Sensitivity of Groundwater Models With Respect To Variations in Transmissivity and Storage

Carl D. McElwee

Kansas Geological Survey, University of Kansas, Lawrence, Kansas 66044

M. Arif Yukler

KFA, Erdol u. Organische Geochemie, Julich, West Germany

Prepared for publication in Water Resources Research

Prepared by the Kansas Geological Survey in cooperation with the U.S. Geological Survey

1977

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ABSTRACT

Sensitivity analysis is the study of a system's response to various disturbances. In this study disturbances of aquifer parameters are considered. In the simulation of an aquifer the investigator must establish tolerances within which the parameters of the physical system may vary without appreciably affecting the model results. An expression for the sensitivity coefficient is obtained in this study by taking the partial derivative of the flow equation with respect to a particular parameter. The sensitivity coefficient is evaluated by standard techniques for several common models. The hydraulic head is obtained for a range of transmissivity (T) and storage coefficient (S) values by applying a first order sensitivity formalism. This procedure should be a valuable tool in calibrating models or establishing tolerances on T and S for a given acceptable error in hydraulic head. In general a ± 20 percent deviation in T or S may be handled adequately by the first order formalism discussed in this work.

INTRODUCTION

Sensitivity analysis is the study of a system's response to various disturbances. These disturbances may be small or large, momentary or permanent; they may be related to initial conditions, boundary conditions, physical parameters, etc. In this study disturbances of aquifer parameters are considered. This analysis yields the parametric sensitivity which is a measure of the change in the system's output resulting from a disturbance in an aquifer parameter.

The mathematical description of groundwater flow requires specific assumptions in order to fit the physical events into a set of equations that can be solved reasonably. The expression of the physical events by mathematical equations, estimation of the aquifer parameters, and the approximation of the equations by their discrete analogs are important sources of error. Because these errors introduce uncertainties in groundwater modeling, future projections cannot be made with absolute accuracy. The validity of the mathematical equations and the errors introduced by numerical methods have been discussed elsewhere. This work shall only be concerned with errors introduced by inaccuracies in aquifer parameters. In the mathematical treatment of dynamic systems it is permissible to speak of the precise values of the physical parameters. However, in the practical simulation of real dynamic systems we are immediately faced with uncertainty as to the exact physical parameters. The investigator must establish tolerances within which the parameters of the physical system may vary without appreciably affecting the model results. These tolerances are often obtained by introducing parameter perturbations in the system and observing the changes in the system's performance. However, the application of sensitivity analysis makes it possible to obtain these tolerances more efficiently.

A partial differential equation for the sensitivity function is obtained in this study by taking the partial derivative of the flow equations with respect to a particular parameter [Tomovic, 1962; Vemuri et al., 1969; McCuen, 1973; Yukler, 1976]. Then these sensitivity equations are solved by computer techniques to obtain parametric sensitivity coefficients. This is a simple and direct method for determining a system's sensitivity. The effect of variations, in transmissivity (T) and storage coefficient (S) for models having different boundary conditions is discussed in this work.

DEFINITION AND USE OF SENSITIVITY COEFFICIENTS

In studying the sensitivity of a groundwater flow system to parameter variations, the following mathematical model is used;

$$F\ (h_{xx}, h_{yy}, h_t;\ T,\ S,\ Q) = 0 \tag{1}$$

where $h_{xx} = \frac{\partial^2 h}{\partial x^2},\ h_{yy} = \frac{\partial^2 h}{\partial y^2},\ h_t = \frac{\partial h}{\partial t},$

h = hydraulic head,

T = transmissivity,

S = storage coefficient, and

Q = discharge

The solution of equation (1) may be written in the form $h = h(x,y,t;T,S,Q)$. Consider the variation of one of the parameters, T for example. Varying this parameter by a small amount, ΔT, the equation becomes

$$F(h^*_{xx},\ h^*_{yy},\ h^*_t;\ T+\Delta T,\ S,\ Q) = 0, \tag{2}$$

where h* is the perturbed head. The solution to equation (2) may be written in the form $h^* = h^*(x,y,t;\ T+\Delta T,S,Q)$. Comparing the solutions of equations (1) and (2), one immediately obtains an indication of the stability of the system, which is expressed by means of the fraction

$$\frac{\Delta h}{\Delta T} = \frac{h^*(x,y,t;T+\Delta T,S,Q)-h(x,y,t;T,S,Q)}{\Delta T}. \qquad (3)$$

If expression (3) has a limiting value as ΔT approaches zero, it may be written as

$$U_T(x,y,t;T,S,Q) = \frac{\partial h}{\partial T} = \lim_{\Delta T\to 0} \frac{\Delta h}{\Delta T}. \qquad (4)$$

The function $U_T(x,y,t;\ T,S,Q)$ will be called the sensitivity coefficient [Tomovic, 1962] for variations in the T value of a groundwater flow system. By applying similar arguments for a variation in storage coefficient (ΔS) one obtains,

$$\frac{\Delta h}{\Delta S} = \frac{h^*(x,y,t;T,S + \Delta S,Q)-h(x,y,t;T,S,Q)}{\Delta S} \qquad (5)$$

and

$$U_S(x,y,t;T,S,Q) = \frac{\partial h}{\partial S} = \lim_{\Delta S\to 0} \frac{\Delta h}{\Delta S}. \qquad (6)$$

U_S is the sensitivity coefficient for variations in the storage coefficient of a groundwater flow system.

It is assumed that the solution of the flow equation (1) depends analytically upon the parameters T and S; and, that T, S, and Q are independent of each other. Now consider a perturbation of the transmissivity, ΔT. Since it has been assumed that the solutions depend analytically on the parameters, the function $h^*(x,y,t;\ T+\Delta T,\ S,\ Q)$ may be expanded into a Taylor series [Tomovic, 1962]. If ΔT is small the second and higher order terms may be neglected,

$$h^*(x,y,t;T+\Delta T,S,Q) = h(x,y,t;T,S,Q) + \frac{\partial h}{\partial T}\Delta T$$

$$= h(x,y,t;T,S,Q) + U_T \Delta T \quad (7)$$

Thus, the new head produced by a perturbation in transmissivity (ΔT) may be calculated from equation (7) if the sensitivity coefficient and the unperturbed head are known. Similarly, if a perturbation in storage coefficient (ΔS) occurs the perturbed head is given by

$$h^*(x,y,t;T,S+\Delta S,Q) = h(x,y,t;T,S,Q) + \frac{\partial h}{\partial S}\Delta S$$

$$= h(x,y,t;T,S.Q) + U_S \Delta S \quad (8)$$

to first order in ΔS.

Equations (7) and (8) show that it would be desirable to calculate U_T and U_S for a given model, if possible. Then the response of the model to various perturbations could be calculated simply from equation (7) or (8) without actually evaluating the model equations again. A differential equation for U_T or U_S may be derived by differentiating the flow equation (1) with respect to T or S. The resulting equation is called the sensitivity equation. In deriving the sensitivity equation, it is assumed that the function h and its partial derivatives $\frac{\partial^2 h}{\partial x^2}$, $\frac{\partial^2 h}{\partial y^2}$, $\frac{\partial h}{\partial t}$, $\frac{\partial h}{\partial T}$, and $\frac{\partial h}{\partial S}$ are continuous so that the order of differentiation is interchangeable [Margenau, 1968]. Examples of sensitivity equations for various models will be given in later sections. In general, the sensitivity equation will be similar in form to the flow equation. If numerical methods are used to solve the model equations, the sensitivity equations usually may be solved simultaneously with little additional effort.

In the following sections the range of validity of the first order equations (7) and (8) will be investigated. The procedure generally will follow these steps:

(1) $h(x,y,t;\ T,S,Q)$, the solution of the flow equation (1) is found for the initial and boundary conditions.

(2) The aquifer parameters T and S are perturbed by some amount and the solution to the flow equation for the perturbed head is obtained.

$$h^*(x,y,t;\ T+\Delta T,S,Q)$$

$$h^*(x,y,t;\ T,S+\Delta S,Q)$$

(3) The sensitivity coefficients U_T and U_S are obtained from the sensitivity equation numerically or by differentiating an analytic solution of the flow equation.

(4) The perturbed head values obtained from equations (7) and (8) are compared to those obtained in step (2).

SENSITIVITY FROM THE THEIS EQUATION

Although the main thrust of this paper is to show the use of sensitivity coefficients with numerical models, much information about the behavior of sensitivity coefficients may be obtained by starting with the Theis equation [Theis, 1935].

$$s = h - h_o = \frac{114.6Q}{T} \int_{\frac{1.87r^2S}{Tt}}^{\infty} \frac{e^{-u}}{u} du \tag{9}$$

In the above equation s is the drawdown, in feet; h_o is the altitude of the piezometric surface before pumping; h is the subsequent altitude of the piezometric surface; Q is the discharge, in gal/min; T is the transmissivity, in gal/day/ft; t is the time, in days; S is the storage coefficient, dimensionless; and r is the radial observation distance from the pumped well, in feet. The sensitivity coefficients may be obtained from equation (9) by applying the definitions given in equations (4) and (6). After applying Leibnitz's rule for differentiating an integral [Hildebrand, 1962] to equation (9) one obtains

$$U_T = \frac{\partial h}{\partial T} = \frac{\partial s}{\partial T} = - \frac{s}{T} + \frac{114.6Q}{T^2} \text{ EXP } [- \frac{1.87r^2S}{Tt}] \tag{10}$$

and

$$U_S = \frac{\partial h}{\partial S} = \frac{\partial s}{\partial S} = - \frac{114.6Q}{TS} \text{ EXP } [- \frac{1.87r^2S}{Tt}]. \tag{11}$$

These equations for the sensitivity coefficients, which may be evaluated quite easily, illustrate some interesting general properties of sensitivity coefficients.

Figure 1 is a plot of drawdown at .0168 days after pumping began versus radial distance from the pumped well. The aquifer is assumed to have a storage coefficient of 0.000948. The well is discharging at a rate of 240,000 gal/day. The three solid lines represent drawdown for three different values of transmissivity. The center solid line is for a transmissivity of 24,000 gal/day/ft. The solid lines, labeled $\pm 20\%$ T, illustrate how the drawdown changes when the transmissivity is changed by the indicated amount. Figure 1 shows that the drawdown is increased by a decrease of the transmissivity and decreased by an increase of the transmissivity for small r.

Figure 2 is an expanded version of figure 1 showing the detailed behavior of the drawdown between 200 and 500 feet from the pumped well. The three curves cross between 300 and 320 feet. This crossing is necessary because equal amounts of water have been pumped in all three examples; thus, the volumes of the three cones of depression must be equal.

It is obvious from figures 1 and 2 that the system is most sensitive to changes in the transmissivity near the well where drawdown is the largest. This radial dependence is shown dramatically in figure 3 where U_T, calculated from equation (10), is plotted versus radial distance from the well. It appears from figure 3 that the sensitivity coefficient U_T diverges at the well. In fact it may be shown from equation (10) that for

$$\frac{1.87r^2S}{Tt} << 1$$

$$U_T \rightarrow \frac{114.6Q}{T^2}\left[1.577216 + \ln\left(\frac{1.87r^2S}{Tt}\right)\right]. \tag{12}$$

Expression (12) shows that U_T should diverge logarithmically at the well. The sensitivity function U_T changes sign in the region 300-320 feet from the well, as it must in order for the cones of depression to have the same volume for differing transmissivities. The magnitude of the sensitivity coefficient U_T is relatively small in the region where U_T is negative.

In figure 1, the dashed lines represent the drawdown calculated from the truncated Taylor series using the sensitivity coefficient

$$h^* - h_o = h - h_o + U_T \Delta T$$

$$s^* = s + U_T \Delta T \qquad (13)$$

for transmissivity varying from 24,000 gal/day/ft by ± 20 percent. Equation (13) states that the magnitude of the change in drawdown will be the same regardless of whether ΔT is positive or negative. This is shown in figure 1 by the fact that, for a given radius, the dashed lines are located symmetrically about the center solid line. However, the solid lines representing the drawdown calculated from the Theis equation for a change in transmissivity of ± 20 percent are not symmetric about the center line. The fact that equation (13) does not show this asymmetry is a result of dropping higher order terms in the Taylor series expansion during the derivation of equation (7). This neglect of higher order terms in equation (13) limits its applicability in cases where ΔT is large.

The error in using equation (13) to calculate the new drawdown when transmissivity is perturbed by ± 20 percent in shown in figure 1 by the separation of the solid and dashed lines for small values of r. The error in this example is considerably less than 5 percent of the total drawdown. Numerical experiments have shown that equation (13) is adequate to calculate perturbed drawdowns for a variation of transmissivity up to 25 percent if an error of 5 percent or less in total drawdown is acceptable. The largest percent error occurs near the well and has a very weak dependence on transmissivity and time. The radial dependence of the percent error, shown in figure 4, indicates that the percent error decreases as the radius increases and approaches zero where the drawdown becomes negligible.

Figure 5 shows the time dependence of U_T for r and t such that U_T is positive, which roughly corresponds to areas of the cone of depression near the well where the drawdown is significant. Notice that for large values of t the dependence of U_T on t is fairly weak though U_T is not constant. The curves labeled ± 20 percent T show how U_T at a radius of one foot changes when the transmissivity is perturbed by ± 20 percent. In this region a larger transmissivity results in decreased sensitivity; and, a smaller transmissivity results in greater sensitivity. The curves labeled r = 1 foot and 1,000 feet, where T = 24,000 gal/day/ft, show the effect of changing r in the evaluation of U_T. These two curves have an identical shape but are displaced from one another along the t axis. The relation of these curves can be seen from equation (10). The critical ratio is r^2/t; as long as this ratio is the same U_T will not change. Thus, the curve for r = 1 foot at $t = 10^{-6}$ days has the same value as the curve for r = 1,000 feet at t = 1 day.

The sensitivity with respect to the storage coefficient, U_S, may be evaluated using equation (11). Figure 6 exhibits the drawdown versus r plot for the same well as described for figure 1 except that the storage coefficient is changed by ± 35 percent while the transmissivity is held constant. The dashed lines represent the drawdown calculated from

$$s^* = s + U_S \Delta S. \quad (14)$$

Although the smallest r shown is 10 feet, the straight line behavior of all the curves continues to the well. It should be noted that a decrease in storage coefficient causes a general deepening of the cone of depression, whereas, an increase in storage coefficient causes a general shallowing of the cone. There is no crossover behavior of the curves as occurs when the transmissivity is changed. This behavior is evident from equation (9) because as long as S/t has the same value the drawdown remains unchanged, provided r, T, and Q are held constant. Thus, increasing S necessitates the evaluation of the drawdown at a greater time to obtain the same value for S/t and the drawdown. This means that the drawdown is progressing less rapidly for a larger S. Similar arguments show that the drawdown progresses more rapidly if a smaller S is used.

The radial dependence of U_S is shown in figure 7. U_S does not diverge at the well as does U_T. Equation (11) and figure 7 show that the radial dependence of U_S is Gaussian. U_S does not change sign because an increase or decrease in S results in a general raising or lowering, respectively, of the cone of depression. The dashed lines in figure 7 show U_S when S is changed by ± 20 percent. These curves indicate that the system is less sensitive for a larger S and more sensitive for a smaller S. This behavior can also be seen from equation (11).

The time dependence of U_S is illustrated in figure 8 for three different r values. As time increases U_S approaches a constant value. Even for r = 1,000 feet U_S is nearly constant after about one day. U_S is practically zero when the drawdown is very small, and nearly constant after the drawdown attains 1-2 ft. The three curves shown in figure 8 are identical except for displacement in time. From equation (11) it may be seen that U_S has the same value when r^2/t remains constant provided Q, T, and S are unchanged. Thus, the t = 1 day point on the curve for r = 1,000 feet is identical to the t = 10^{-4} days point on the curve for r = 10 feet.

It is seen from figure 6 that even a 35 percent change in S does not produce as much change in the drawdown near the well as a 20 percent change in the transmissivity. However, comparing figures 2 and 6, it is evident that the change in drawdown at 300 feet due to a 35 percent change in S is much greater than that due to a 20 percent change in transmissivity. The effect of a change in S is significant over a larger area than that due to a change in T.

The asymmetry of the outer solid lines about the center line in figure 6 illustrates the effect of higher order terms neglected in equation (14). The dashed lines, calculated from equation (14) using only first order terms in ΔS, are symmetric about the center line. The error caused by using equation (14) is given by the difference in the solid and dashed curves for ± 35 percent S. The radial dependence of this error is plotted in figure 9. The error is roughly constant where the drawdown exceeds one foot and decreases to zero as the drawdown becomes negligible for larger values of r. The error at a given point has a very weak dependence on time and S once the drawdown is non-negligible. This result can be inferred from figure 9 because changing time or S merely results in shortening or lengthening the constant error segment. Numerical experiments indicate that the new drawdown is adequately represented by equation (14) as long as ΔS is less than or approximately 35 percent of S.

SENSITIVITY FROM A NUMERICAL RADIAL MODEL

The equation describing confined water flow through porous media in cylindrical coordinates is [Jacob, 1940]

$$\frac{T}{r}\frac{\partial}{\partial r}\left[r\frac{\partial h(r,t)}{\partial r}\right] = S\frac{\partial h(r,t)}{\partial t}, \tag{15}$$

where the z coordinate has been eliminated by considering h to be an average hydraulic head over the z dimension. Equation (15) along with appropriate boundary conditions and an initial function for h poses a complete description of the radial model to be discussed. If one considers a well of radius r_w pumped at a constant rate Q, these conditions define the radial derivative of h at the well face.

$$\left[\frac{\partial h(r,t)}{\partial r}\right]_{r=r_w} = -\frac{Q}{2\pi T r_w} \tag{16}$$

At some radius R, one of two common boundary conditions will be imposed. The condition of constant head defined by

$$h(R,t) = \text{Constant} \tag{17}$$

is frequently used in the literature. When no water is allowed to flow through a boundary a barrier boundary exists defined by

$$\left[\frac{\partial h(r,t)}{\partial r}\right]_{r=R} = 0. \tag{18}$$

An initially flat head function is assumed.

$$h(r,0) = \text{Constant} \tag{19}$$

The radial model defined by equations (15) through (19) can be put in Crank-Nicolson numerical form and solved by use of the Thomas algorithm [Von Rosenberg, 1969; Carnahan, et al., 1969]. Ehlig and Halepaska [1976] give a good discussion of this procedure. As long as R is large (i.e. for times small enough that the boundary is not important) and r_w is small, the numerical results agree very well with the Theis equation (9).

At this point the objective is to see the effect of the outer boundary on the sensitivity coefficients U_T and U_S. An equation for the sensitivity function U_T or U_S may be obtained by differentiating equation (15) with respect to T or S, respectively.

$$\frac{T}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_T}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\ \frac{\partial h}{\partial r}\right) = S\ \frac{\partial U_T}{\partial t} \tag{20}$$

$$\frac{T}{r}\frac{\partial}{\partial r}\left(r\ \frac{\partial U_S}{\partial r}\right) = \frac{\partial h}{\partial t} + S\ \frac{\partial U_S}{\partial t} \tag{21}$$

These equations have the same form as equation (15) except for terms involving derivatives of h. However, h is known from the numerical solution of equations (15) through (19). Therefore, equations (20) and (21) can also be put in Crank-Nicolson form and solved by use of the Thomas algorithm when the appropriate boundary and initial conditions on U_T or U_S have been specified.

The boundary condition on U_T or U_S at the well face can be obtained by differentiating equation (16) with respect to T or S.

$$\left[\frac{\partial U_T}{\partial r}\right]_{r\ =\ rw} = \frac{Q}{2\pi r_w T^2} \tag{22}$$

$$\left[\frac{\partial U_S}{\partial r}\right]_{r\ =\ rw} = 0 \tag{23}$$

The condition on U_T or U_S at R is given by differentiating either equation (17) or equation (18) with respect to T or S.

$$U_T(R,t) = 0 \qquad (24)$$

Constant head boundary

$$U_S(R,t) = 0 \qquad (25)$$

$$\left[\frac{\partial U_T}{\partial r}\right]_{r=R} = 0 \qquad (26)$$

Barrier boundary

$$\left[\frac{\partial U_S}{\partial r}\right]_{r=R} = 0 \qquad (27)$$

The initial conditions on U_T and U_S are,

$$U_T(r,0) = 0 \qquad (28)$$

$$U_S(r,0) = 0. \qquad (29)$$

The values for U_T and U_S, obtained by numerical solution of equations (20) through (29) for times such that the boundary at R does not appreciably affect the drawdown, agree very well with those values obtained from the Theis equation. Therefore, the discussion in the previous section is equally valid here for small times and will not be repeated. The boundary condition at R on U_T and U_S represents a new constraint and introduces effects that were not observed in the infinite Theis model. The exhibition of these boundary effects is the main objective of this section.

The values of the parameters Q, T, and S for the radial numerical model were chosen to be the same as those used in the Theis equation of the preceding section. However two additional parameters are needed, the radius of the well and the radius of the outer boundary (R). The radius of the well was taken as 1 foot and an outer boundary of 10,000 feet was used. The numerical results for U_T, obtained by choosing a constant head boundary or a barrier boundary at 10,000 feet, are shown in figure 10 along with U_T calculated from the Theis equation. For times less than about 10 days there is no difference in the three curves for r = 1 foot and for r = 1,000 feet.

The constant head boundary at 10,000 feet produces a U_T as shown by the dot-dash curves in figure 10. U_T was calculated by a numerical solution of equations (20), (22), (24), and (28). The Theis infinite model results are shown as a solid curve. For times greater than about 20 days the water level is static due to the constant head boundary and U_T obtains a constant value in time. Figure 11 shows the radial dependence of U_T at steady state. Notice that U_T is a straight line when plotted versus ln r. This fact is easily obtainable from the analytic formula for the steady state head distribution [Jacob, 1950],

$$h = h_w + \frac{Q}{2\pi T} \ln(\frac{r}{r_w}) \tag{30}$$

where h_w is the water level in the well and the other symbols have been previously defined. At $r = R$, $h = h_o$ (h_o is the initial head before pumping).

$$h_o = h_w + \frac{Q}{2\pi T} \ln(\frac{R}{r_w}) \tag{31}$$

Using equation (31) to eliminate h_w in equation (30), one obtains

$$s = h - h_o = \frac{Q}{2\pi T} \ln(\frac{r}{R}). \tag{32}$$

Differentiating equation (32) with respect to T gives the steady state form of U_T.

$$U_T = \frac{\partial h}{\partial T} = \frac{\partial s}{\partial T} = - \frac{Q}{2\pi T^2} \ln \frac{r}{R} = - \frac{s}{T} \tag{33}$$

This equation explains the straight line behavior in figure 11 and shows that U_T goes to zero at the outer boundary where the drawdown is zero.

It should be noted that U_T at steady state is positive for all r. This is to be contrasted with the U_T shown in figure 3 for the Theis equation. U_T being positive is a direct consequence of the fact that all the water being pumped is supplied by the constant head boundary; no water is being supplied from storage. Increasing or decreasing T results in a general raising or lowering, respectively, of the hydraulic head at steady state. The error in using equation (7) to calculate the new hydraulic head when T is perturbed by ΔT is less than 5 percent of the total drawdown at steady state when ΔT is ± 20 percent of T.

The numerical solution for U_T with a barrier boundary condition at 10,000 feet is shown by the dashed curves in figure 10. After about 10 days the cone of depression has reached the barrier boundary and the water level declines uniformly with time. When this happens U_T becomes constant in time. Thus, any correction in hydraulic head calculated from equation (7) for a given ΔT is constant in time after about 10 days. The previous statement has been verified by numerical experiments using different transmissivities. The error in using equation (7) to calculate the new hydraulic head after changing T was less than 5 percent of the total drawdown for ΔT = ± 20 percent of T for the times considered. As time continues to increase after about 10 days, the drawdown increases linearly with time while the error in using equation (7) remains numerically constant; thus, the error in terms of percent of total drawdown decreases with time.

In figure 12 the radial dependence of U_T (for a barrier boundary) is shown for time considerably greater than 10 days, i.e. for time such that U_T is constant in time. Notice that U_T is negative for r greater than about 5,500 feet. This is to be constrasted with a positive U_T for a constant head boundary. Because no water can flow into the system with a barrier boundary, all water pumped must come from storage. Thus, for two systems with differing T and the same Q, the cones of depression must contain the same volume at any given instant. The low T system will have a larger drawdown near the well and a smaller drawdown far from the well because the lower T impedes the flow to the well. This explains why U_T has both negative and positive areas. A change in T will produce greater drawdown in one area and less drawdown in another area.

Figure 13 compares U_S calculated from the Theis equation (solid curve) with U_S calculated numerically for a constant head boundary (dot-dash curve) or a barrier boundary (dashed curve) at 10,000 feet. From earlier discussion it will be remembered that the water level does not change much after 20 days for the constant head boundary condition (approximately steady state). From figure 13 we see that U_S for the constant head boundary is approximately zero after about 100 days. This behavior of U_S is to be expected since the solution at steady state is independent of S because no water is coming from storage at steady state. Because U_S is growing smaller as steady state is approached, equation (8) should be valid for the same range of ΔS as discussed for the Theis equation.

All values of U_S plotted in figure 13 are positive indicating that an increase or decrease of S results in a general raising or lowering, respectively, of the hydraulic head. U_S for the barrier boundary condition increases dramatically after a few days time as shown in figure 13. Figure 14, which is an expanded time scale plot of U_S for the barrier boundary condition, shows that U_S increases linearly with time after about 10 days. Numerical results indicate that U_S is the same for all values of r after about 10 days. The linear increase of U_S with time is undoubtedly due to the fact that the hydraulic head decreases uniformly with time after about 10 days. If S is changed by ΔS the difference in hydraulic head of the perturbed and unperturbed systems becomes larger and larger as time continues past a few days. The correction to the hydraulic head $U_S \Delta S$ in equation (8) increases linearly with time after a few days. In a few words, the system becomes increasingly sensitive to S as the barrier boundary exerts a greater influence on the drawdown. The drawdown of the perturbed system and the drawdown calculated using equation (8) will increase linearly with time but at slightly different rates because equation (8) is not exact. This difference means that the error committed by using equation (8) grows linearly with time. Therefore, equation (8) can not be used indiscriminately at large values of time for situations having an important barrier boundary.

In summary, the sensitivity formalism may be used with numerical models having a variety of boundary conditions. However, the first order formalism given here may have excessive error when applied to a system at large values of time where a barrier boundary is exerting considerable influence on the drawdown. In general, a $\pm$ 20 percent deviation in T or S should be handled adequately by the first order formalism; i.e., the perturbed drawdown calculated from equation (7) or (8) should be an adequate approximation to the model drawdown for the new T or S value.

SENSITIVITY FROM A 2-DIMENSIONAL X-Y MODEL

The equation describing confined water flow through porous media in cartesian coordinates is [Stallman, 1956]

$$T\left[\frac{\partial^2 h(x,y,t)}{\partial x^2} + \frac{\partial^2 h(x,y,t)}{\partial y^2}\right] + Q(x,y,t) = S\,\frac{\partial h(x,y,t)}{\partial t}\ . \tag{34}$$

Again h is an average hydraulic head in the z dimension. Along with equation (34) an initial condition for h and boundary conditions for h are needed. Many times a flat surface is chosen for the initial head.

$$h(x,y,0) = \text{Constant} \tag{35}$$

The boundary may be a rather arbitrary function of x and y denoted by

$$B(x,y) = 0. \tag{36}$$

The boundary conditions must be specified on this curve. A constant head boundary is specified by

$$[h(x,y,t)]_{B(x,y)} = \text{Constant}. \tag{37}$$

If a constant flow of water is required on the boundary, then

$$\left[\frac{\partial h(x,y,t)}{\partial n}\right]_{B(x,y)} = \text{Constant}, \tag{38}$$

where $\frac{\partial}{\partial n}$ denotes the partial derivative in a direction normal to the boundary. If the constant is zero in equation (38), the result is a barrier boundary. In general the boundary could contain both types of boundary conditions given by equations (37) and (38).

For complicated geometries, equations (34) through (38) must be solved numerically. The procedure for applying Crank-Nicolson differencing to equation (34) and for solving the resulting equations by ADI or iterative methods is well documented in the literature [von Rosenberg, 1969]. An iterative procedure suggested by Halepaska, <u>et</u> <u>al</u>. [1972] has been used for this work.

The sensitivity equation for U_T or U_S is obtained by differentiating the flow equation (34) with respect to T or S, respectively.

$$T\left(\frac{\partial^2 U_T}{\partial x^2} + \frac{\partial^2 U_T}{\partial y^2}\right) + \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = S\,\frac{\partial U_T}{\partial t} \tag{39}$$

$$T\left(\frac{\partial^2 U_S}{\partial x^2} + \frac{\partial^2 U_S}{\partial y^2}\right) = \frac{\partial h}{\partial t} + S\,\frac{\partial U_S}{\partial t} \tag{40}$$

It should be remembered that continuous derivatives are assumed so that the order of differentiation may be interchanged. Also T, S, and Q are assumed to be independent parameters, i.e.

$$\frac{\partial Q}{\partial T} = 0,\ \frac{\partial S}{\partial T} = 0,\ \frac{\partial Q}{\partial S} = 0.$$

The boundary conditions for U_T and U_S can be found by differentiating equations (37) and (38).

Constant head boundary:

$$[U_T(x,y,t)]_{B(x,y)} = 0 \tag{41}$$

$$[U_S(x,y,t)]_{B(x,y)} = 0 \tag{42}$$

Constant flow boundary:

$$\left[\frac{\partial U_T(x,y,t)}{\partial n}\right]_{B(x,y)} = 0 \tag{43}$$

$$\left[\frac{\partial U_S(x,y,t)}{\partial n}\right]_{B(x,y)} = 0 \tag{44}$$

Except for the terms involving h, the sensitivity equations are similar to the flow equation. Because h is known from the solution of the flow equation, the same numerical techniques used on the flow equation can be applied to the sensitivity equations. Usually, little additional programming effort is required for the solution of the sensitivity equations once the solution technique for the flow equation has been established. The computing effort needed to solve either sensitivity equation (39) or (40) is roughly the same as that required to solve the flow equation.

For comparison with the previous sections based on the Theis equation and the radial model, equations (39) through (44) have been evaluated numerically for a square aquifer 20,000 feet on a side having a barrier or constant head outer boundary and a discharging well at the center. The values of T, S, and Q were chosen to be the same as those used in the Theis equation. For early times before the boundaries are affecting the drawdown, the results are basically the same as those obtained from the Theis equation. At later times when the boundaries are important, the results are similar to those reported for the radial model. In particular, the shapes of the curves for U_T and U_S and the error introduced by using equation (7) or (8) to calculate the perturbed head remained the same. Minor differences could be traced to the lack of radial grid symmetry, a different grid spacing, or a difference in specifying the flow into the well location.

DISCUSSION AND SUMMARY

Although this paper has been concerned with defining the sensitivity coefficients and applying sensitivity analysis to simple systems, sensitivity analysis should be of the most value in two dimensional X-Y models with arbitrary boundaries and several wells. By solving the flow equation and the sensitivity equations once for each time step, the head can be obtained for a range of T and S values simply by using equations (7) and (8). This technique should be a valuable tool in calibrating models or establishing tolerances on T and S for a given acceptable error in hydraulic head. Type curve solution for T and S is the standard method of analyzing pumping test data; however, sensitivity analysis could be a useful tool in analyzing the data, especially where an automated computer fit of the data is required.

The ideas presented in this work also can be applied to unconfined flow systems [Yukler, 1976]. In that application, one solves for the sensitivity with respect to hydraulic conductivity and specific yield. Equations similar to (7) and (8) may be developed to calculate the elevation of the water table for a range of values in the hydraulic conductivity and specific yield. The equations are somewhat more difficult to solve due to the non-linearity of the flow equation; but no conceptual difficulty is encountered. The results are very similar to those presented for the confined system and are not presented here.

The work presented here has dealt only with systems having spacially constant aquifer parameters T and S. This condition is somewhat restrictive because most sophisticated models allow T and S to vary spacially. This is an area of study where additional work could be done.

In summary, sensitivity analysis may be used to calculate the change in hydraulic head produced by changing one of the aquifer parameters within certain limits. In general a $\pm$ 20 percent deviation in T or S should be handled adequately (error less than 5 percent of drawdown) by the first order formalism discussed in this work. However, an important barrier boundary could produce greater errors at large values of time. Sensitivity analysis should be of the most use in systems with complicated geometry and several wells.

NOTATION

$B(x,y)$	arbitrary outer boundary of Cartesian model.
h	piezometric surface or hydraulic head at any time, feet.
h_o	original piezometric surface or hydraulic head, feet.
h^*	piezometric surface or hydraulic head at any time produced by a change in transmissivity or storage, feet.
Q	pumpage, gal per day or gal per min.
R	outer radius of model, feet.
r	radial distance, feet.
r_w	radius of a well, feet.
S	Storage coefficient, dimensionless.
s	drawdown, feet.
s^*	drawdown produced by a change in transmissivity or storage, feet.
T	transmissivity, gal per day per foot.
t	time, days.
U_S	sensitivity coefficient for storage variations, feet.
U_T	sensitivity coefficient for transmissivity variations, square feet days per gal.
x,y	Cartesian coordinates, feet.
Δh	a change in hydraulic head produced by ΔT or ΔS, feet.
ΔS	a change in storage coefficient, dimensionless.
ΔT	a change in transmissivity, gal per day per foot.

ACKNOWLEDGMENT

This paper was prepared as part of the cooperative program of groundwater research and investigations between the Kansas Geological Survey and the U. S. Geological Survey.

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FIGURE LEGENDS

Figure 1. Effect of varying transmissivity on the drawdown for small values of radius.

Figure 2. Effect of varying transmissivity on the drawdown for radii between 200 and 500 feet.

Figure 3. Radial dependence of sensitivity U_T.

Figure 4. Radial dependence of error in using equation (13).

Figure 5. Effect of radius and transmissivity on the time dependence of U_T.

Figure 6. Effect of varying storage coefficient on the drawdown.

Figure 7. Radial dependence of sensitivity U_S.

Figure 8. Time dependence of sensitivity U_S.

Figure 9. Radial dependence of error in using equation (14).

Figure 10. Effect of the boundary at 10,000 feet on U_T.

Figure 11. Radial dependence of U_T at steady state for a constant head boundary at 10,000 feet.

Figure 12. Radial dependence of U_T at large time for a barrier boundary at 10,000 feet.

Figure 13. Effect of the boundary at 10,000 feet on the time dependence of U_S.

Figure 14. Barrier boundary at 10,000 feet, time dependence of sensitivity U_S.

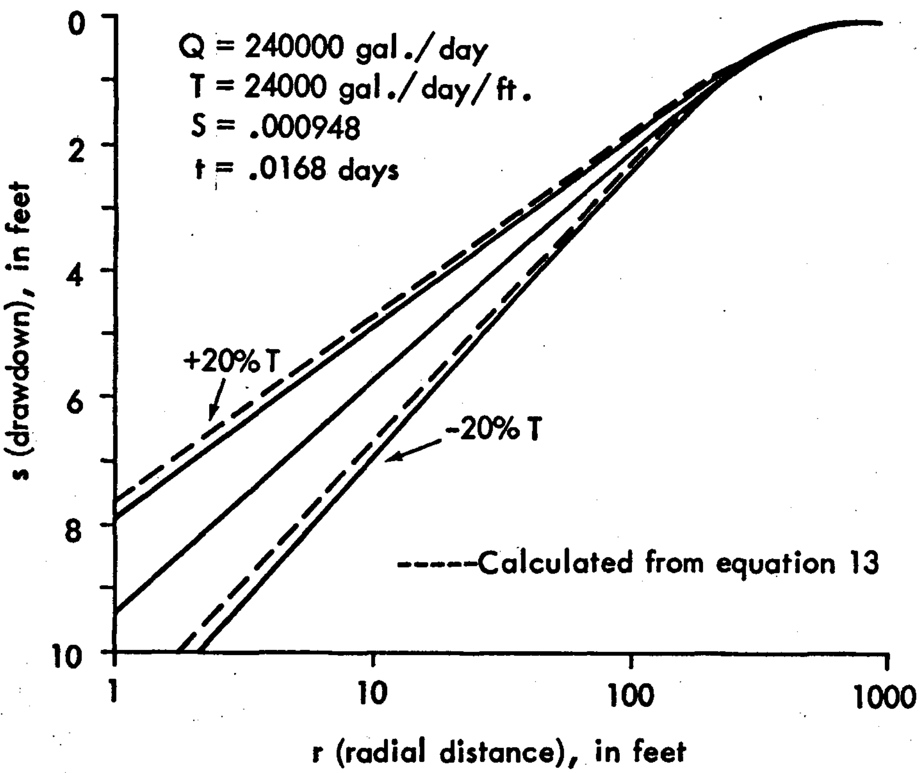

Figure 1. Effect of varying transmissivity on the drawdown for small values of radius.

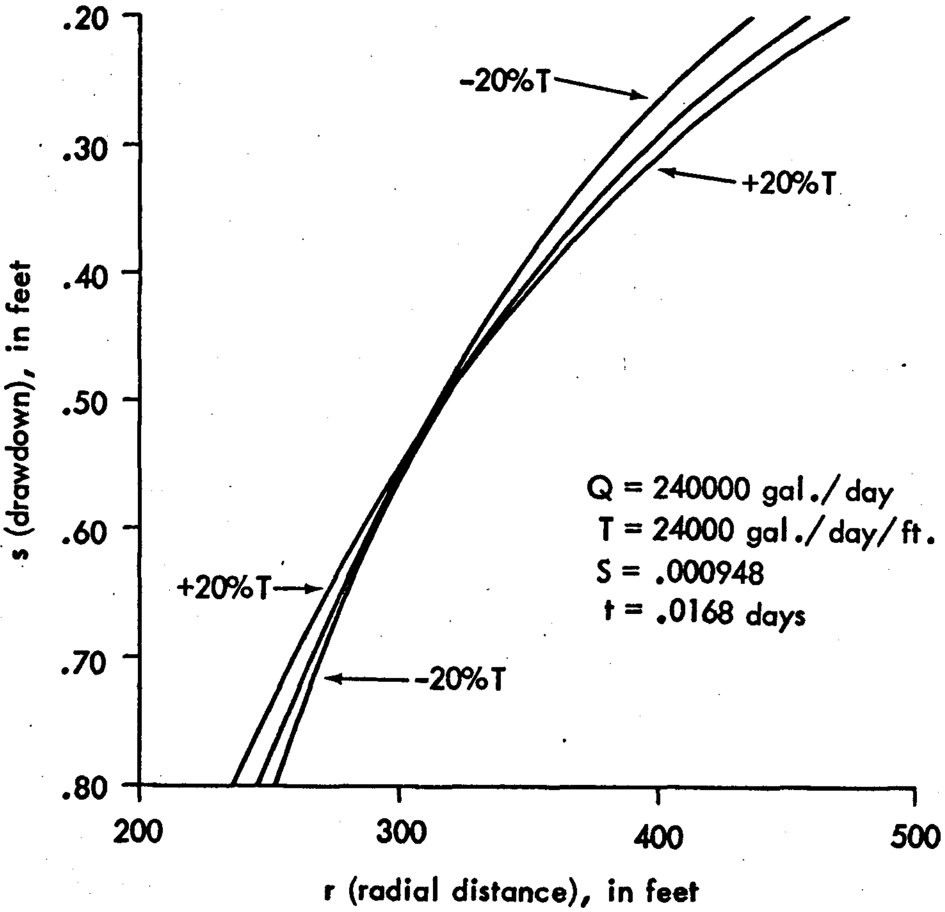

Figure 2. Effect of varying transmissivity on the drawdown for radii between 200 and 500 feet.

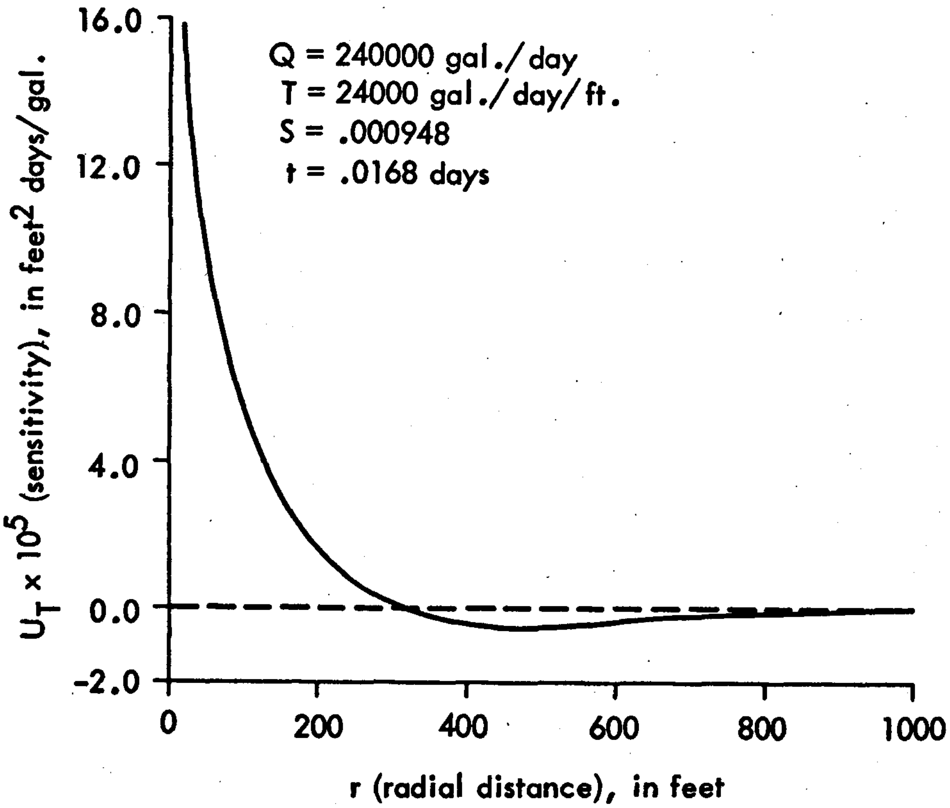

Figure 3. Radial dependence of sensitivity U_T.

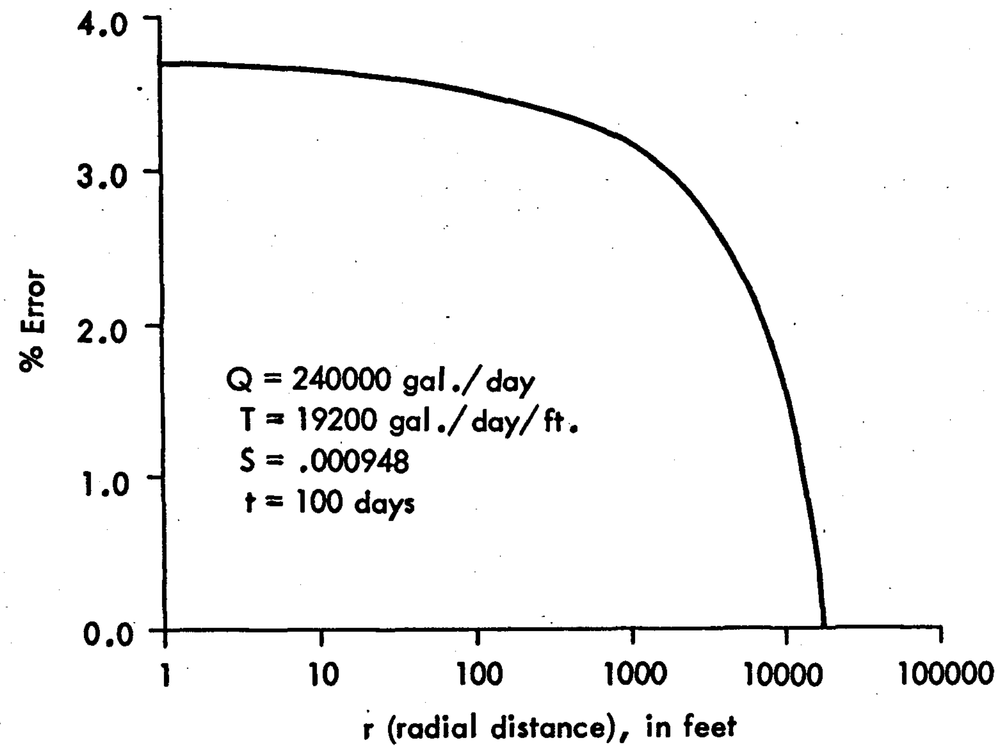

Figure 4. Radial dependence of error in using equation (13).

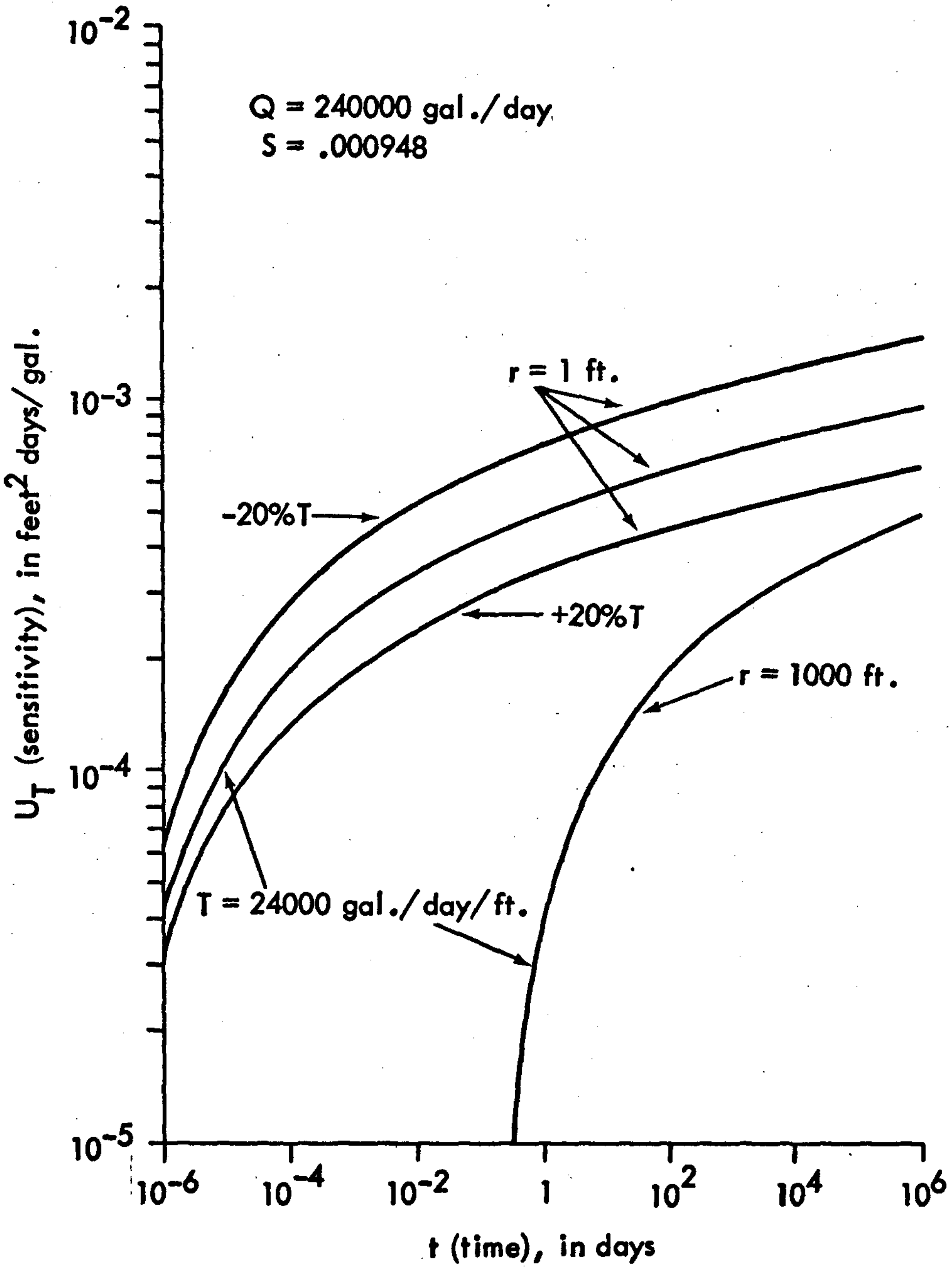

Figure 5. Effect of radius and transmissivity on the time dependence of U_T.

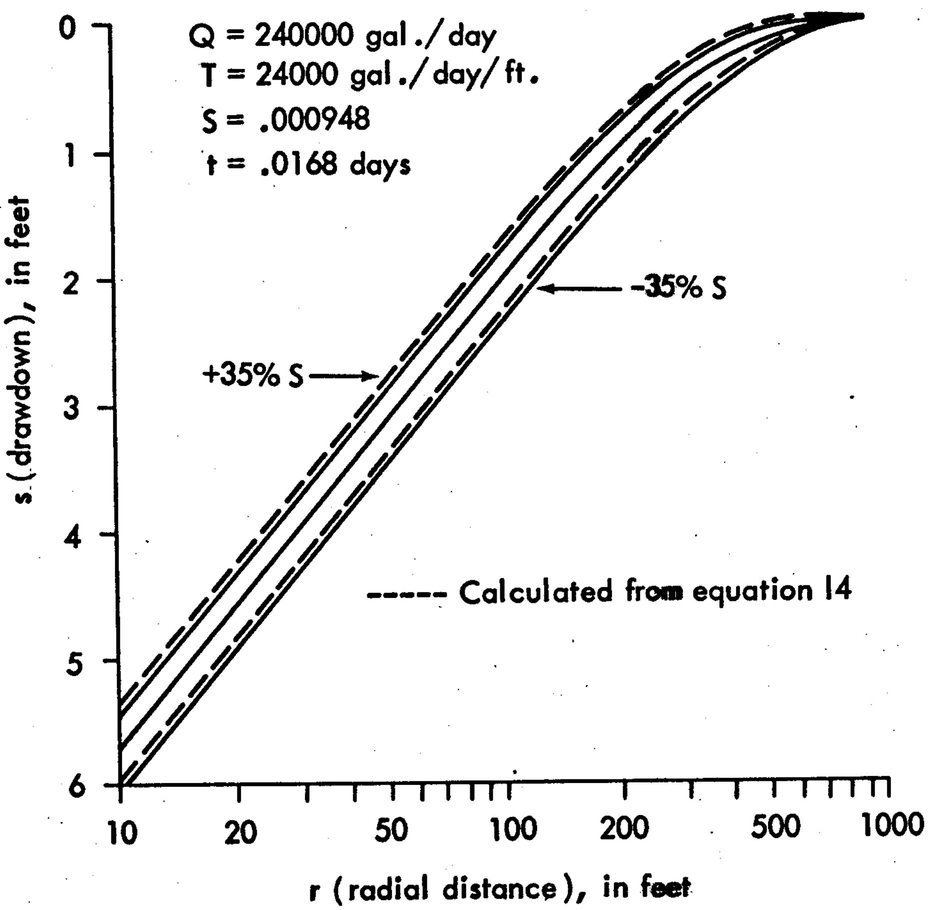

Figure 6. Effect of varying storage coefficient on the drawdown.

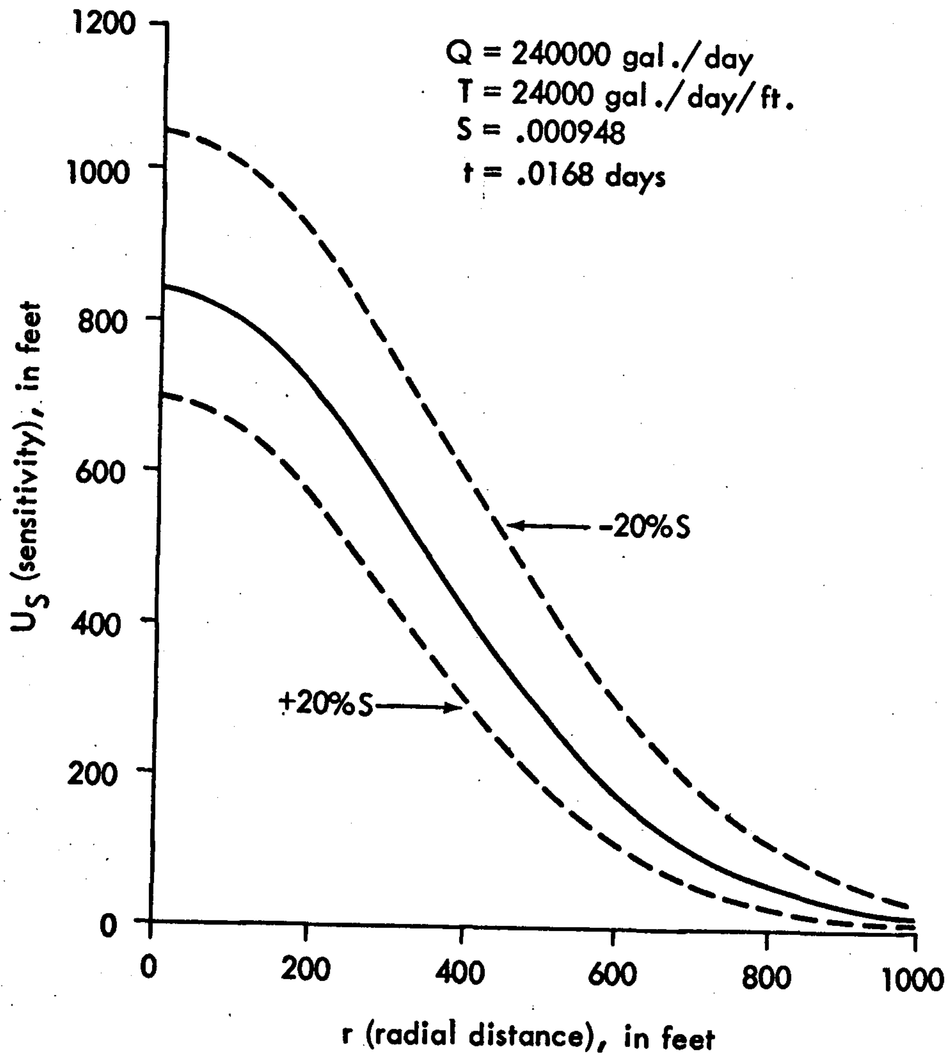

Figure 7. Radial dependence of sensitivity U_S.

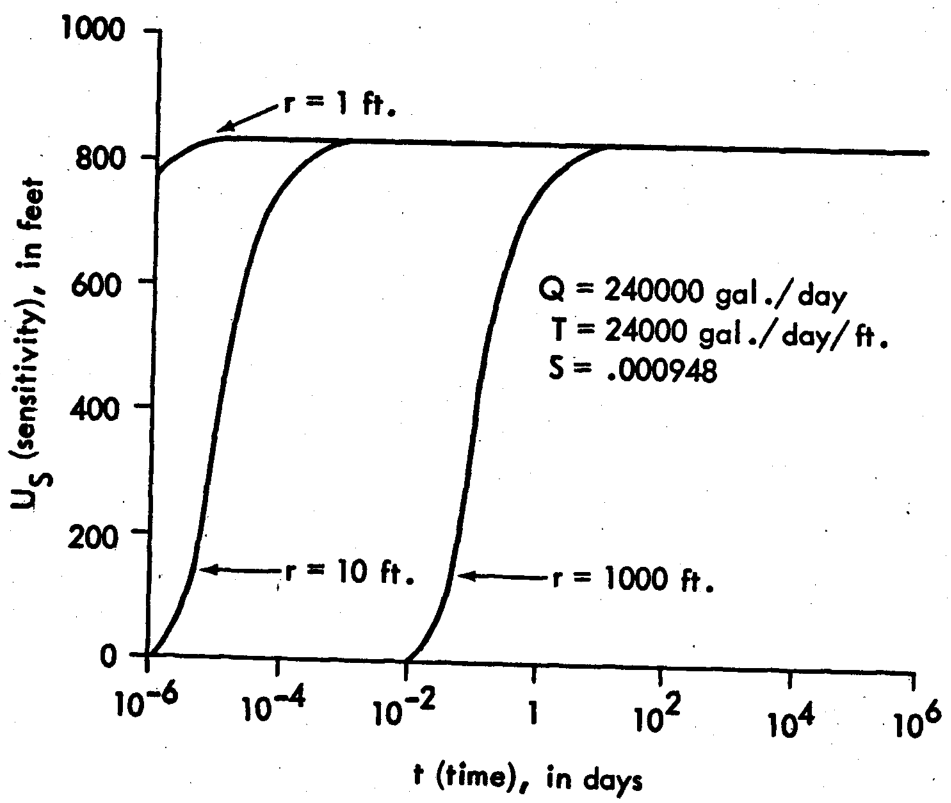

Figure 8. Time dependence of sensitivity U_S.

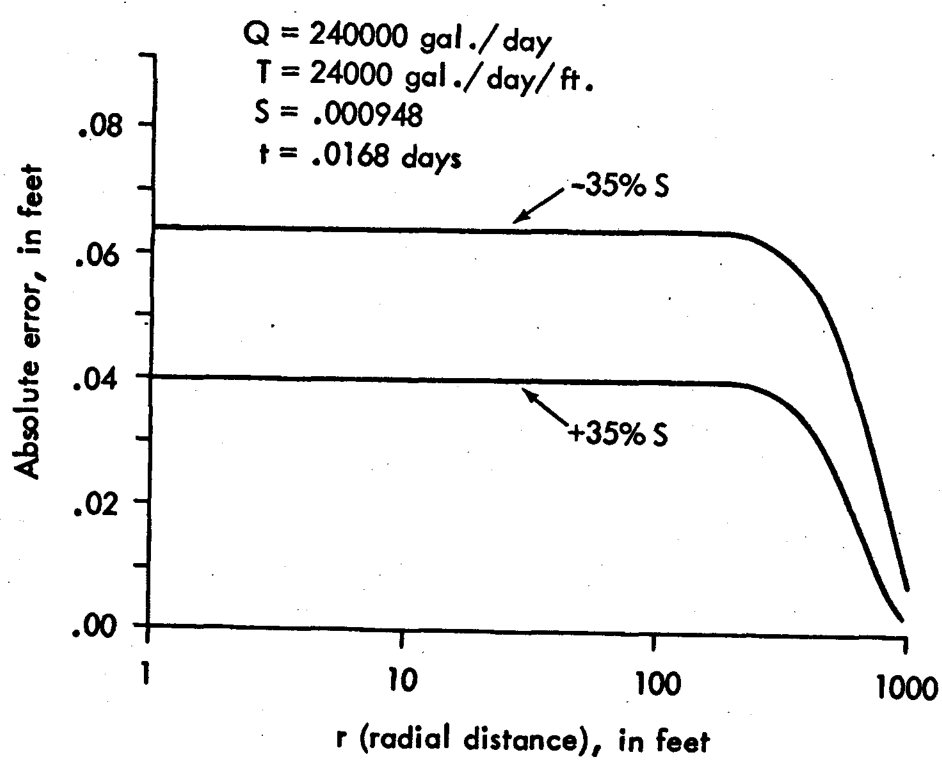

Figure 9. Radial dependence of error in using equation (14).

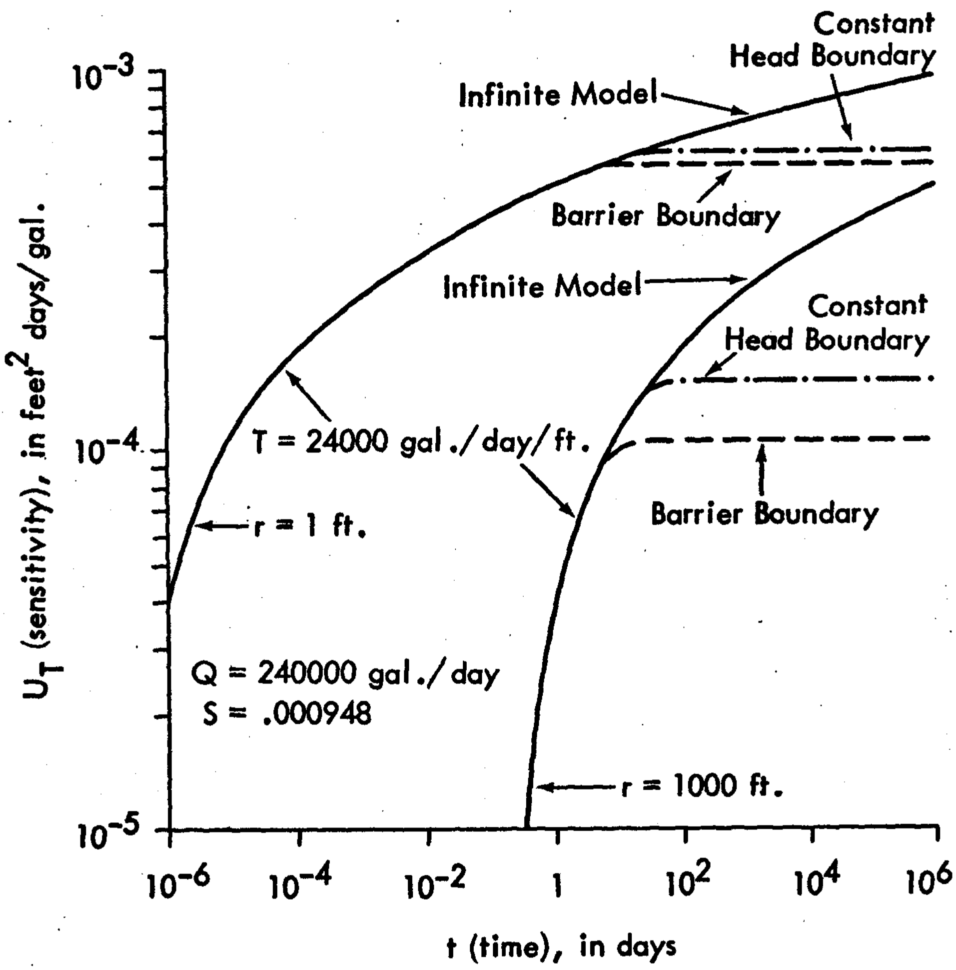

Figure 10. Effect of the boundary at 10,000 feet on U_T.

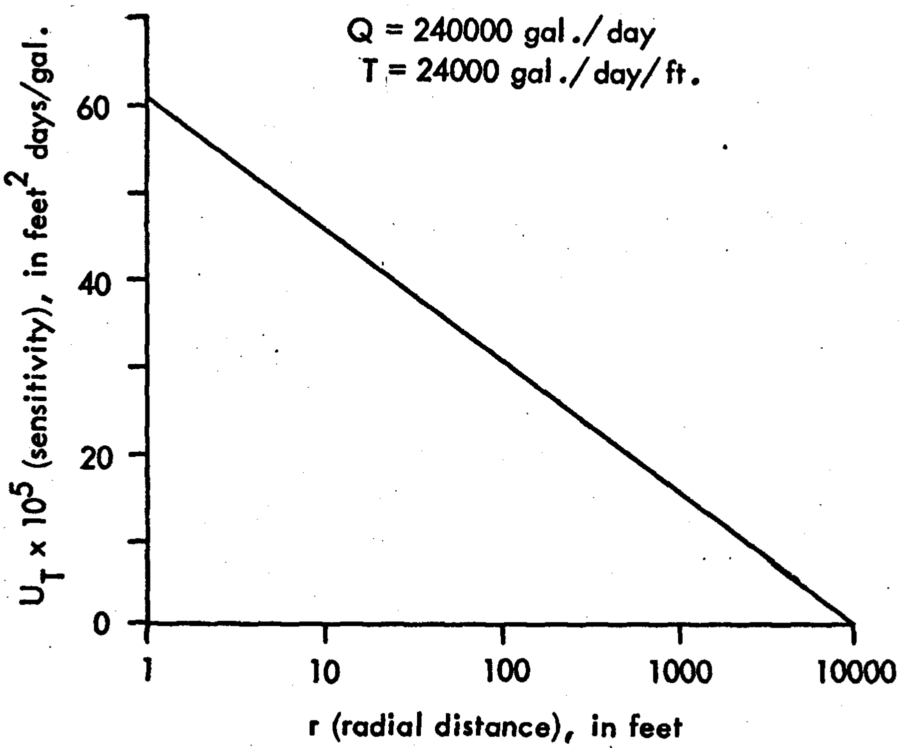

Figure 11. Radial dependence of U_T at steady state for a constant head boundary at 10,000 feet.

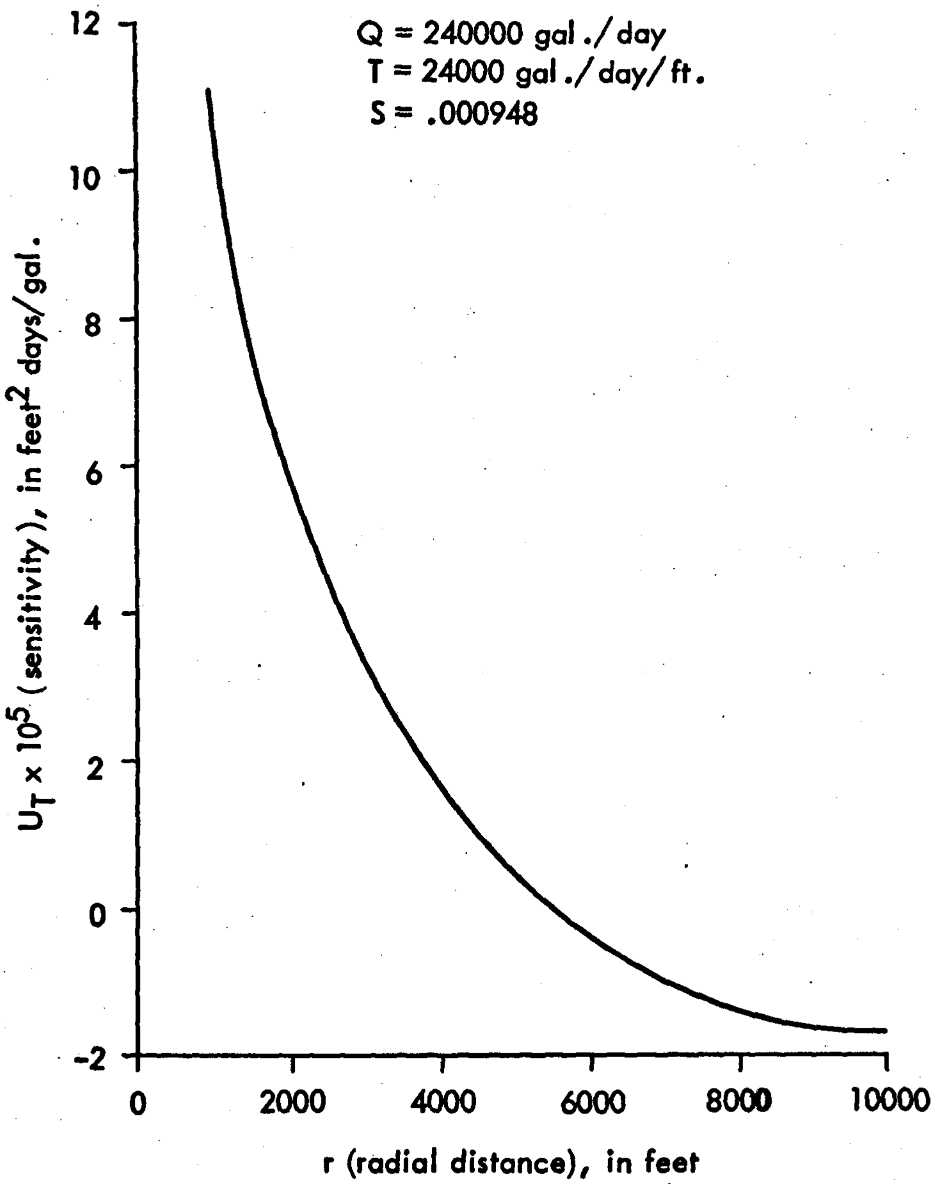

Figure 12. Radial dependence of U_T at large time for a barrier boundary at 10,000 feet.

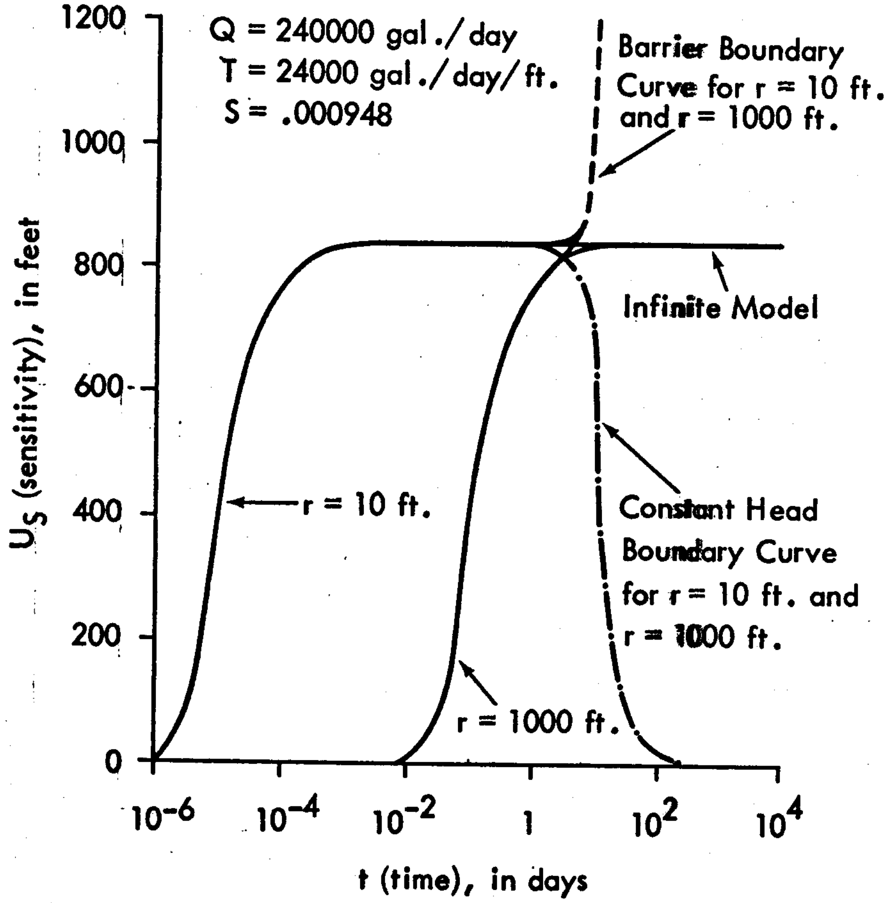

Figure 13. Effect of the boundary at 10,000 feet on the time dependence of U_S.

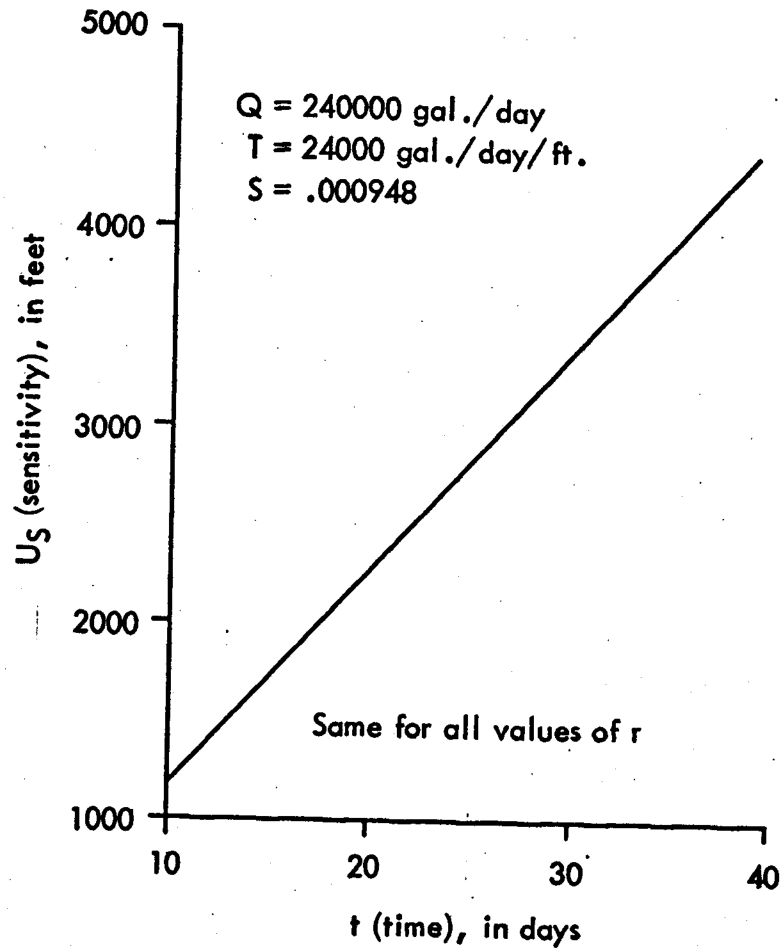

Figure 14. Barrier boundary at 10,000 feet, time dependence of sensitivity U_S.