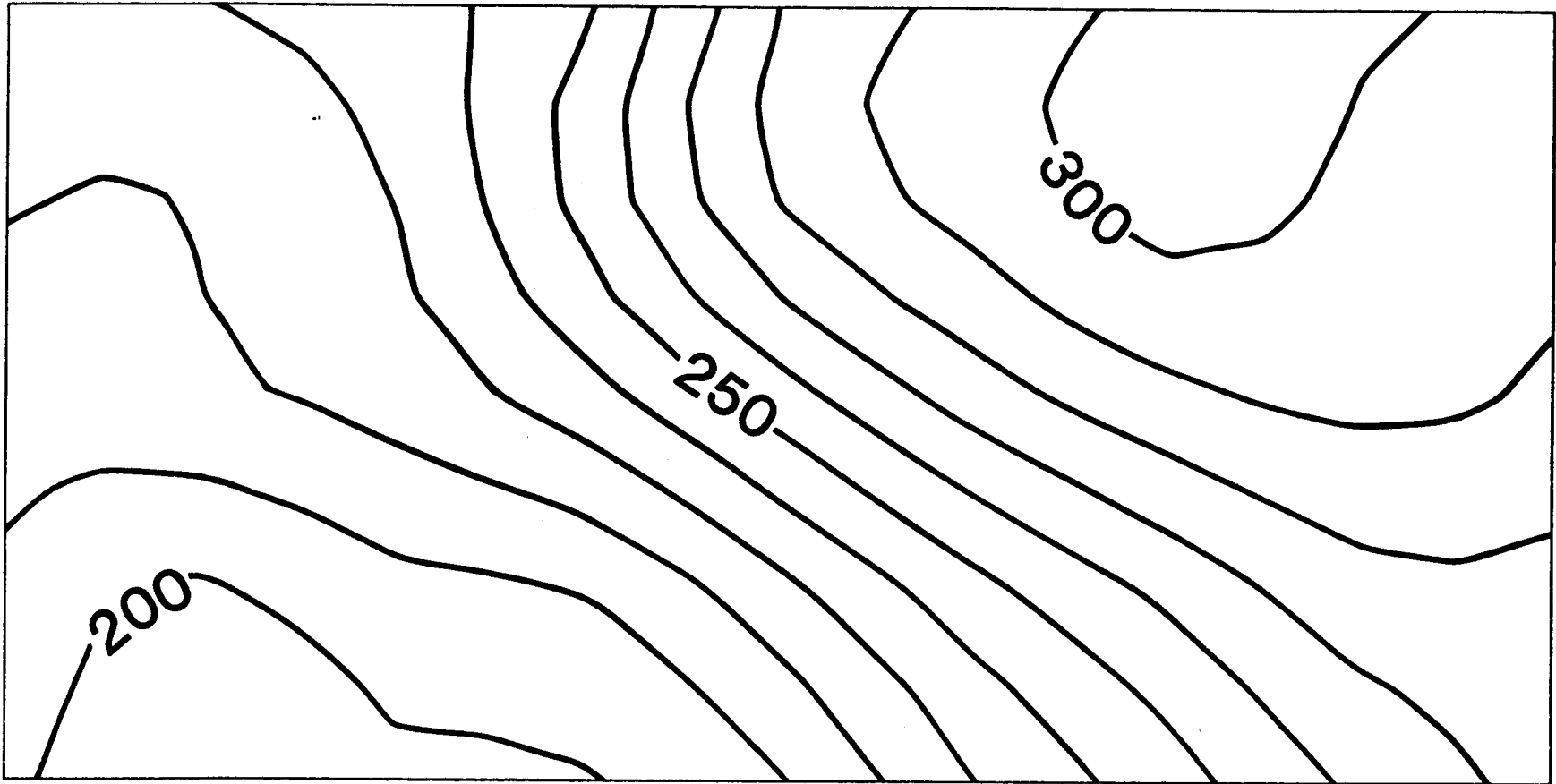
**Direct Percent Error**



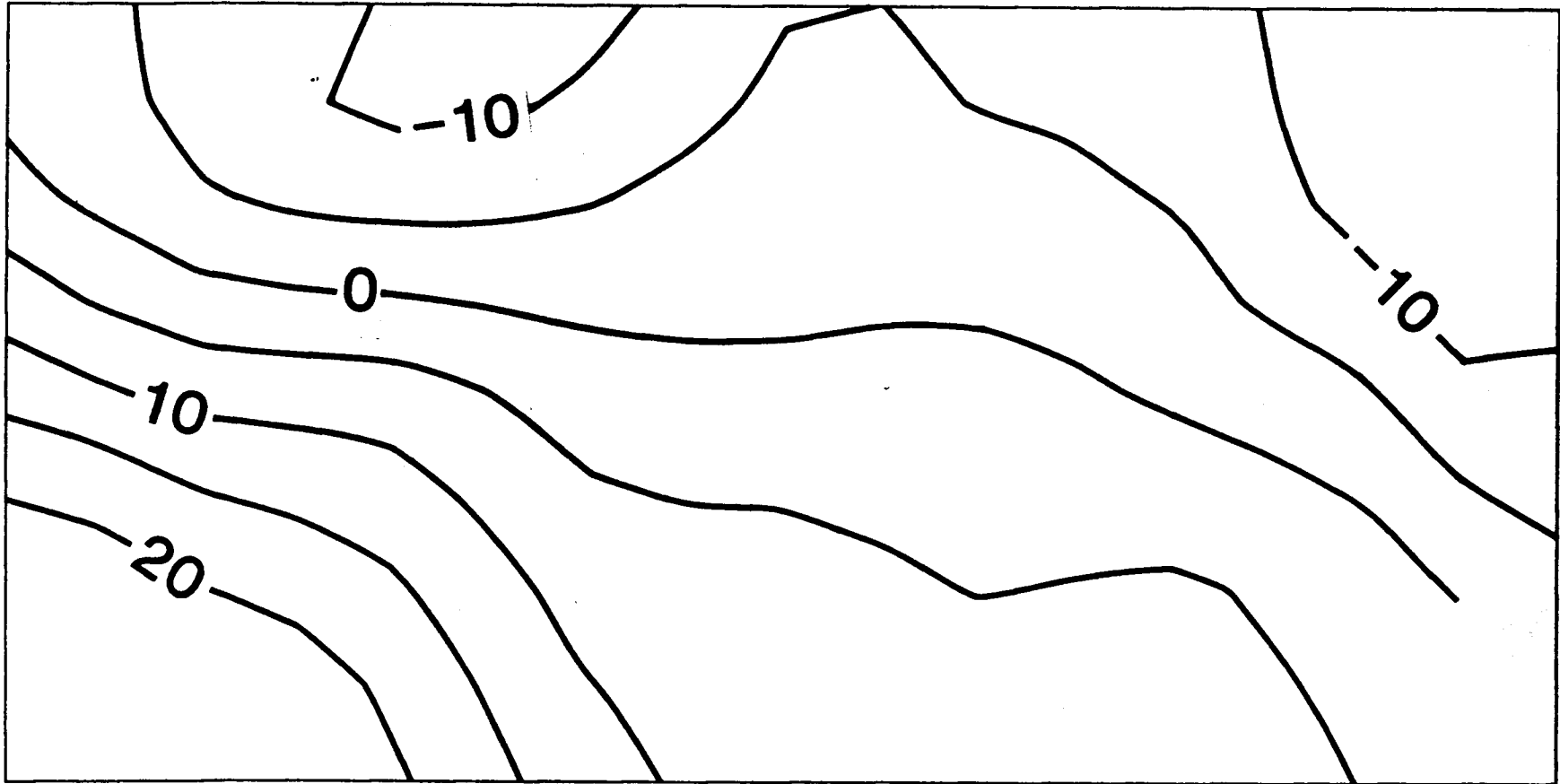
**Indirect Transmissivity Distribution**



**Indirect Percent Error**



**Indirect Transmissivity Distribution  
Modified**



**Indirect Percent Error  
Modified**