LEAKY AQUIFER PARAMETER IDENTIFICATION BY SENSITIVITY ANALYSIS

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Prepared for Presentation at the
Midwestern Meeting of AGU
September 25-27
St. Louis, Missouri

Prepared by the Kansas Geological Survey

1978

ABSTRACT

Traditional analysis of aquifer parameters involves fitting experimental pump test data to a standard theoretical curve by visual comparison and solving the appropriate formulae for the match point. A disadvantage to the visual method of curve fitting is the lack of an estimate of error in the fit. By use of an automated curve fitting procedure, the pump test data may be fitted to a standard curve and the goodness of fit may be estimated from a computed standard deviation. One of the more frequently used curves is that for non-steady radial flow in an infinite homogeneous, isotropic, leaky aquifer (Hantush and Jacob, 1955). The purpose of this paper is to present the methodology for constructing an algorithm which will successfully fit time-drawdown data to the theoretical leaky aquifer curve and produce values for storage, tramsmissivity, and leakage. In the fitting procedure, sensitivity analysis is used to predict the change in drawdown due to changes in the aquifer parameters. After the leaky aquifer equation has been evaluated by numerical means, the sensitivity coefficients for storage, tramsmissivity, and leakage can be obtained by standard techniques and can then be combined with a least squares fitting technique to produce an algorithm for fitting the time-drawdown data to the theoretical leaky aquifer curve. The method is not complex and is easily programmed for computer implementation. The method is completely objective and eliminates human error in the fitting procedure. Convergence properties appear to be good in that the initial guesses for storage, transmissivity, and leakage may vary by about an order of magnitude above or below the correct value.

Introduction

Solution of the inverse problem, that is the determination of aquifer parameters from pumptest data, has classically been performed by visual curve matching (slide #1). One of the frequently encountered hydrologic situations is that of the leaky artesian aquifer as defined by Hantush and Jacob. The curve matching technique for this situation has been discussed by Walton and others. The purpose of this paper is to discuss an automated method for solving the inverse problem which uses sensitivity analysis to implement a least squares fitting procedure in order to obtain the best values of transmissivity, storage coefficient, and leakage.

The Leaky Artesian Aquifer Equation

In solving the equation of flow for the leaky artesian aquifer, Hantush and Jacob assumed a horizontal aquifer of homogeneous and isotropic character, infinite in areal extent and of uniform thickness throughout. The aquifer is bounded below by an aquiclude and above by an aquitard. The rate of vertical leakage is assumed to be proportional to the head difference between the free surface of the unconfined source bed and the piezometric surface of the artesian aquifer. Furthermore, it is assumed that no water is removed from storage in the aquitard and that no lowering of the free surface occurs. Hence, at some time, the drawdown in the piezometric surface due to pumping reaches steady state. The leaky artesian aquifer equation as originally stated (slide #2) defines the specific leakage as the ratio of vertical permeability of the aquitard P', to its saturated thickness, m'. The variable B is defined as the square root of the ratio of the aquifer transmissivity, T, to the specific leakage. The remaining parameters are identical to those in the classical

Theis equation (slide #3). For convenience the parameter $\frac{1}{B}$ is replaced by L which we defined as the modified coefficient of leakage. This modification allows the leakage to be set to zero easily. Otherwise B would have to be set to infinity. For L equal to zero the equation reduces to the classic Theis equation.

Sensitivity Analysis

When treating groundwater flow mathematically, it is permissible to speak of precise values of physical parameters. When dealing with real situations however, one is immediately faced with the uncertainty of these parameters. Tolerances for parameter uncertainty must be established, such that model results are insensitive to these errors. Parametric sensitivity analysis (slide #4) has been used in this work to establish these tolerances. The assumption is that the solution to the flow equation h may be written as a function of x, y, t, T, S, L, and Q. Suppose that we choose to determine the sensitivity with respect to the transmissivity T. Varying this parameter by a small increment ΔT produces a new solution h*, which is a function of x, y, t, $T + \Delta T$, S, L, and Q. If the quantity $\Delta h/\Delta T$ approaches a limiting value as ΔT approaches zero (slide #5), we may write $\partial h/\partial T$ and describe this result as the sensitivity coefficient with respect to transmissivity, U_T. Similar arguments result in sensitivity coefficients with respect to storage U_S, and the modified leakage U_L. We may expand h* as a Taylor Series in each parameter for which we have determined the sensitivity. If we assume that the parameter variation is small, second and higher order terms may be neglected in each expansion. The result is a linear equation which will allow computation of a new head resulting from a small perturbation of the aquifer parameters (slide #6).

We see that if it is possible to evaluate U_T, U_S, and U_L for a given model, the response of that model to perturbations of the aquifer parameters can be obtained from the results of the Taylor expansions, without having to evaluate the model equations again (slide 6a).

Sensitivity coefficients were obtained for the leaky artesian aquifer equation by two methods. U_T and U_S were obtained in analytical form by simply differentiating the drawdown equation with respect to T or S. Leibnitz's rule for differentiating an integral was used (slide 6b). U_L was obtained by using a finite difference scheme.

Properties of the Sensitivity Coefficients

Shown here (slide #7) is the radial dependence of U_T. The function diverges logarithmically near the well. U_T changes sign at some value of radius. This corresponds to the fact that when T is changed, the cone of depression deepens in some areas and shallows in other areas.

In the next slide (slide #8) the time dependence of positive values of U_T are shown for various values of r and T. Note the curves representing values of T = 24331 ft^2/day and ±20% of that value for r = 100 ft. Here U_T is inversely proportional to T. After about three or four days the values of U_T become time independent for all curves. This phenomenon corresponds to the fact that, at some value of time, Q is totally derived from the leakage, and a steady state will result.

In this slide (slide #9) we see the radial dependence of U_S. Note that U_S does not diverge at the well. U_S does not change sign, since purturbation of S results in a general raising or lowering of the cone of depression. The three curves represent values of S = .002 and S ±20%. Note that U_S is inversely proportional to S.

The time dependence of U_S is presented in the following slide (slide #10). Three values of r are represented by the curves. Each curve reaches its maximum value for U_S at a different time, the time being directly proportional to r. As time passes, all curves approach a zero value which corresponds to the steady state value. The differing nature of the curves is related to the fact that until steady state is attained, there is a dual source supplying the pumpage, namely water released from storage and leakage. The curves roll over and approach zero as leakage becomes more important. The maximum value U_S is inversely proportional to r and is zero outside the steady state cone of depression. At large values of time, the system is totally insensitive to S since leakage supplies the total pumpage.

The radial dependence of the sensitivity with respect to leakage, U_L, is presented in this slide (slide #11). The three curves represent a value of L = .0004 and L ±20%. The sensitivity coefficient U_L does not diverge at the well and approaches zero at large values of r. For time equal to .1 day, U_L id directly proportional to L. However, it has been found that at larger values of time, near equilibrium, U_L is inversely proportional to L.

The time dependence of U_L is shown for two values of r in this slide (slide #12). All curves grow with time until a steady state is achieved when leakage supplies the total Q. Note the set of curves labelled

$L = .0004\ ft^{-1}$ and L ±20%. For early times, less than .6 days, U_L is directly proportional to L while for t greater than .6 days, U_L is inversely proportional to L. As stated above, Q is supplied by two sources in the leaky artesian aquifer: water taken from storage in the aquifer and water supplied by the leakage through the aquitard. This duel source mechanism results in the changing dependence on L.

The Least Squares Fit

The object is to obtain "best" values for aquifer parameters S, T, and L by means of a least squares fitting technique. In order to apply this procedure we define the squared error function (slide #13) where s_e is an experimental drawdown and s* is the updated drawdown computed from the truncated Taylor series. Expanding the error function, taking partial deviatives with respect to the perturbed parameters, and setting the partials to zero yields a set of three simultaneous equations which must be solved in order to obtain the best fit.

The set of linear equations to be solved are shown here in matrix form (slide #14). Here, s_g is the expected drawdown. Solution of this set is straight forward since all quantities are known or can be computed. The values obtained for ΔT, ΔS, and ΔL, are used to update the previous values of T, S, and L. If the delta values are less than some predetermined error parameter, the procedure terminates. If the delta values are larger than the error parameter, the updated values of T, S, and L are used to generate a new set of sensitivity coefficients and s_g values. The system of equations is again solved for ΔT, ΔS, and ΔL. This procedure is repeated until the delta values satisfy the error parameter. This procedure is diagrammed in the following slide (slide #15). Convergence may not occur if initial values of T, S, and L are especially bad. However, numerical experiments indicate that good convergence may be obtained for T, S, and L, even if the initial values are under-estimated or over-estimated by about two orders of magnitude from their true values.

Application

Here, as in the solution of other hydrologic problems, considerable knowledge of the geology is necessary. As the problem is defined by Hantush and Jacob, $s = s(Q, r, t, S, T, P', m')$, while the regression model presented here assumed the form $s = s(Q, r, t, S, T, L)$, where $L^2 = \frac{P'/m'}{T}$. It is obvious that from a purely mathematical viewpoint there is no unique solution for P' and m', since any values satisfying a particular ratio would also satisfy the same particular value of L. Hence, in order to make P' and m' unique, P' or m' must be obtained from physical measurements. The most easily obtained parameter is m', which may be estimated from a drillers log. Given this information, P' is uniquely determined.

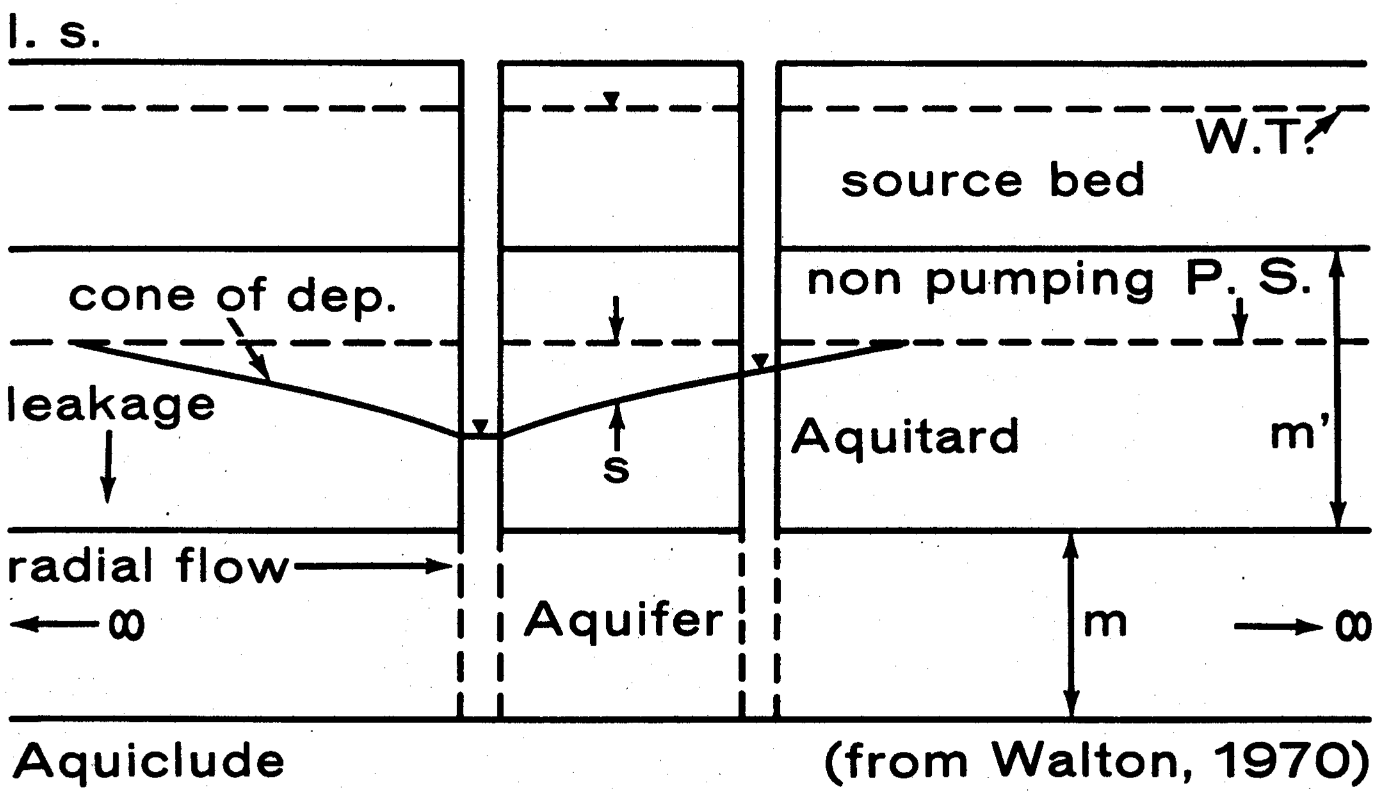
Several sets of data have been analyzed. A particular example is an evaluation of an aquifer test made in Dieterich, Illinois (slide #16) which was taken from Walton. Shown here is a comparison of the results of data reduction by visual matching and by computer regression. Differences as percentage of the visual matched values are shown. Note the large differences for T, while the two values for both S and L are moderately close. The aquifer parameters shown here were used, along with the time values from Walton, to compute time drawdown curves which were compared to the experimental curve (slide #17). Note that all the data can be approximated by one curve. Many points are superimposed on each other. Standard deviations were computed for each data set relative to the experimental data.

Visually matched values had a standard deviation of .163 feet, while the regressed values had a standard deviation of .125 feet for the best fit values of S, T, and L. This example is typical of several analyses we have performed.

Summary

This work deals only with the simple leaky artesian aquifer. The general method and application to the Theis equation have been presented by McElwee.

The algorithm presented here is simple to use, inexpensive, and yields accurate results quickly. Its convergence properties are good. Initial estimates of S, T, and L may vary by two orders of magnitude above or below the correct values. The accuracy of the fit may be judged from the standard deviation. For a good fit, this value is a few tenths of a foot. If the value exceeds this range, the data probably represents a different hydrologic situation. The automated curve fitting procedure presented here should be a useful addition to the aquifer analysis techniques available to hydrologists.


l. s.
W.T.
source bed
non pumping P. S.
cone of dep.
leakage
Aquitard
s
m'
radial flow
∞
Aquifer
m
∞
Aquiclude
(from Walton, 1970)

The Leaky Aquifer Equation

$$s = \frac{Q}{4 \pi T} \int_{\frac{r^2 S}{4 T t}}^{\infty} \frac{1}{y} \operatorname{Exp} -\left(y + \frac{r^2}{4 B^2 y}\right) dy$$

(Hantush and Jacob, 1955)

where:

- s = drawdown (ℓ)
- Q = discharge (ℓ^3/t)
- T = transmissivity (ℓ^2/t)
- S = storage coefficient
- t = time (t)
- r = radial observation distance (ℓ)
- p' = vertical permeability of aquitard (ℓ/t)
- m' = saturated thickness of aquitard (ℓ)
- p'/m' = specific leakage (t^{-1})

$$B = \sqrt{T/(p'/m')} \quad (\ell)$$

The Modified Leaky Aquifer Equation

$$s = \frac{Q}{4\pi T} \int_{\frac{r^2 S}{4Tt}}^{\infty} \frac{1}{y} \,\mathrm{Exp}\, -\left(y + \frac{L^2 r^2}{4y}\right) dy$$

where:

$$L = \sqrt{\frac{p'/m'}{T}} = \frac{1}{B} = \text{modified coefficient of leakage } (\ell^{-1})$$

and all other variables retain the definitions in slide #1.

Sensitivity Coefficients

Let h (x, y, t; T, S, L, Q) be a solution to the flow equation. If T is changed by some ΔT, we have h* (x, y, t; T + ΔT, S, L, Q), such that

$$\frac{\Delta h}{\Delta T} = \frac{h^* (x, y, t; T + \Delta T, S, L, Q) - h (x, y, t; T, S, L, Q)}{\Delta T}$$

Taylor Series Expansion

If ΔT, ΔS, and ΔR are small quantities we have approximately:

$$h^* (x, y, t; T + \Delta T, S, L, Q) = h (x, y, t; T, S, L, Q) + U_T \Delta T$$

$$h^* (x, y, t; T, S + \Delta S, L, Q) = h (x, y, t; T, S, L, Q) + U_S \Delta S$$

$$h^* (x, y, t; T, S, L + \Delta L, Q) = h (x, y, t; T, S, L, Q) + U_L \Delta L$$

Sensitivity Coefficients Defined for this Study

Sensitivity with respect to transmissivity:

$$U_T = U_T\ (x,\ y,\ t;\ T,\ S,\ L,\ Q) = \frac{\partial h}{\partial T} = \lim_{\Delta T \to 0} \frac{\Delta h}{\Delta T}$$

with respect to storage

$$U_S = U_S\ (x,\ y,\ t;\ T,\ S,\ L,\ Q) = \frac{\partial h}{\partial S} = \lim_{\Delta S \to 0} \frac{\Delta h}{\Delta S}$$

with respect to leakage:

$$U_L = U_L\ (x,\ y,\ t;\ T,\ S,\ L,\ Q) = \frac{\partial h}{\partial L} = \lim_{\Delta L \to 0} \frac{\Delta h}{\Delta L}$$

Results of Leibnitz Differentiation Applied to the Leaky Aquifer Equation

$$U_S = \frac{\partial s}{\partial S} = - \frac{Q}{4 \pi T} \frac{1}{u} \mathrm{Exp}\ (-u-L^2\ r^2/4u) \quad \frac{r^2}{4\ T\ t}$$

$$U_T = \frac{\partial s}{\partial T} = - s/T + \frac{Q}{4 \pi T} \frac{1}{u} \mathrm{Exp}\ (-u-L^2\ r^2/4u) \quad \frac{r^2\ S}{4\ T^2\ t}$$

where:

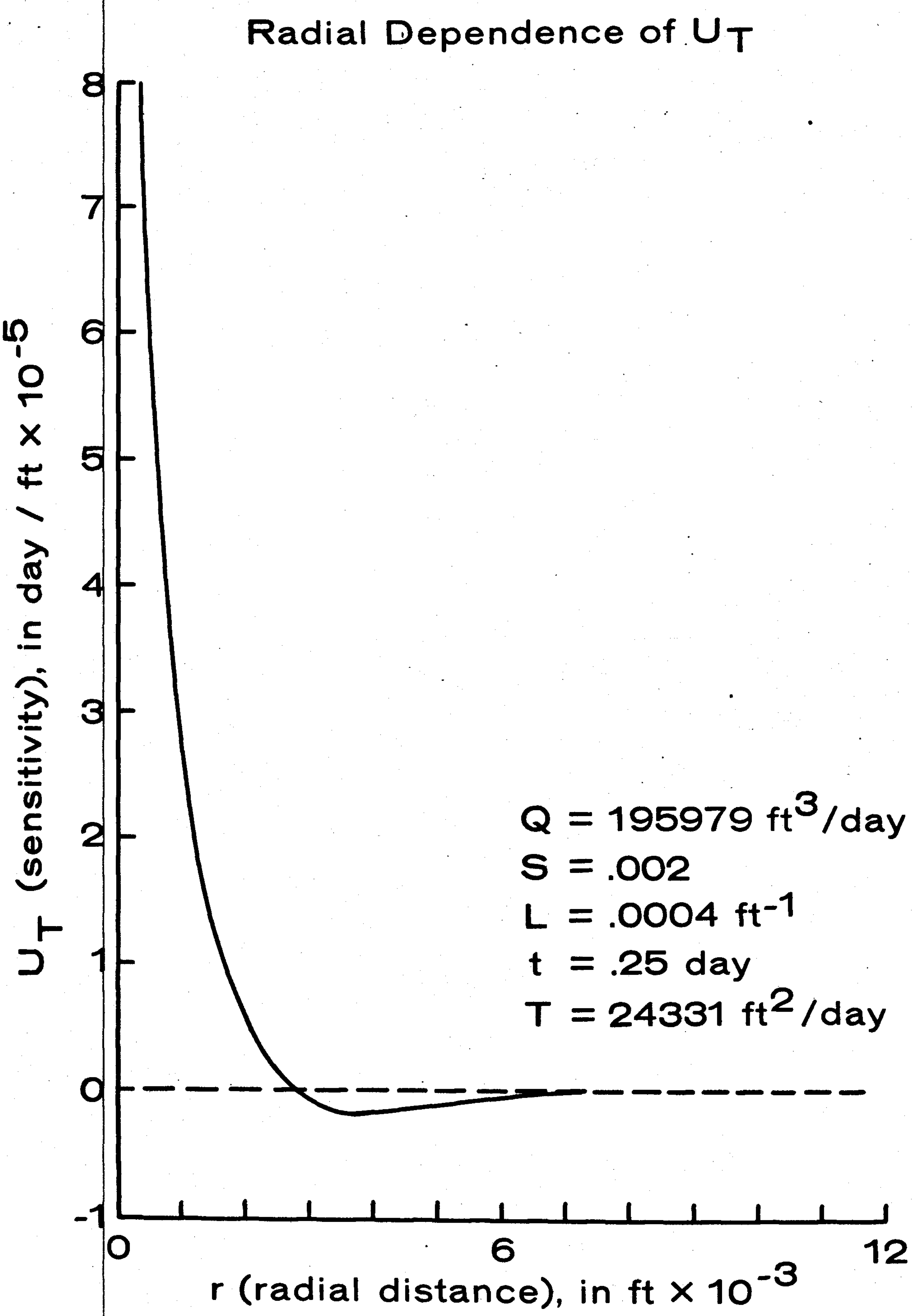
$$u = \frac{r^2\ S}{4\ T\ t}$$

Formulation of Sensitivity with Respect to L

$$\frac{\partial s}{\partial L} \approx \frac{s\ (L + .01L) - s\ (L - .01L)}{.02L}$$

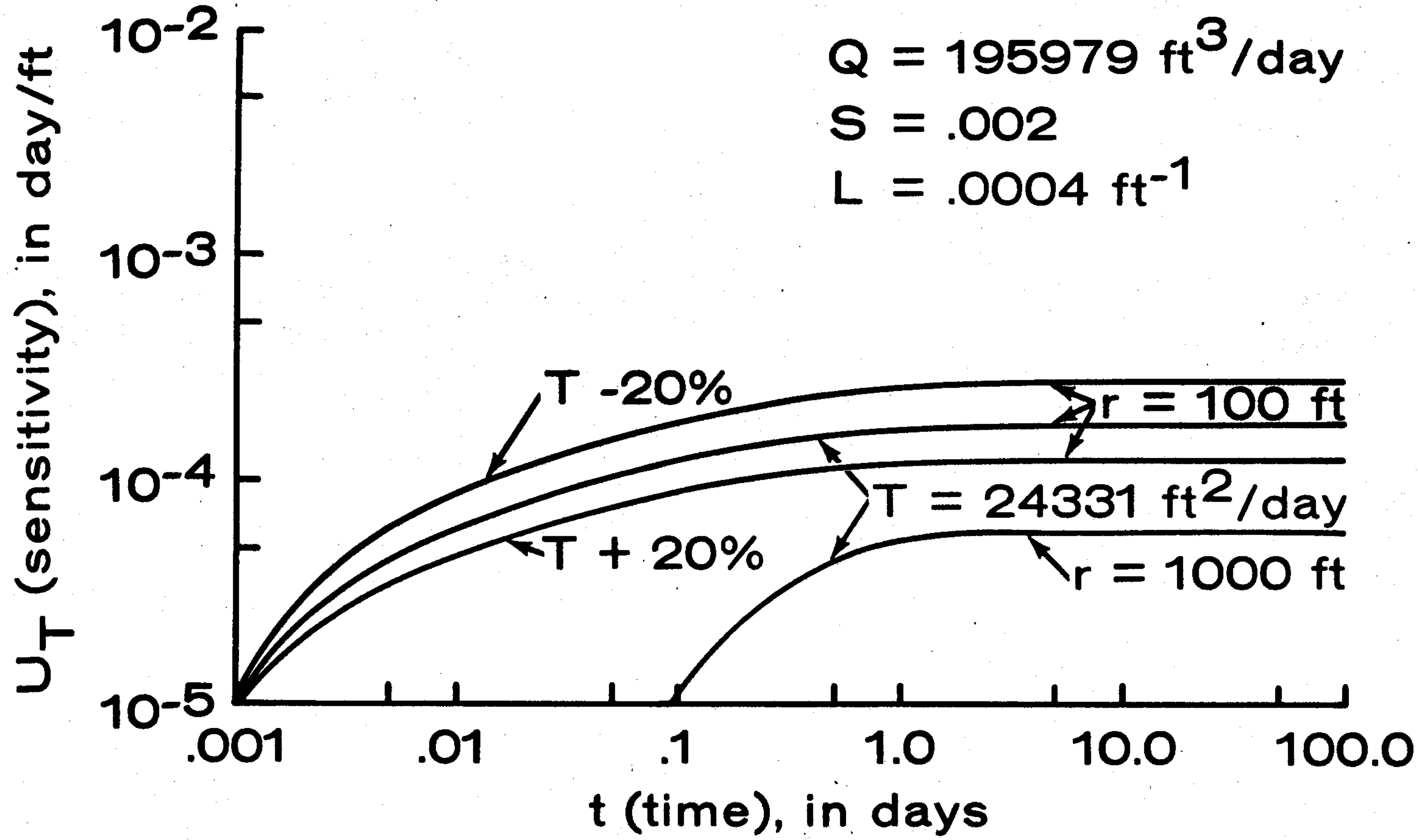
where:

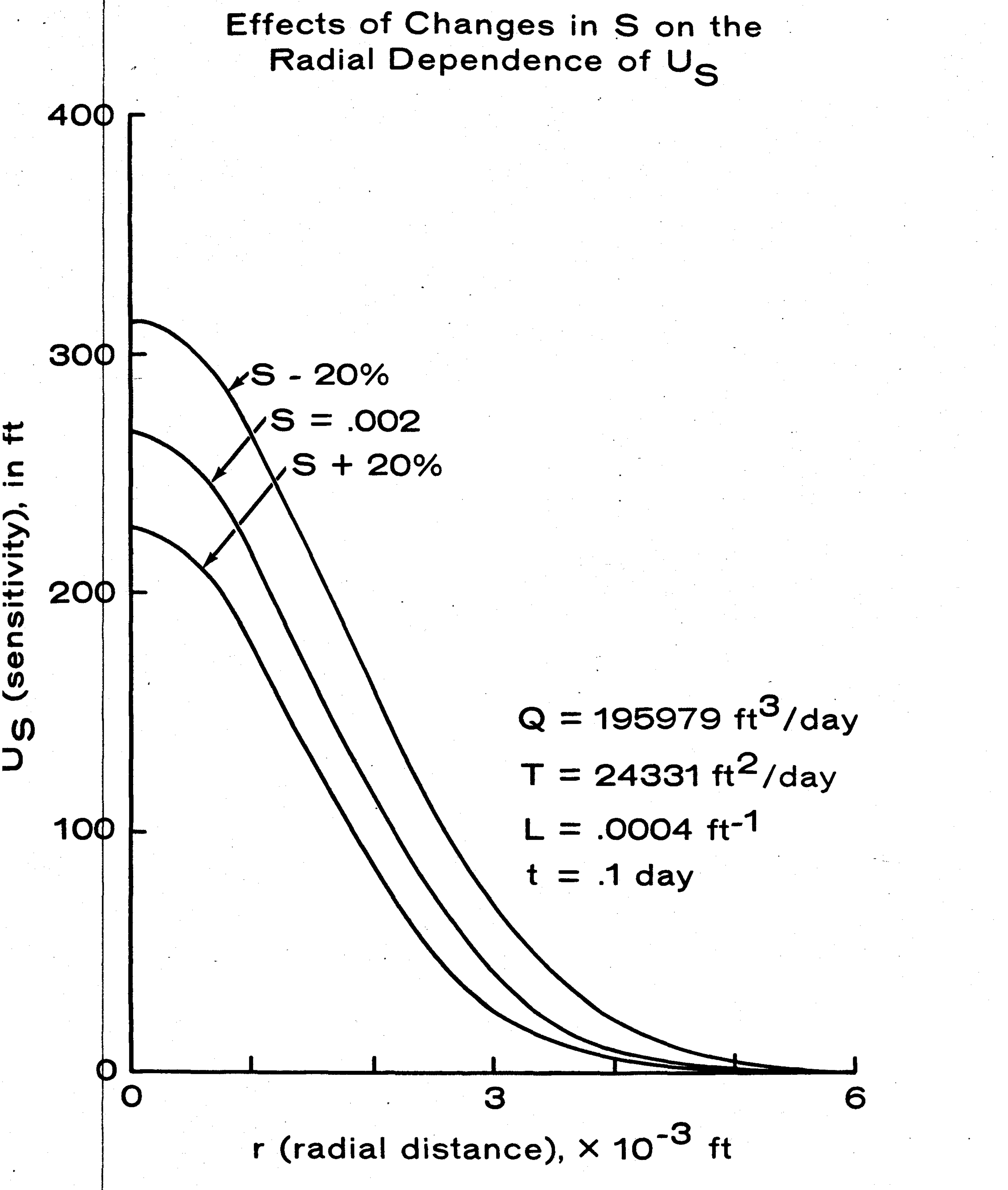
$$s\ (L + .01\ L) = \frac{Q}{4\ \pi\ T} \int_{U}^{\infty} \frac{1}{y}\ \mathrm{Exp}\ \left(-y - \frac{r^2\ (1.01\ L)^2}{4y}\right)\ dy$$



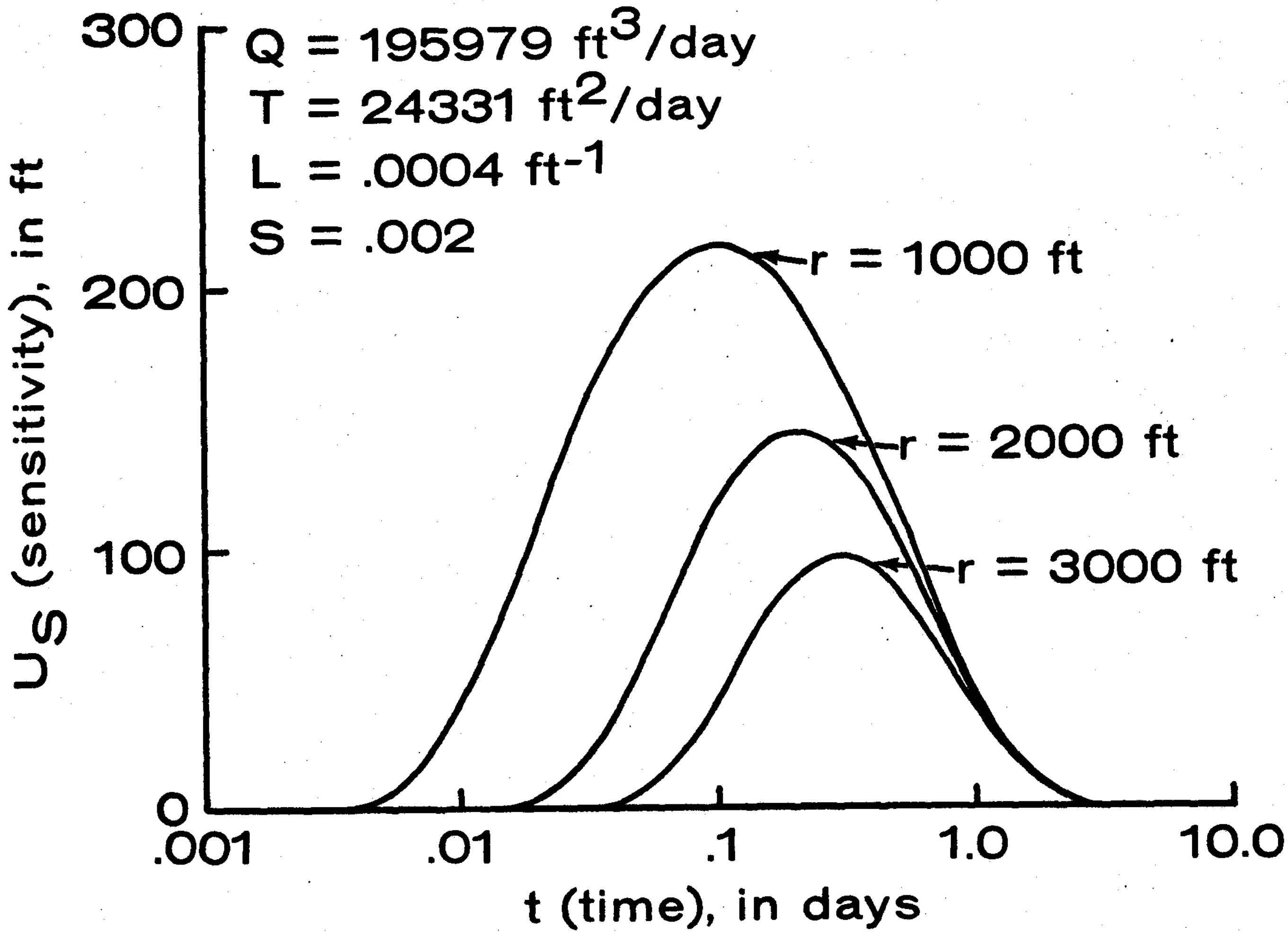
Radial Dependence of U_T
U_T (sensitivity), in day / ft × 10^{-5}
8
7
6
5
4
3
2
1
0
-1
0
6
12
r (radial distance), in ft × 10^{-3}
Q = 195979 ft^3/day
S = .002
L = .0004 ft^{-1}
t = .25 day
T = 24331 ft^2/day

Effect of Radius and Transmissivity on the Time Dependence of U_T

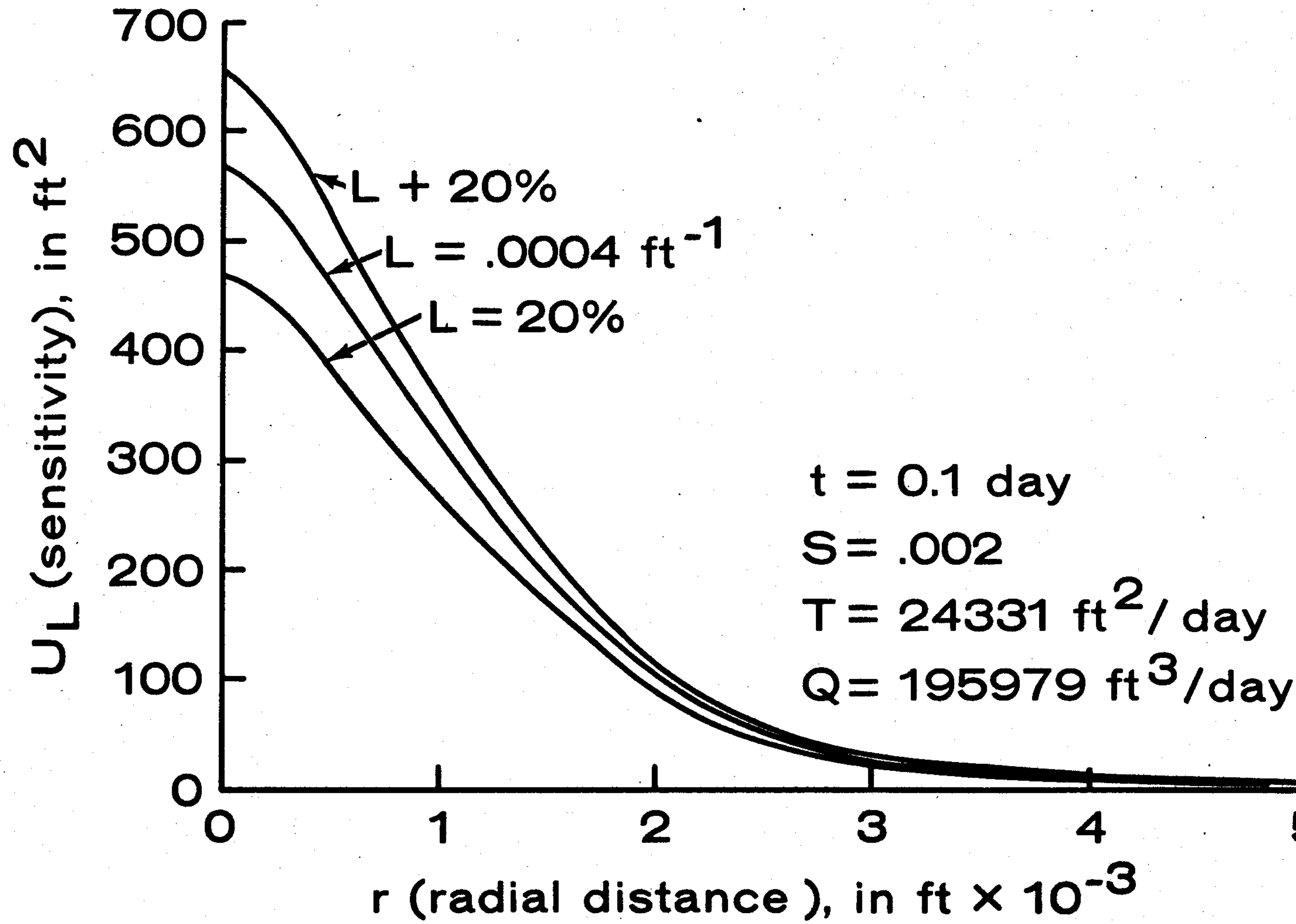


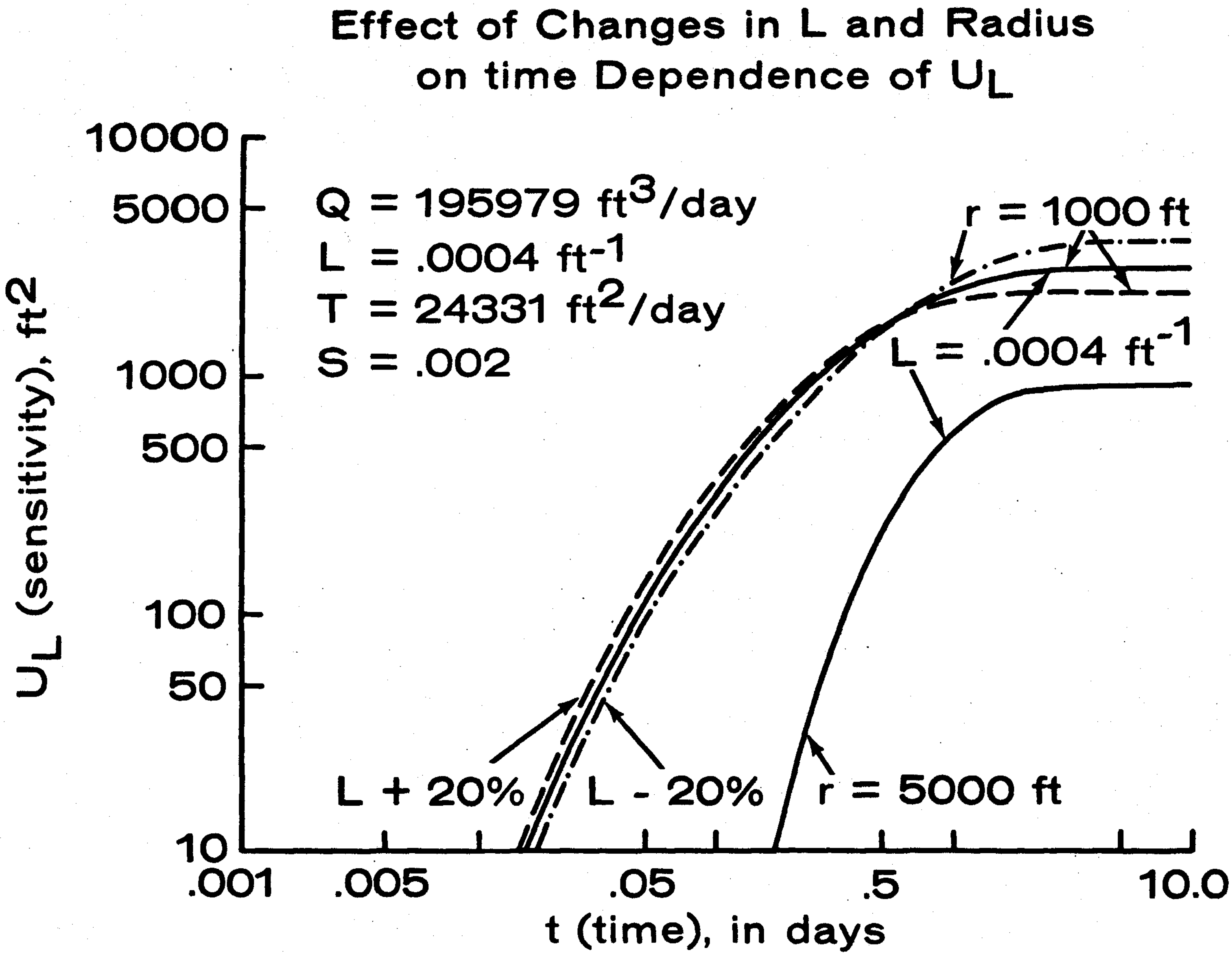

Effects of Changes in S on the
Radial Dependence of U_S
400
300
200
100
0
0
3
6
U_S (sensitivity), in ft
r (radial distance), $\times 10^{-3}$ ft
S - 20%
S = .002
S + 20%
$Q = 195979\ ft^3/day$
$T = 24331\ ft^2/day$
$L = .0004\ ft^{-1}$
t = .1 day

Effects of Radius on the Time Dependence of U_S



Effect of Changes in L on the Radial Dependence of U_L





Effect of Changes in L and Radius
on time Dependence of U_L
Q = 195979 ft^3/day
L = .0004 ft^{-1}
T = 24331 ft^2/day
S = .002
r = 1000 ft
L = .0004 ft^{-1}
L + 20%
L - 20%
r = 5000 ft
U_L (sensitivity), ft^2
10000
5000
1000
500
100
50
10
.001
.005
.05
.5
10.0
t (time), in days

Establishing the Error Formula

The new drawdown produced by a change in T, S, and L is

$$s^* = s + U_T\ \Delta T + U_S\ \Delta S + U_L\ \Delta L$$

The squared error in s* compared to the experimentally measured drawdown, s_e is

$$E = \sum_i [s_e\ (t_i) - s^*\ (t_i)]^2$$

$$= \Sigma [s_e(t_i) - s(t_i) - U_T(t_i)\Delta T - U_S(t_i)\Delta S - U_L(t_i)\Delta L]$$

For minimizing E we require that

$$\frac{\partial E}{\partial \Delta T} = \frac{\partial E}{\partial \Delta S} = \frac{\partial E}{\partial \Delta L} = 0$$

Matrix Equation to be Solved for Delta Values

$$\begin{bmatrix} \sum_i U^2_L & \sum_i U_L U_S & \sum_i U_B U_T \\ \sum_i U_S U_L & \sum_i U^2_S & \sum_i U_S U_T \\ \sum_i U_T U_L & \sum_i U_T U_S & \sum_i U^2_T \end{bmatrix} \begin{bmatrix} \Delta L \\ \Delta S \\ \Delta T \end{bmatrix} = - \begin{bmatrix} \sum_i U_L (s_g - s_e) \\ \sum_i U_S (s_g - s_e) \\ \sum_i U_T (s_g - s_e) \end{bmatrix}$$

This equation is solved for the delta values which are used to update the aquifer parameters:

$$L_{i+1} = L_i + \Delta L_i$$

$$S_{i+1} = S_i + \Delta S_i$$

$$T_{i+1} = T_i + \Delta T_i$$

Iteration Procedure

1) Initial guess for T, S, and L

2) Calculate expected drawdown values

3) Calculate sensitivity coefficients U_T, U_s, and U_L

4) Determine ΔT, ΔS, and ΔL

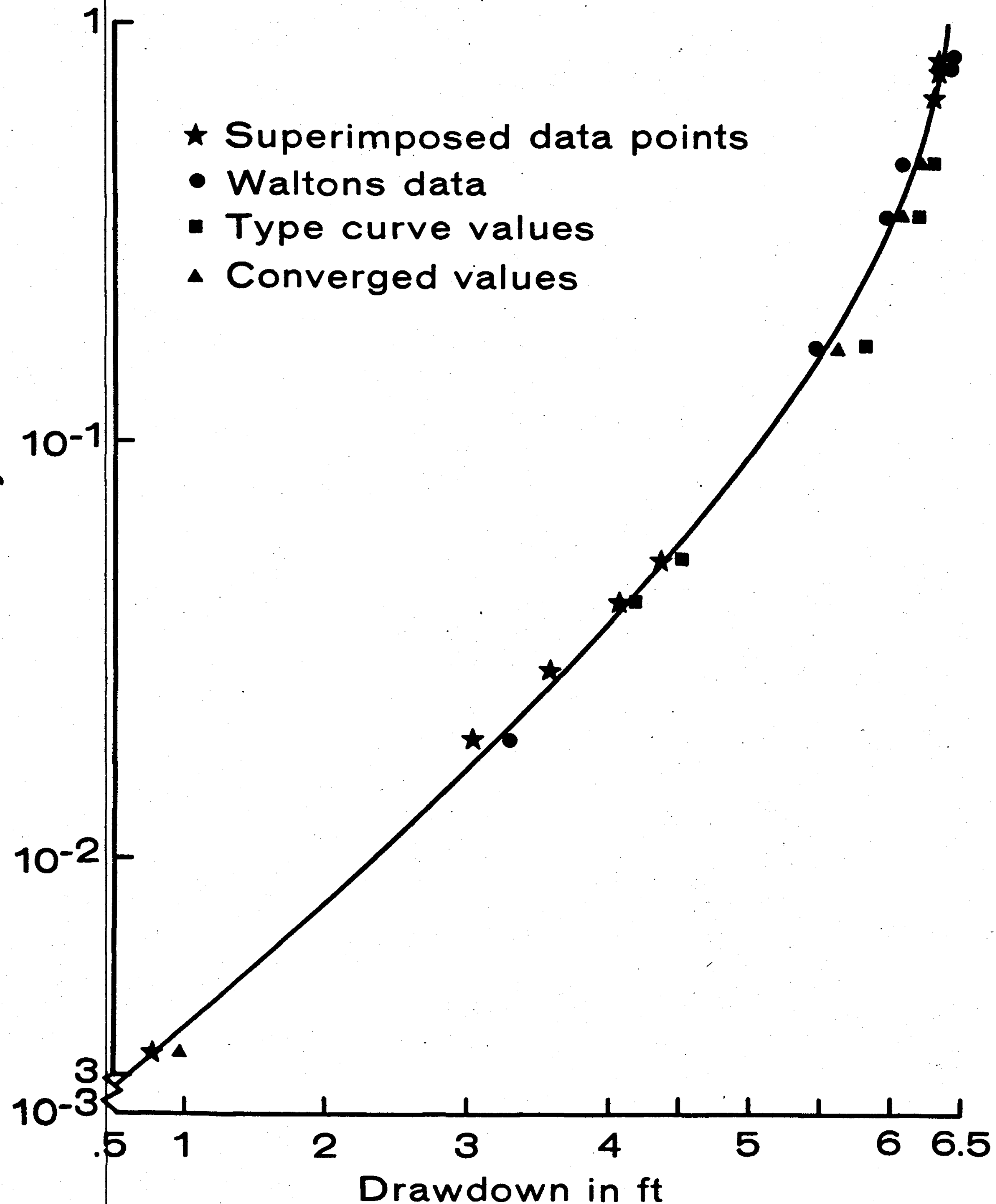
5) Upgrade values for T, S, and L

6) If ΔT, ΔS, and ΔL are less than ε, stop.

7) If maximum number of iterations is not exceeded, go to step 2.

Comparison of Aquifer Parameters

Parameter	Type Curve	Regression	Percent Difference
T (ft^2/day)	100	248	148
S (unitless)	.00015	.00017	13
L (ft^{-1})	.002	.0015	25



Superimposed data points
Waltons data
Type curve values
Converged values
1
10-1
10-2
10-3
3
.5
1
2
3
4
5
6
6.5
Drawdown in ft