The Use of Sensitivity Analysis for Automated Pumping Test Evaluation

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THE USE OF SENSITIVITY ANALYSIS FOR AUTOMATED PUMPING TEST EVALUATION

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ABSTRACT

The Theis equation has played an important role in groundwater hydrology since its introduction. Comparison of field pumping-test data with this theoretical curve by graphical means has been a standard method of determining aquifer transmissivity and storage. The purpose of this paper is to present a technique to evaluate the sensitivity of the Theis equation with respect to transmissivity and storage and to automatically fit aquifer pumping-test data to the Theis equation. After the Theis equation has been evaluated, the sensitivity coefficients can be obtained with little additional work. These sensitivity coefficients are used with a least squares fitting technique to develop an algorithm for fitting the Theis equation to the aquifer test data. The automated fit for pumping-test data developed in this work should be a useful tool for the groundwater hydrologist. We have used it on many aquifer tests with excellent results. It is easy to use, quick, and inexpensive. The computer automated fit has the advantage that it is always objective. As a measure of the error in fitting, the standard deviation in drawdown is calculated for the "best" transmissivity and storage. For the aquifer tests we have analyzed, the standard deviation in drawdown was no more than a few tenths of a foot. If it is much larger than this, one has either poor data or a hydrologic situation which can not be represented by the Theis equation. The algorithm for fitting has good convergence properties. It will converge for a wide range of initial guesses for storage and transmissivity. This work deals only with the Theis equation. However, sensitivity analysis and least squares fitting could be applied to more hydrologically complicated situations.

INTRODUCTION

Comparison of experimental pumptest data with the Theis curve by graphical means has been a standard method of determining aquifer transmissivity and storage. The purpose of this paper is to present a technique to automatically fit experimental pumptest data to the Theis equation obtaining the "best" transmissivity and storage in the least squares sense.

THE THEIS EQUATION

The Theis equation describes radial confined groundwater flow in a uniformly thick horizontal, homogeneous, isotropic aquifer of infinite areal extent. (Slide #1) In the Theis equation script s is drawdown, Q is the discharge, capital T is the transmissivity, small t is the time, capital S is the dimensionless storage coefficient. and r is the radial observation distance from the pumped well.

SENSITIVITY ANALYSIS

In the mathematical treatment of groundwater flow it is permissible to speak of the precise values of the physical parameters. However, in the practical simulation of a real aquifer we are immediately faced with uncertainty as to the exact physical parameters. The investigator must establish tolerances within which the parameters of the physical system may vary without appreciably affecting the model results. These tolerances are often obtained by introducing parameter perturbations in the system and observing the changes in the system's performance.

However, the application of sensitivity analysis makes it possible to obtain these tolerances more efficiently (Slide #2). The solution of the flow equation for the hydraulic head may be written as a function of x, y, t, T, S, and Q. Consider the variation of one of the aquifer parameters, T for example. Varying this parameter by a small amount, delta T, the perturbed hydraulic head is given by h*. The quantity $\Delta h/\Delta T$ gives one an indication of the stability of the system. If $\Delta h/\Delta T$ has a limiting value as delta T approaches zero, it may be written as $\frac{\partial h}{\partial T}$. $\frac{\partial h}{\partial T}$ or U_T will be called the sensitivity coefficient for variations in transmissivity. By applying similar arguments for a variation in storage coefficient one obtains U_S. U_S is the sensitivity coefficient for variations in the storage coefficient. (Slide #3) Now consider a perturbation of the transmissivity, delta T. The function h* may be expanded into a Taylor series. If delta T is small the second and higher order terms may be neglected. Thus, the new head produced by a perturbation in transmissivity may be calculated from this expansion if the sensitivity coefficient and the unperturbed head are known. Similarly, if a small perturbation in storage coefficient occurs the perturbed head is also given by a Taylor expansion to first order in delta S.

These results show that it would be desirable to calculate U_T and U_S for a given model, if possible. Then the response of the model to various perturbations could be calculated simply from the Taylor expansions without actually evaluating the model equations again.

The sensitivity coefficients (Slide #4) may be obtained from the Theis equation by differentiating with respect to T or S. After applying Leibnitz's rule for differentiating an integral, one obtains these expressions for U_T and U_S.

PROPERTIES OF THE SENSITIVITY COEFFICIENTS

The radial dependence of U_T is shown in this slide (Slide #5). U_T diverges logarithmically at the well. U_T changes sign at some value of r, as it must in order for the cones of depression to have the same volume for differing transmissivities.

The next slide (Slide #6) shows the time dependence for positive values of U_T. Notice that for large time the dependence of U_T on time is fairly weak. The curves labeled ± 20 percent T show how U_T at a radius of one foot changes when the transmissivity is perturbed by ± 20 percent. In this region U_T is inversely proportional to transmissivity. The two curves for r = 1 foot and 1,000 feet, have an identical shape but are displaced from one another along the time axis.

The Gaussian radial dependence of U_S is shown in the next slide (Slide #7). U_S does not diverge at the well. Also, U_S does not change sign because an increase or decrease in S results in a general raising or lowering, respectively, of the cone of depression. The dashed lines in this slide show U_S when S is changed by ± 20 percent. U_S is inversely proportional to S.

The time dependence of U_S is illustrated in the next slide (Slide #8) for three different r values. As time increases U_S approaches a constant value. Even for r = 1,000 feet U_S is nearly constant after about one day.

LEAST SQUARES FIT

The objective is to use the sensitivity formalism to obtain a least squares fit of experimental pump test data to the Theis equation and thus obtain the "best" estimate for S and T (Slide #9). The new drawdown, s*, after changing T and S by delta T and delta S respectively, is given by a Taylor expansion.

Let script s_e represent the experimentally measured drawdowns. Suppose it is possible to guess a reasonable S and T and let script s denote the drawdowns calculated from the Theis equation with these parameters. One would like to change the original guess by delta S and delta T in such a way that a better fit of the experimental data results. This is done by minimizing E, the error function. The t_i represents a discrete time at which an experimental measurement is made for the drawdown. The error is defined as the sum over all measurements of the squared difference in script s_e and script s.

The error is minimized by taking the first derivatives with respect to delta T and delta S, setting them equal to zero, and solving the resulting equations for delta T and delta S (Slide #10). For conciseness we have defined the variables shown at the bottom of this slide. Notice that all these quantities are known from experimental data or can be calculated from previously derived equations.

These values for delta T and delta S can be used to update the first guess for T and S (Slide #11). This better estimate for T and S is then used in the least squares procedure again to obtain new values of delta T and delta S. In general this can be continued until delta T and delta S become so small as to be insignificant, at which time the iteration is terminated. The "best" fit after the i^{th} iteration is obtained by using the equations in step 5. The procedure may not converge if the initial guess for T and S is especially bad. However, numerical experiments indicate that good convergence may be obtained even if the initial guess is off considerably. These numerical experiments show that the initial guesses for T and S must not be larger than about twice their actual value; however, T and S may be underestimated by about three orders of magnitude without preventing convergence.

APPLICATION TO A TYPICAL PUMPING - TEST

This slide (Slide #12) shows data, taken from Walton, for a typical pumping test. The drawdown was measured to the nearest tenth of a foot over a maximum pumping time of 500 minutes. The well was pumped at 220 gpm and the data was obtained from an observation well 824 feet away. Initial guesses for T and S were chosen as shown on the slide. The program converged to the "best fit" values in six iterations. The convergence criteria requires the change in storage and transmissivity since the last iteration to be less than or equal to .1%. The "best fit" values are T = 9,909 gpd/ft and S = 2.095×10^{-5}. These compare very well with those found graphically by Walton. The "best fit" drawdowns are shown at the right of the table. The standard deviation in drawdown is .09 ft.. This is a measure of the magnitude of error at an "average" data point. It compares very closely with the field accuracy of the pumping - test data. These results are typical for the many pumping tests we have analyzed with these techniques.

SUMMARY

The automated fit for pumptest data developed in this work should be a useful tool for the groundwater hydrologist. It is simple to use, quick, and inexpensive. The automated fit has the advantage that it is always objective. As a measure of the error in fitting, the standard deviation in drawdown is calculated for the "best" T and S. Usually the standard deviation in drawdown is no more than a few tenths of a foot. If it is much larger than this one has either poor data or a hydrologic situation which cannot be represented by the Theis equation.

The algorithm for fitting has good convergence properties. It will converge provided the initial guess for storage and transmissivity is less than about twice their actual values. Convergence is generally achieved even if the initial guess is too small by three orders of magnitude.

This work deals only with the Theis equation. However, sensitivity analysis and least squares fitting could be applied to more hydrologically complicated situations.

Slide #1

The Theis Equation

$$s = \frac{Q}{4\pi T}\int_{\frac{r^2 S}{4Tt}}^{\infty} \frac{e^{-u}}{u}\,du$$

s is drawdown (L)

Q is the discharge (L^3/T)

T is the transmissivity (L^2/T)

t is the time (T)

S is the dimensionless storage coefficient

r is the radial observation distance (L)

Slide #2

Sensitivity Coefficients

Let $h(x, y, t; T, S, Q)$ be a solution to the flow equation.

If T is changed by ΔT we have $h^*(x, y, t; T + \Delta T, S, Q)$

$$\frac{\Delta h}{\Delta T} = \frac{h^*(x, y, t; T + \Delta T, S, Q) - h(x, y, t; T, S, Q)}{\Delta T}$$

Define the sensitivity coefficient as

$$U_T(x, y, t; T, S, Q) = \frac{\partial h}{\partial T} = \lim_{\Delta T \to 0} \frac{\Delta h}{\Delta T}$$

Similarly with respect to storage

$$U_S(x, y, t; T, S, Q) = \frac{\partial h}{\partial S} = \lim_{\Delta S \to 0} \frac{\Delta h}{\Delta S}$$

Slide # 3

Taylor Series Expansion

If ΔT or ΔS is small we have approximately

$$h^* (x, y, t; T + \Delta T, S, Q) \cong h (x, y, t; T, S, Q) + U_T \Delta T$$

or

$$h^* (x, y, t; T, S + \Delta S, Q) \cong h (x, y, t; T, S, Q) + U_S \Delta S$$

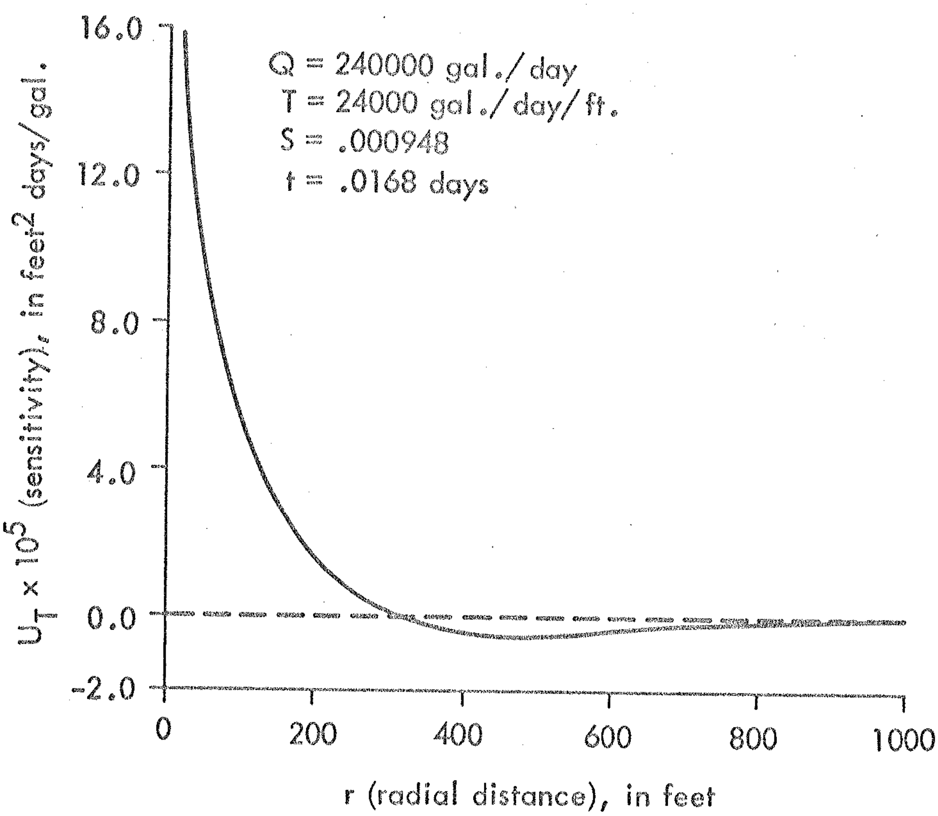
Slide #4

The sensitivity coefficients may be obtained by differentiating the Theis equation.

$$U_T = \frac{\partial s}{\partial T} = -\frac{s}{T} + \frac{Q}{4\pi T^2} \, \mathrm{EXP}\left[-\frac{r^2 S}{4 T t}\right]$$

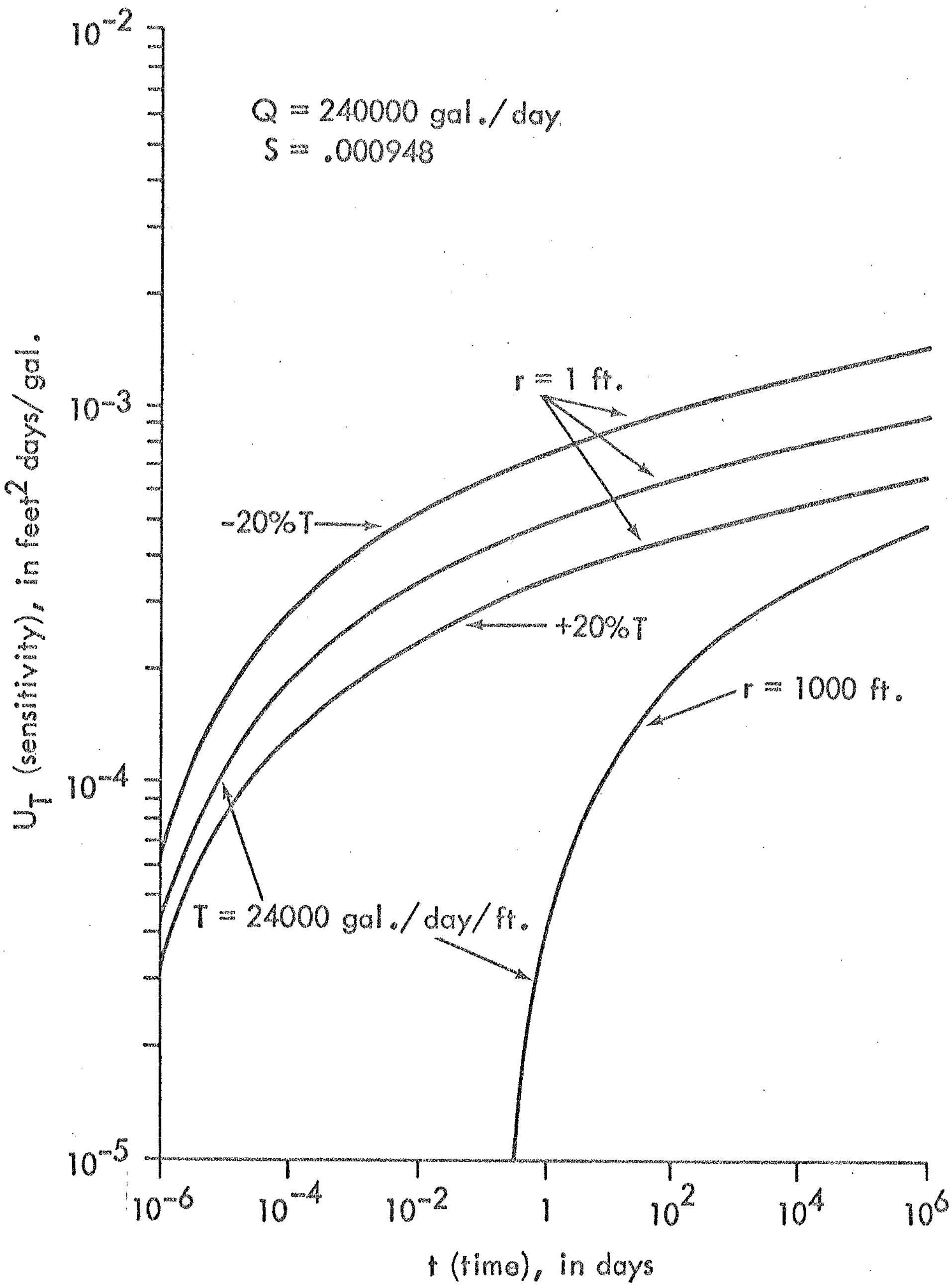
$$U_S = \frac{\partial s}{\partial S} = -\frac{Q}{4\pi T S} \, \mathrm{EXP}\left[-\frac{r^2 S}{4 T t}\right]$$

Slide #5



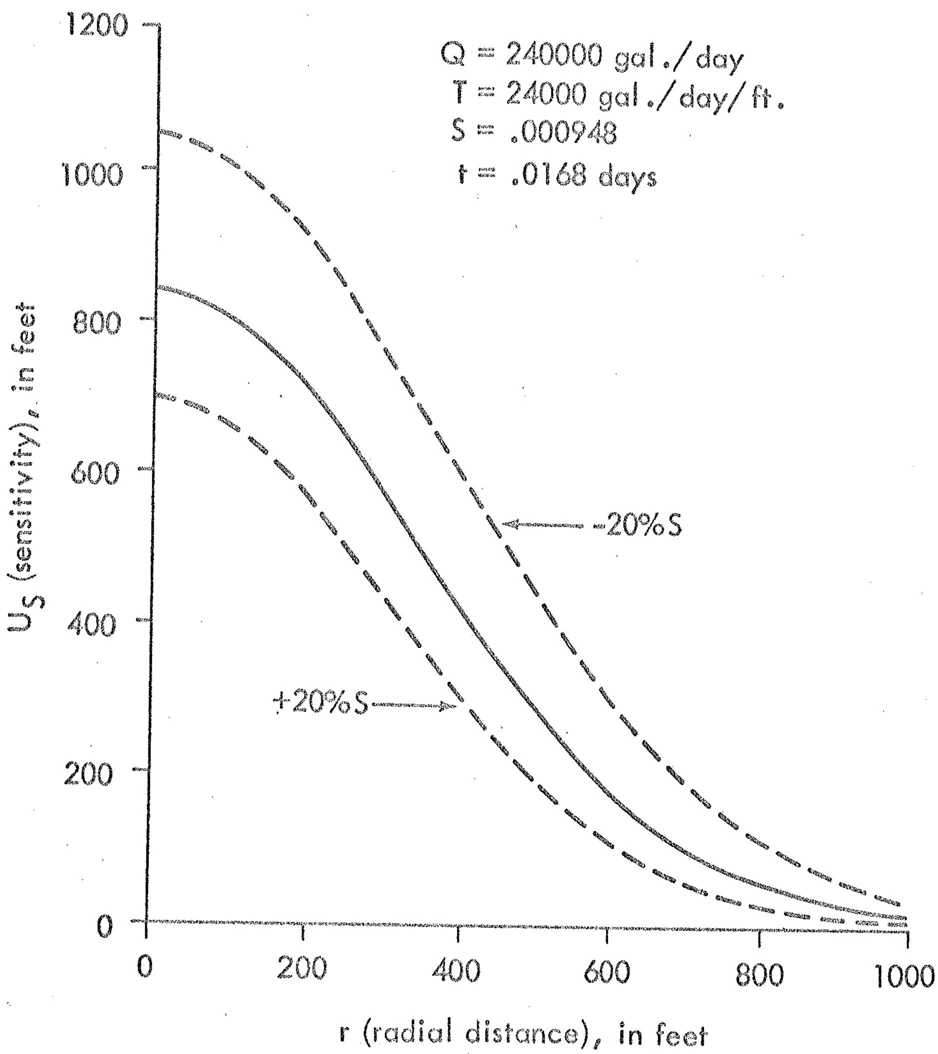
Q = 240000 gal./day
T = 24000 gal./day/ft.
S = .000948
t = .0168 days
$U_T \times 10^5$ (sensitivity), in feet^2 days/gal.
16.0
12.0
8.0
4.0
0.0
-2.0
0
200
400
600
800
1000
r (radial distance), in feet

Slide #6



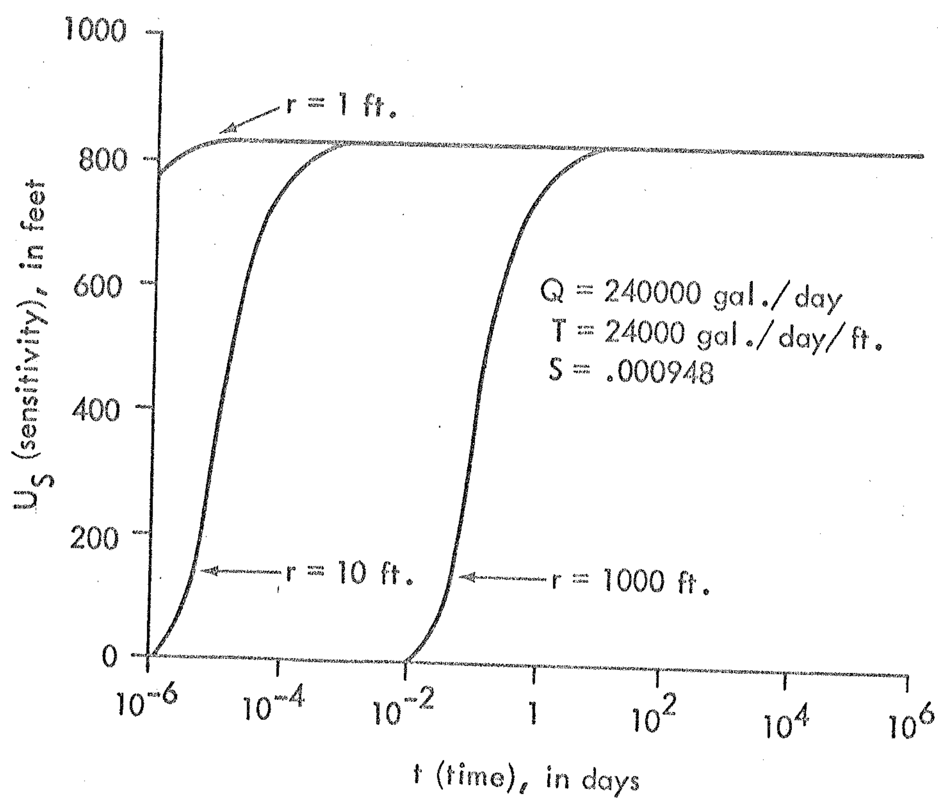
Q = 240000 gal./day
S = .000948
r = 1 ft.
-20%T
+20%T
r = 1000 ft.
T = 24000 gal./day/ft.
U_T (sensitivity), in feet^2 days/gal.
t (time), in days
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-4}
10^{-2}
1
10^{2}
10^{4}
10^{6}

Slide #7



Q = 240000 gal./day
T = 24000 gal./day/ft.
S = .000948
t = .0168 days
-20%S
+20%S
1200
1000
800
600
400
200
0
U_S (sensitivity), in feet
0
200
400
600
800
1000
r (radial distance), in feet

Slide #8



1000
800
600
400
200
0
U_S (sensitivity), in feet
r = 1 ft.
r = 10 ft.
r = 1000 ft.
Q = 240000 gal./day
T = 24000 gal./day/ft.
S = .000948
10^{-6}
10^{-4}
10^{-2}
1
10^{2}
10^{4}
10^{6}
t (time), in days

Slide #9

The new drawdown produced by a change in T and S is

$$s^* = s + U_T \Delta T + U_S \Delta S$$

The squared error in s^* compared to the experimentally measured drawdown, s_e is

$$E = \sum_i \left[s_e(t_i) - s^*(t_i) \right]^2$$

$$= \sum_i \left[s_e(t_i) - s(t_i) - U_T(t_i)\,\Delta T - U_S(t_i)\,\Delta S \right]^2$$

Minimization of E requires

$$\frac{\partial E}{\partial \Delta T} = 0 \quad \text{and} \quad \frac{\partial E}{\partial \Delta S} = 0$$

Slide # 10

Solving the two equations for ΔT and ΔS yields

$$\Delta S = \frac{(SSUT)(SUSDIF) - (SUTUS)(SUTDIF)}{(SSUS)(SSUT) - (SUTUS)^2}$$

$$\Delta T = \left[SUTDIF - (SUTUS)\ \Delta S \right] / SSUT$$

Where

$$SSUS = \sum_i U_S^2 (t_i)$$

$$SSUT = \sum_i U_T^2 (t_i)$$

$$SUTUS = \sum_i U_S (t_i)\ U_T (t_i)$$

$$SUSDIF = \sum_i U_S (t_i) \left[\phi_e (t_i) - \phi (t_i) \right]$$

$$SUTDIF = \sum_i U_T (t_i) \left[\phi_e (t_i) - \phi (t_i) \right]$$

Slide # 11

ITERATION PROCEDURE

1. Initial guess for T and S.
2. Calculate expected drawdown values.
3. Calculate sensitivity coefficients U_T and U_S.
4. Determine ΔT and ΔS.
5. Upgrade values for T and S.

$$T^{i+1} = T^{i} + \Delta T^{i}$$

$$S^{i+1} = S^{i} + \Delta S^{i}$$

6. If ΔT and ΔS are less than δ stop.
7. If maximum number of iterations is not exceeded go to step 2.

Slide # 12

TIME-DRAWDOWN DATA for well 1 at Gridley, Illinois. (From Walton, 1962)

Time after pumping started, min	Drawdown, ft	"Best fit" Drawdown, ft
3	0.3	0.35
5	0.7	0.83
8	1.3	1.48
12	2.1	2.17
20	3.2	3.18
⋮	⋮	⋮
200	8.5	8.62
260	9.2	9.28
320	9.7	9.80
380	10.2	10.23
500	10.9	10.92

Q = 220 gpm and r = 824 ft.
Standard deviation in drawdown is .09 ft.

Initial guess
T = 20,000 gpd/ft
$S = 1.0 \times 10^{-5}$

"Best fit" values
T = 9,909 gpd/ft
$S = 2.095 \times 10^{-5}$

Walton values
T = 10,100 gpd/ft
$S = 2 \times 10^{-5}$