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ANALYSIS OF ERROR IN GROUNDWATER MODELLING

by

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Submitted to the Department of Geology and the Faculty of the Graduate School of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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To my understanding wife Ayşe Hülya Yükler.

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ACKNOWLEDGEMENTS

The writer wishes to express his appreciation to Dr. John C. Halepaska who patiently taught the writer the essential concepts in hydrogeology and computer modelling of groundwater flow basins. The writer is also grateful to Dr. Carl D. McElwee for his great help in the preparation of this thesis. My special appreciation goes to Kansas Geological Survey and its director Dr. William W. Hambleton and the Chief of groundwater section Mr. Howard O'Connor.

The writer is grateful to Drs. Ernest E. Angino, Carl D. McElwee, Fawwaz S. Ulaby, John C. Halepaska, Louis F. Dellwig, and William M. Merrill for serving in the dissertation committee. I also thank Dr. William M. Merrill for his efforts in the correction of the thesis.

Thanks are also extended to Mrs. Carol Stanclift and Mrs. Kathe Rhodes of the U.S. Geological Survey for typing the manuscript.

INTRODUCTION

Theories and equations governing the flow of groundwater in porous media have been established and aquifer flow models with spatially distributed parameters are being employed to an increasing degree in the management of groundwater resources. The major difficulty in applying such models to real world situations results from the lack of sufficient field aquifer data and from the assumptions made in the derivation of the flow equations. Thus, the errors in the data and the assumptions made introduce uncertainties in the resulting models of the groundwater flow systems.

In this study an attempt is made to determine the range through which theories and equations believed to govern the flow of groundwater under different geologic conditions and fluid properties are valid. Also a direct method is developed using sensitivity analysis to determine the quantitative errors in groundwater modelling resulting from the incorrect estimation of aquifer parameters.

A computer model is prepared to describe the flow of groundwater in the west-central Kansas aquifer. Sensitivity equations derived in this study are applied to the aquifer. The quantitative errors in the aquifer parameters are determined from the sensitivity coefficients. Aquifer parameters are adjusted to make projections into the future with greater confidence.

REASONS FOR THIS STUDY

The use of computers and computer techniques in the management of groundwater resources began in the middle and late 1960's. Since then solutions of equations derived from well established theories of groundwater flow have been sought using computers. Numerical techniques were developed to express the equations in a format suitable for solution using the computers.

Mathematical formulation of the flow of groundwater requires that certain assumptions be made in order to fit the physical relationships into an equation. The expression of the relationships by mathematical equations, estimation of the aquifer parameters, and the approximation of the equations by their discrete analogs using numerical analysis methods are important sources of error. Lack of quantitative expressions for these errors introduce uncertainties in groundwater modelling. Thus, projections into the future cannot be made with absolute accuracy. Uncertainties recognized in the design of models of groundwater flow systems led several investigators to study the sensitivity of the equations of groundwater flow. Nevertheless, general methods to calculate the errors have not been developed.

This study is made to determine the above mentioned errors both qualitatively and quantitatively. Two major sources of error are considered in this work. First, the validity of the theories and of assumptions made in the derivation of the equations are evaluated; then the errors due to incorrect estimation of aquifer parameters are determined using sensitivity analysis.

The process of building a model is shown in Figure 1. Figure 2 shows how a working model is deduced from a real world situation. The "filtering processes" are taken as assumptions made to simplify the real world system. In this work the ranges of the validity of certain filtering processes is examined critically.

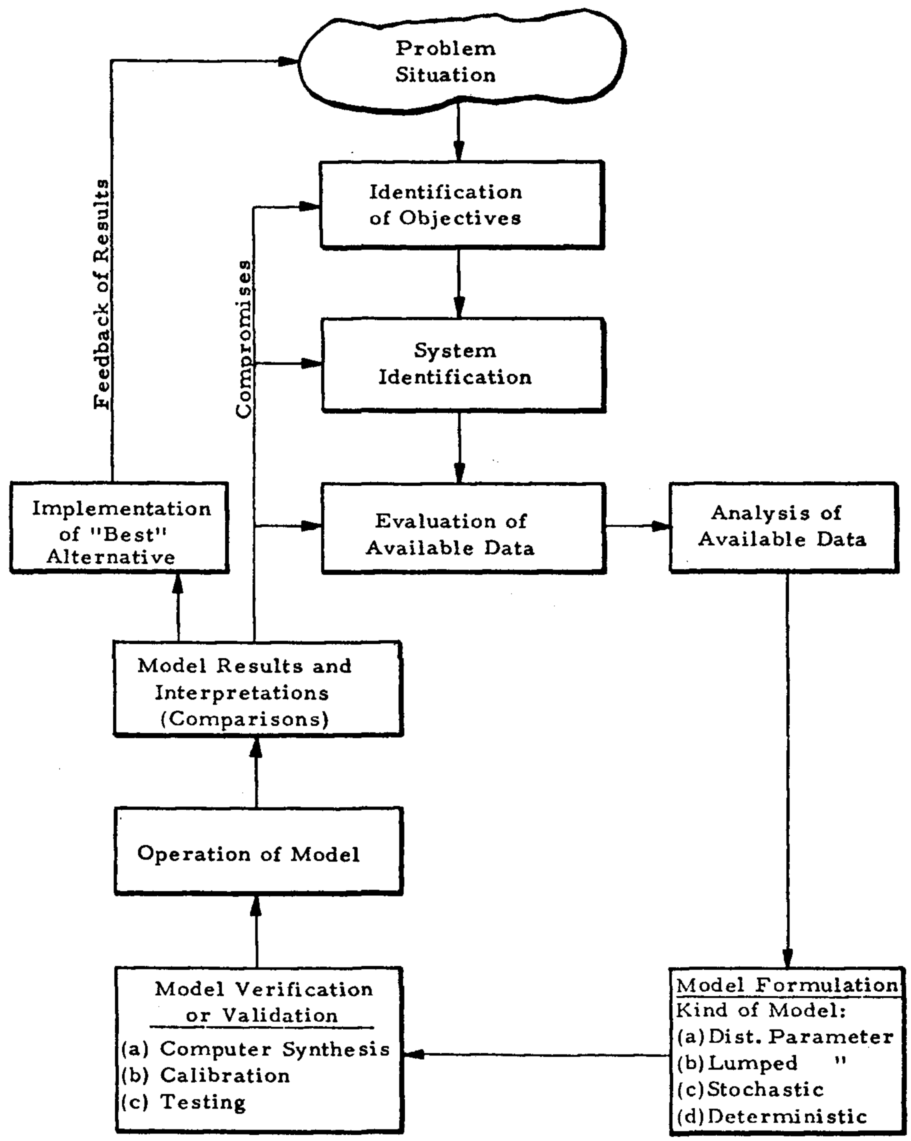


Figure I. Steps in the development and application of a simulation model(Riley,1970).

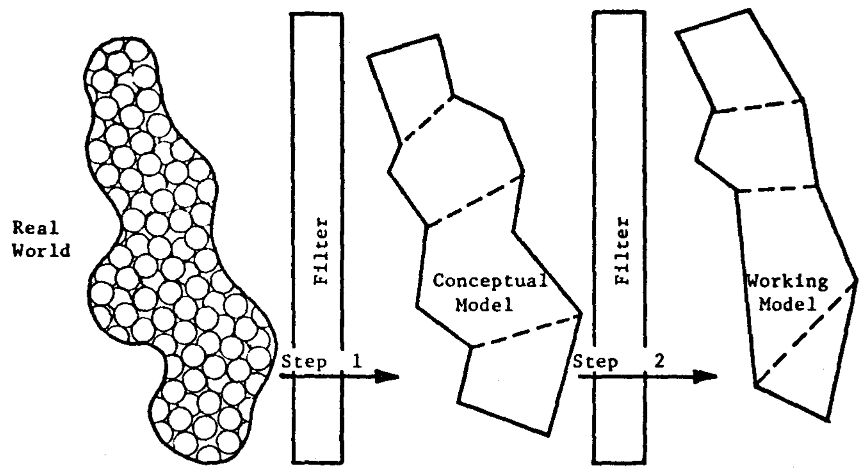


Figure 2. Steps in the development of a model of a real world system (Riley,1970).

GENERAL DISCUSSION OF GROUNDWATER FLOW

TERMS AND DEFINITIONS

Darcy's law -- It states that the velocity of moving groundwater is directly proportional to the head difference between two points and inversely proportional to the path length. In its simplest form:

$$\text{velocity } (v) = -K \, \partial h/\partial r$$

Hydraulic conductivity (K) -- It is a function of the specific weight, γ, of the fluid, of its dynamic viscosity, μ, and the average pore diameter, d, of the porous medium. It is expressed by the equation:

$$K = \frac{Cd^2\gamma}{\mu}$$

where C is a dimensionless constant which takes into account effects of stratification, packing, arrangement of grains, size distribution, and porosity.

Hydraulic head (h) -- It is the sum of the pressure head ($\frac{p}{\gamma}$) or the height of the column of water in the piezometer above its bottom and the elevation of the bottom of the piezometer (z) above an arbitrary datum of elevation (Hantush, 1964).

$$h = \frac{p}{\gamma} + z$$

Permeability (k) -- It is a measure of the capacity of a material to transmit a fluid through its interstices. It is independent of fluid properties and related to the average pore diameter. It is expressed as:

$$k = Cd^2$$

Porosity (θ) -- It is the ratio of pore volume to the bulk volume.

Specific yield (Sy) -- It is the ratio of the volume of water that a rock or soil will yield by gravity to its own volume. In other words, it represents the effective porosity.

Storativity (S) -- The ability of an aquifer to store water is expressed by a constant called its storativity. It is the volume of water stored under a unit surface area by a unit rise of head. It is also called storage coefficient.

Transmissivity (T) -- It is the ability of a uniformly thick bed or aquifer to transmit water. It is expressed as the product of hydraulic conductivity and the saturated thickness of the transmissive formation.

The symbols used in the derivation of the equation of flow of groundwater are given after the statement of the final equation (Appendix A).

PREVIOUS STUDY

Darcy (1856) was the first person to state clearly the mathematical law that governs the flow of groundwater through porous media. Dupuit (1863) developed the formula for the flow of water into a well. Forchheimer (1886) introduced the concept of equipotential surfaces and their relation to streamlines. In 1906, Thiem modified Dupuit's formula to compute the hydraulic characteristics of an aquifer by pumping a well and observing the effects in other wells in the vicinity. In 1935, Theis introduced an equation, based on an analogy with heat flow, for nonsteady state flow to a well. Jacob (1940), however, derived the same equation through hydraulic considerations alone. Then he presented an equation for three-dimensional flow of the type we are now considering (1950).

Flow problems in which the hydraulic head varies in a vertical direction, such as the flow in unconfined aquifers, leaky aquifers, or aquifers of nonuniform thickness received great attention after the derivation of the basic theory of groundwater flow. Hantush (1964), derived the flow equation for unconfined aquifers, including water-table movement. He defined an average head, which is the ratio of head integrated along the saturated thickness to saturated thickness, and obtained a simpler equation. Although he did not use the Dupuit-Forchheimer assumptions (Jacob, 1950), there is a great resemblance between his equation and the equations derived using Dupuit-Forchheimer assumptions. Hantush (1964) assumed that the storage coefficient could be neglected in unconfined aquifers and that the specific yield is independent of time. Boulton (1955) developed a theory of unsteady radial flow to a pumped well, allowing for delayed yield so that he could permit the specific yield to change with time. The studies of Neumann (1970, 1972, 1974) and Streltsova (1972, 1973) showed that the elastic properties of unconfined aquifers could be much more pronounced than those of deep-seated confined aquifers composed of similar materials. Streltsova (1973) found that it is the free surface and the flux that have an exponential form of change in time, and not specific yield. Therefore, specific yield can be taken to be a constant.

As computers became available, digital computer techniques were applied as a tool in hydrology. Numerical techniques for solving partial differential equations were developed (Smith, 1965; Zienkiewicz, 1967; Carnahan, et. al., 1969; Von Rosenberg, 1969).

The use of simulative flow models of groundwater basins became an established management tool in late 60's and early 70's. These models are "parametric" in that they contain unknown parameters which have to be estimated from observed data in some fashion.

Among the most important sources of uncertainity in model preparation is the lack of precise measurements of storativity and transmissivity. Methods for their estimation are given by Theis (1935), Wenzel (1942), Jacob (1940), and others. Their determination under different geologic conditions was studied by Horner, Gray, Russell, David, Hawking (in the presence of an impermeable boundary), Bixel, Larkin, Van Pollen, Weller (laterally varying hydraulic conductivity), Lefkovits, et al. (in layered reservoirs), Warren and Root, Pollard, Freedman and Natalson, and Samara (in naturally fractured formations) (Matthews and Russell, 1967).

Adjustment of these parameters is required until the model matches some measured history of hydraulic head change. Current methods of calibrating groundwater flow models are either direct or indirect. The indirect approach is essentially a trial and error procedure (Jacquard and Jain, 1965; Jahns, 1966; Vemuri and Karplus, 1969; Vemuri, et al., 1969; Coats, et al., 1970; Dupuy, et al., 1971; Slater and Durrer, 1971). The direct approach is to treat the model parameters as dependent variables in an inverse problem (Stallman, 1956; Nelson, 1968; Nelson and McCollum, 1969; Kleinecke, 1971; Emsellem and Marsily, 1971).

Lack of quantitative expressions of errors introduced into groundwater modelling due to simplification of the physical events and due to insufficient or erroneous data led several investigators to study the sensitivity of their mathematical models of groundwater flow (Vemuri, et al., 1969; Slater and Durrer, 1971; Lovell, et al., 1972; McCuen, 1973; Neumann, 1973). A simple and general method to accomplish this, however, has not been introduced.

HYDROLOGIC CYCLE AND GROUNDWATER MOVEMENT

The ever-changing distribution of water in the atmosphere, on the earth's surface, and in the subsurface reflects a complex dependent system called, collectively, the hydrologic cycle (Fig. 3). The parameters involved in the hydrologic cycle are: precipitation, surface runoff, storage of water below and above the surface layer, and evapotranspiration. There are also man made disturbances to the hydrologic cycle through the different land use patterns and industrialization.

In this study, movement of water in the saturated zone, or aquifer, below the land surface is analyzed. There are basically two kinds of aquifers: confined (artesian) and unconfined (water table) (Fig. 4). These two cases will be treated separately.

CONFINED AQUIFERS

Water in a confined aquifer is separated from the atmosphere by impermeable material. Confined water is also called artesian water. Most examples show a synclinal basin with water entering the system in an intake area above the level of ground-water discharge. In nature, however, the variety of conditions is endless. Thus, confined aquifers are aquifers in which the hydraulic head is greater than the bottom of the confining bed. A well drilled into a confined aquifer shows water levels above the confining layer.

If the porous medium were perfectly rigid and filled with a frictionless and incompressible fluid, the effects of well discharge would be observed as an instantaneous lowering of water levels in all the wells and

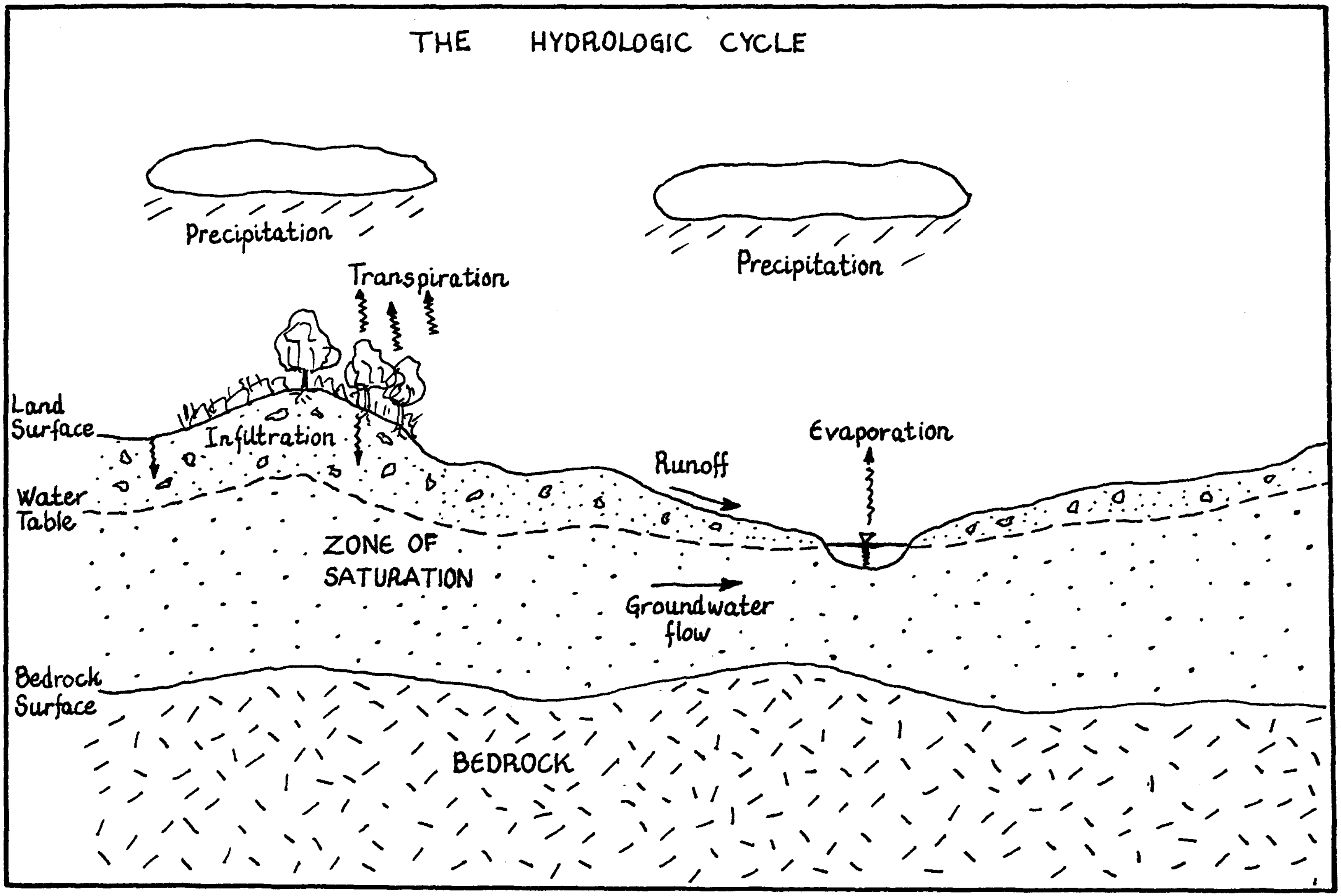


Figure 3. Illustration of simplified hydrologic cycle.

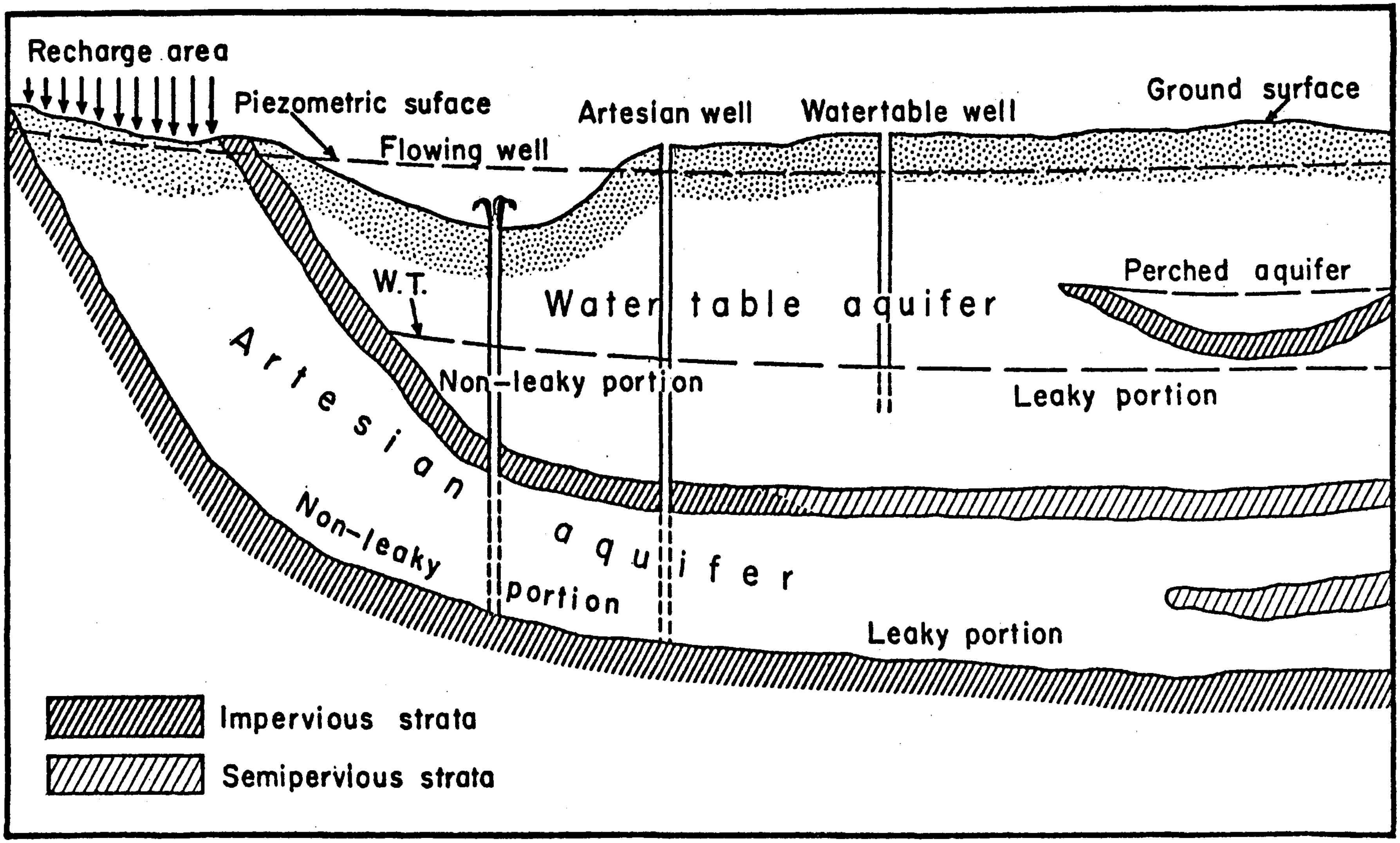


Figure 4. Illustration of different aquifers (Hantush,1964).

on the boundaries of the system. This is not observed in artesian systems. Meinzer (1928) was one of the first to note that water flowing into an area of withdrawal is much less than the water which is being pumped. Meinzer concluded that the water being pumped was from three sources: (1) water migrating into the aquifer, (2) water being forced out of the aquifer by compaction of the aquifer, and (3) expansion of water due to lowering of the pressure in the aquifer. Since a confining layer separates the aquifer from layers above, the weight of the overlying layers and the force of atmospheric pressure must be in equilibrium with the sum of stresses on the skeleton of the aquifer and the pressure of water filling the pores. Therefore, as water is pumped the pressure that the water exerts on the skeleton through the pores will decrease causing an increase in the stresses acting on the skeleton.

Water levels measured in wells are conveniently studied by means of maps and graphs. If full details of well construction and aquifer geometry are known, water-level contour maps can be classified more precisely as hydraulic head maps. Water migrates from areas of high hydraulic head to areas of low hydraulic head. These maps show the direction and rate of water migration qualitatively. In groundwater modelling the direction and the rate of water migration is sought quantitatively. Thus, the mathematical description of the above three sources must be obtained.

The velocity of groundwater is determined by Darcy's law. The rate of change of mass of water per unit volume can be found from conservation of mass. The difference between the flux of water leaving the system and coming into the system is equal to the compressibility of the aquifer and the water multiplied by the change in the hydraulic head with time. The flux of the water is equal to the velocity of the water times the area of the system perpendicular to the direction of flow. Replacing the velocity term by Darcy's law yields the groundwater flow equation for the confined case (Appendix A). This section is summarized from Davis and DeWiest's (1967) text book on hydrogeology.

UNCONFINED AQUIFERS

Aquifers in which water is in direct contact vertically with the atmosphere through open spaces in upper permeable material are called unconfined aquifers. The upper surface of the unconfined aquifer along which the hydrostatic pressure is equal to the atmospheric pressure is called the water table. Application of the conservation of mass principle to the unconfined case requires special attention to the change of the volume of the system due to water table movement caused by natural or artificial recharge and/or discharge. The appropriate derivation of the continuity equation is to integrate the whole equation from the base of the aquifer to the water table. However, the equation becomes difficult to solve. Dupuit and Forchheimer overcame this difficulty by assuming that the flow is horizontal and the drawdown is little compared to the thickness of the aquifer.

In unconfined flow a change of storage is due to raising or lowering of the water table, thus, filling or dewatering the voids above or below the preceding position. This volume of water is expressed as a fraction of the total volume through which the water table is displaced. It is called specific yield. Its complement is specific retention. Therefore, the sum of specific yield and specific retention gives the porosity of the medium. Even in an homogeneous aquifer there is some lag in the appearance of the water thus withdrawn from storage.

Storage in the unconfined aquifer due to compressibility of water and the aquifer skeleton is neglected by some hydrologists. Neumann (1972, 1974) and Streltsova's (1973) works demonstrate that this storage cannot always be neglected. Therefore, both the storage of the aquifer and the specific yield have to be considered.

There is an unsaturated zone above the water table in which voids are occupied by air and by water. The flow in this zone is two phase flow. Several investigators have analyzed the effect of this unsaturated zone. Neumann's and Streltsova's work also showed that this effect is not of major importance. They concluded that the unsaturated zone can be ignored by determining a constant specific yield.

EQUATIONS OF GROUNDWATER FLOW

In hydrodynamics, special attention is given to conservation of mass, energy, and momentum. However, conservation of energy and momentum are generally assumed to be negligible compared to conservation of mass in the derivation of equations for groundwater flow. Thus, a mathematical description of fluid flow in a porous medium is obtained from the following physical principles: (1) the law of conservation of mass, (2) Darcy's law.

The derivation of the following equations is described in detail in Appendix A. Here, they are written in their final form for both the confined and unconfined cases.

CONFINED FLOW SYSTEM

The groundwater flow equation in two dimensions is,

$$\frac{\partial}{\partial x} T \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} T \frac{\partial h}{\partial y} + Q = S \frac{\partial h}{\partial t} \tag{1}$$

where the parameters and the variables are as described under the heading "Terms and Definitions."

UNCONFINED FLOW SYSTEM

The equation is derived using Dupuit-Forchheimer assumptions,

$$\frac{\partial}{\partial x} Kh \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Kh \frac{\partial h}{\partial y} + Q = Sy \frac{\partial h}{\partial t} \tag{2}$$

ANOTHER APPROACH FOR UNCONFINED FLOW SYSTEM

Free surface or water table movement is defined in the x and y directions leading to a pseudo-three-dimensional flow equation,

$$\frac{\partial}{\partial x} Kh \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Kh \frac{\partial h}{\partial y} + K\left(\frac{\partial h}{\partial x}\right)^2 + K\left(\frac{\partial h}{\partial y}\right)^2 + Q = S \frac{\partial h}{\partial t} + Sy \frac{\partial h}{\partial t} \tag{3}$$

SOLUTION OF THE EQUATION OF FLOW OF GROUNDWATER

The partial differential equations derived are solved by the Crank-Nicolson scheme (Von Rosenberg, 1969). This scheme is chosen because it has been proved to be unconditionally stable and convergent (Von Rosenberg, 1969). Therefore, there is no restriction on the time step and on the space interval. Application of this scheme to a two-dimensional flow equation, however, yields five unknowns at every time step (Appendix B). There is no simple algorithm to solve these five unknowns. Iterative methods have been introduced to solve this kind of problems. In this study an extrapolated time routine introduced by Halepaska, et al. (1971), is used. Thus, the unknowns are reduced to three at every time step. This is a tri-diagonal matrix and is solved using Thomas' algorithm.

VALIDITY OF ASSUMPTIONS MADE IN THE DERIVATION OF GROUNDWATER FLOW EQUATIONS

Henry Darcy (1856) was the first person to state clearly the mathematical law that describes the flow of groundwater through porous media. Since then the problem of transient flow in a deformable medium, especially in confined flow cases, has been tackled by hydrologists. Theis (1935) pointed out the analogy between groundwater flow and heat flow, and mathematical formulation of the theory first was given by Jacob (1940). Jacob developed the classical two-dimensional parabolic equation of flow, taking into account the compressibility of both medium and water. Then Jacob (1950) extended his development to a three-dimensional basin. In all these derivations the assumptions made can be stated as follows:

a. Darcy's flow law is valid
b. Storativity is independent of space and time
c. Transmissivity is dependent on space, but independent of time
d. The flow can be described perfectly in the three orthogonal directions, or x, y, and z axes.

VALIDITY OF DARCY'S LAW

In deriving Darcy's law, the flow is assumed to be laminar; that is, the velocity of flow is proportional to the first power of the hydraulic gradient. This assumption has long been verified by observation (Jacob, 1950; Todd, 1959; Hantush, 1964). The usual index used to determine the tendency of flow to be laminar is the Reynolds number, defined as follows:

$$R = \frac{vd}{\nu}$$

where v is the velocity of flow or bulk velocity, d is the mean grain diameter, and ν is the kinematic viscosity of the fluid. (The kinematic viscosity is equal to μ/ρ, μ being the dynamic viscosity and ρ the density). Experiments show that a departure from laminar flow begins at values of R between 1 and 10, depending upon the range of grain sizes and shapes (Jacob, 1950). Fortunately, almost all natural ground-water motion has Reynolds numbers less than unity, and thus Darcy's law is applicable (Hantush, 1964). In rock aquifers, in unconsolidated aquifers with steep hydraulic gradients, in aquifers containing large-diameter solution channels, and in the immediate vicinity of open bodies of water, such as streams and wells where steep gradients are seen, velocities may be relatively high and flow may be expected to deviate from Darcy's law.

Jacob, 1940, suggested that one way of accounting for turbulence in the immediate vicinity of a well is to find a valid basis for estimating the exponent of the well-loss term, LQ^n. L is a constant determined empirically and Q is the discharge. Since drawdown at the well is defined as the sum of the formation-loss (i.e., the loss of head in the sand from the outer boundary to the face of the well) and the well-loss, a better estimate of drawdown can be achieved with a good estimate of exponent n.

In materials composed of extremely fine grains, such as colloidal clays (in which the pores may be reduced to a few molecular diameters), or sands that are not completely saturated with water, the law of flow is reported to deviate from Darcy's law (Elnagger, et al., 1971). Various equations have been proposed to express the relationship between the velocity and the hydraulic gradient, especially in clay soils, but none have been accepted.

As a result we can say that the work on the validity of Darcy's law is still incomplete and requires a better understanding of transitional and turbulent flow. Once a sufficient understanding is reached, mathematical expressions of the flow when Darcy's law is not valid can be written. Darcy's law is assumed valid in this study since neither serious limitations on its applicability in groundwater flow systems have not yet been stated nor has a better mathematical expression of flow, in systems where Darcy's law is suspected to be invalid, been introduced.

DEPENDENCE OF STORATIVITY WITH RESPECT TO SPACE AND TIME

Mathematical formulation of groundwater flow systems was first provided by Jacob (1940). He developed, for a homogeneous and isotropic horizontal formation, the classical two-dimensional parabolic equation, taking into account the compressibility of both medium and water. Later he extended his development to a three-dimensional basin (Jacob; 1950). The development of Jacob's equation was based on the assumption that horizontal deformation is absent, so in effect this approach couples a two- or three-dimensional hydrologic model to a one-dimensional deformation model.

If flow is unconfined, compressibility of sand and water are relatively unimportant compared to changes of water volume that accompany vertical displacements of the water-table under unsteady conditions (Jacob, 1950; Hantush, 1964). In the confined case, however, the compressibility of sand and water is important because the upper confining layer exerts an overburden pressure in the system. With the assumption that horizontal deformations are negligible, the vertical component of compressive stress, σ_z, and the water pressure, p, are in equilibrium with the overburden load. This leads to the definition of storage coefficient as follows:

Storage coefficient (S) = $\gamma(\theta\beta + \alpha)$

where θ is the porosity

α is the compressibility of the solid volume

β is the compressibility of water

Biot (1941, 1955) linked a three-dimensional deformation model with a three-dimensional fluid flow model resulting in four coupled equations known as the Biot system. However, this system is considered to be impractical due to its complexity (Gambolati and Freeze, 1973). Cooper (1966) considered the grain velocity and in a way, expressed Darcy's law in terms of the relative velocity of the fluid relative to the grains, thus allowing for deformation of the system. Gambolati and Freeze (1973) and later Gambolati (1973) developed a different mathematical formulation of the flow equation, taking grain velocity into account. The assumptions made in this derivation are:

1. Compressibility of the solid volume is a function of pressure

   $\alpha = \alpha(p)$

2. Specific weight of water (hence the density) is a function of pressure and can be expressed as:

   $\gamma = \gamma_0 \exp[\beta(p_0 + p)]$

3. Classical compressibility α is replaced by the compressibility defined in the laboratory, c_b, where c_b is defined as:

   $c_b = \left(p \frac{d\alpha}{dp} + \alpha\right) \Big/ \left(1 + \alpha p\right)$.

Gambolati (1973) concluded that the diffusion equation is good when the formation compacts less than 5% of the initial thickness. His findings support Smith's (1971) statement that, in cases where the grain velocity exceeds water velocity, that is, where the rate of consolidation exceeds the rate of movement of the percolating fluid, the diffusion equation is not valid. Under these conditions subsidence will occur. We can also add that Gambolati assumed an instantaneous boundary pressure variation which causes great compressive stress on the system. In nature, however, pressure variation occurs as a continuous function through a relatively long time interval. Therefore, it may be possible to assume less severe limitations on compaction and maintain the validity of the diffusion equation.

Effects of variations in the storativity of groundwater flow systems are investigated and the errors introduced in the systems are determined quantitatively with the help of sensitivity analysis on subsequent pages.

DEPENDENCE OF TRANSMISSIVITY WITH RESPECT TO TIME

The product of the thickness of the aquifer, b, and the average value of the hydraulic conductivity in a vertical section of the aquifer defines the transmissivity of the aquifer, i.e., the ability of the aquifer to transmit water. In an unconfined aquifer the thickness of the aquifer varies with time as the water-table moves (the water-table is the upper bound of the aquifer). Transmissivity is a function of space and time in unconfined flow. However, transmissivity is assumed to be only a function of space and not of time in confined flow equations.

Hydraulic conductivity, K, depends on the specific weight, γ, of the fluid, on its dynamic viscosity, μ, and on the permeability, k, of the porous medium. The dependence of K on γ, μ, and k makes it dependent upon time, temperature, and direction. Variation in hydraulic conductivity is a common occurrence in unconsolidated sedimentary deposits where the horizontal conductivity exceeds the vertical conductivity 2 to 10 times or more (Hantush, 1964). Under these conditions Darcy's law can be stated for the three directions (x,y,z) as follows (Collins, 1961):

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = - \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial z} \end{bmatrix}$$

Here the nine quantities, $K_{i,j}$, (i = x,y,z, j = x,y,z) form the elements of a tensor. The K-matrix is found to be symmetric (Collins, 1961). If the axes x,y,z are chosen such that they correspond to the principal axes of the porous medium, off-diagonal terms become zero. Then we have only the terms on the diagonal. In the derivation of equations this assumption is made.

Parswell (1967) studied the thermal influence on flow from a compressible porous medium. He concluded that over the range from 0 to 100°C change in density can generally be neglected, whereas changes in viscosity are of importance. As the temperature increases viscosity decreases and hydraulic conductivity increases. This leads to increased rate of flow, causing greater increments of stress to be transferred to the matrix per unit time. He plotted volume change vs temperature rise in the system. It can be seen that for low and medium temperature rises (1% and 2% variations when initial temperature is 25°C) the change in volume is less than 0.3% and 1%, respectively. Since more rapid rises are not expected in a very short time with natural conditions, mainly climatic effects, this much change can be ignored over a large area.

If we write the flow equation as follows;

$$-\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right] = \rho\ S_s\ \frac{\partial h}{\partial t}$$

where S_s is the specific storage of the aquifer,
and expand the bracketed terms and put the resulting expressions of the velocity component and the density gradients in terms of hydraulic gradient (Jacob, 1950), we obtain

$$\rho\left[\frac{\partial}{\partial x} K \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} K \frac{\partial h}{\partial y} + \frac{\partial}{\partial z} K \frac{\partial h}{\partial z}\right] + K\rho_o^2\beta g\left[\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2 - \left(\frac{\partial h}{\partial z}\right)\right] = \rho\ S_s\ \frac{\partial h}{\partial t}$$

If we look at the problem in two-dimensions,

$$\rho\left[\frac{\partial}{\partial x} K \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} K \frac{\partial h}{\partial y}\right] + K\rho_o^2\beta g\left[\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right] = \rho\ S_s\ \frac{\partial h}{\partial t}$$

If both sides are multiplied by the saturated thickness, b, and divided by ρ,

$$\left[\frac{\partial}{\partial x} Kb \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Kb \frac{\partial h}{\partial y}\right] + \frac{Kb\rho_o^2\beta g}{\rho}\left[\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right] = S\ \frac{\partial h}{\partial t}$$

can be obtained. Jacob states that the second bracketed term is negligible for systems with small drawdown. The magnitude of the terms $[(\partial h/\partial x)^2 + (\partial h/\partial y)^2]$ are determined from the equation derived for the unconfined case (equation 3), and it is found that they are typically very small.

APPROXIMATIONS FOR THE WATER TABLE MOVEMENT

Two basically different derivations of equations governing the flow of water in an unconfined aquifer case have been encountered in the literature.

1. Hantush's derivation by integrating the velocity between the bottom and top of the saturated zone.
2. Derivations using the Dupuit-Forchheimer assumptions.

The major difficulty is to account for the water-table movement. Hantush (1964) included the water table movement in his derivation by integrating the continuity equation between the base of the aquifer and the free surface; thus, he defined an average head which is the ratio of head integrated along the saturated thickness to saturated thickness. Although there is no theoretical basis for this definition of an average head value, he assumed it is justified by the observed water levels in wells which are also average head values.

On the other hand, Dupuit-Forchheimer assumptions (Jacob, 1950) assume horizontal and uniform flow over the depth of saturation. Very little or no movement of the free surface is allowed in the vertical direction. This forces the system to approach the confined case. The derivations with these assumptions in two dimensions ignore the flux due to free surface movement.

Therefore, Hantush's derivation averages the effect of the free surface movement over the aquifer, while Dupuit-Forchheimer assumptions completely ignore the free surface movement in the vertical direction in the continuity equation.

In this study an equation is derived which is a better approximation for the water-table movement (Appendix A). The free surface or water table movement is defined in the x and y directions leading to a pseudo-three-dimensional flow equation, equation (3). The water-table configuration is determined both from equation (2), derived using Dupuit-Forchheimer assumptions, and from equation (3). Then the results are compared to find the magnitude of the errors due to negligence of the free surface movement.

A square unconfined aquifer is considered with sides of 20,000 feet. A discharging well is located at the center and the initial water-table surface is 150 feet above the base of the aquifer. The change in the water table with time in this aquifer is studied under two different sets of aquifer parameters.

i. Hydraulic conductivity (K_1) = 355 gal/day-ft^2
   specific yield (Sy_1) = 0.24

ii. Hydraulic conductivity (K_2) = 480 gal/day-ft^2
   Specific yield (Sy_2) = 0.15

The well discharge is 240,000 gal/day and no flow is permitted across the boundaries.

The water levels computed from equation (2) and equation (3) are given in table 1. The water table determined by equation (3) is higher at the early times. Since equation (3) includes the water-table movement, the volume of water drawn by pumpage is supplied by not only the decline in hydraulic head values but also the lowering of the free surface. Thus, less drawdown or higher water table is computed by equation (3). As the time increases the cone of depression reaches the boundary earlier in the case where free surface movement is ignored and the whole system begins to respond to the pumpage. However, the cone of depression has not reached the boundary in the

	i. K_1=355.2 gal./day-ft^2 Sy_1=0.24				ii. K_2=480.0 gal./day-ft^2 Sy_2=0.15			
Time (in hours)	Head (h) From Eq.2	Head (h*) From Eq.3	Error (in ft.) h*-h	% Error (h*-h)/s	Head (h) From Eq.2	Head (h*) From Eq.3	Error (in ft.) h*-h	% Error (h*-h)/s
1.0	149.9592	149.9592	0.0000	0.00	149.9363	149.9363	0.0000	0.00
3.0	149.8820	149.8820	0.0000	0.00	149.8225	149.8225	0.0000	0.00
7.0	149.7429	149.7430	0.0001	0.04	149.6387	149.6388	0.0001	0.03
15.0	149.5146	149.5148	0.0002	0.04	149.3877	149.3881	0.0004	0.07
31.0	149.1943	149.1951	0.0008	0.10	149.1172	149.1186	0.0014	0.16
63.0	148.8362	148.8388	0.0026	0.20	148.8836	148.8868	0.0032	0.29
127.0	148.5177	148.5235	0.0058	0.39	148.6833	148.6861	0.0028	0.21
255.0	148.2441	148.2516	0.0075	0.43	148.4902	148.4925	0.0023	0.15
511.0	147.9820	147.9900	0.0080	0.40	148.3066	148.2990	-0.0076	-0.45
1023.0	147.7313	147.7293	-0.0020	-0.09	148.1343	148.0865	-0.0478	-2.56
2047.0	147.4959	147.4456	-0.0503	-2.01	147.9669	147.8318	-0.1351	-6.65
4095.0	147.2689	147.1095	-0.1594	-5.84	147.7760	147.7339	-0.0421	-1.89
8191.0	147.0150	146.9757	-0.0393	-1.32	147.5905	147.5551	-0.0354	-1.47

h;head or water level values from eq.2

h*;head or water level values from eq.3

Table I. Comparison of the head values,at the well, with different sets of aquifer parameters.

case where free surface movement is considered and higher drawdown values are obtained at the well. Then the difference between the head values of the former and latter cases is negative (table 1). This result indicates that the sign of the error due to negligence of the free surface changes when the boundary effects become important in the system.

The error in water levels is found by subtracting the water level values computed by equation (3) from water levels determined from equation (2). The error in feet is found to be less than 0.16 feet in the system with aquifer parameters (i) and less than 0.14 feet in the system with aquifer parameters (ii) (table 1). The percentage error is computed by dividing the error in water levels by the drawdown for the Dupuit-Forchheimer approximations. The percentage errors are higher in the system with aquifer parameters (i) at the early times compared to system with aquifer parameters (ii), with maximum percentage errors 0.43% and 0.29%, respectively. However, the percentage errors are higher in the latter case after the boundary effects become important in the system. Nevertheless, extra errors are introduced into the system when boundary effects become important as will be seen in the later chapters. If the performance of the system is studied up to the time when the boundary effects become important, we can conclude that the errors due to negligence of the water-table movement are very small.

SENSITIVITY ANALYSIS OF GROUNDWATER FLOW SYSTEMS

DEFINITION OF SENSITIVITY ANALYSIS

Sensitivity analysis is a study of the sensitivity of a system's response to various disturbances in the system. These disturbances may have a widely different character: they may be small or large, momentary or permanent; they may be related to initial conditions, to coefficients, etc. In this study disturbances related to aquifer parameters are considered. This is called parametric sensitivity which is a measure of the change in output resulting from a disturbance in an aquifer parameter.

Deriving the system's sensitivity to a single parameter has traditionally involved incrementing the parameter by a small amount and determining the resulting change in output. This is commonly known as the perturbation method. However, the determination of the sensitivity to each parameter of a complex system by perturbation methods involves an excessive amount of time. Furthermore, most complex simulation models are expressed in finite difference form and it is impossible to derive a closed form sensitivity function (McCuen, 1973). As a result of these computational difficulties parametric sensitivity has not been widely used with complex systems.

Direct, closed forms of sensitivity functions are obtained in this study by taking the partial derivative of the flow equations with respect to a particular parameter. Then these sensitivity equations are solved in the computer and parametric sensitivity coefficients are determined. This method is a simple and direct method for determining a system's sensitivity. Furthermore, this method is faster and more accurate than the perturbation techniques.

THE SENSITIVITY EQUATION AND ITS IMPLICATIONS

In the mathematical treatment of dynamic systems it is permissible to speak of the precise value of the physical parameters. However, the investigator's problem is to establish tolerances within which the values of the components of the physical system may vary without appreciably affecting the model results. This is basically a cost and engineering problem.

Basic equations and corresponding numerical forms used in this study are subjected to variations in aquifer parameters to quantitatively assess the degree of the system's sensitivity with respect to these variations. The effect of variations in aquifer parameters, transmissivity (T) and storage coefficient (S), under different conditions is analyzed.

In studying the sensitivity of the groundwater flow system to parameter variations, the following mathematical model is used;

$$F(h_{xx}, h_{yy}, h_t;\ T, S, Q) = 0 \qquad (4)$$

where,

$$h_{xx} = \frac{\partial^2 h}{\partial x^2},\ h_{yy} = \frac{\partial^2 h}{\partial y^2},\ h_t = \frac{\partial h}{\partial t}$$

We shall write the solution of equation (4) in the form $h=h\ (x,y,t;\ T,S,Q)$. If, for simplicity, only one parameter, T, is allowed to vary by an amount, ΔT, the equation becomes

$$F(h^*_{xx},\ h^*_{yy},\ h^*_t;\ T+\Delta T, S, Q) = 0 \qquad (5)$$

We shall write the solution of equation (5) in the form $h^* = h\ (x,y,t;\ T + \Delta T,S,Q)$. Comparing the solutions of the equations (4) and (5), we immediately obtain an indication of the stability of the system, which we express by means of the fraction

$$\Delta u = \frac{h^*(x,y,t;\ T+\Delta T,S,Q) - h(x,y,t;\ T,S,Q)}{\Delta T}$$

$$= \frac{h^*-h}{\Delta T} \qquad (6)$$

If expression (6) has a limiting value as ΔT approaches zero we get

$$\lim_{\Delta T \to 0} \frac{h^*-h}{\Delta T} = \frac{\partial h}{\partial t} = u(x,y,t;\ T,S,Q) \qquad (7)$$

We shall call the function $u(x,y,t;\ T,S,Q)$ the sensitivity coefficient, for variations in T values, of the groundwater flow system.

Using the above procedure we can derive the sensitivity equation of the groundwater flow system. The sensitivity equation will be derived for two essential cases: the confined flow system and the unconfined flow system.

SENSITIVITY ANALYSIS OF CONFINED FLOW SYSTEMS

SENSITIVITY WITH RESPECT TO TRANSMISSIVITY VARIATIONS

The equation that describes the confined groundwater flow system is;

$$\frac{\partial}{\partial x} T \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} T \frac{\partial h}{\partial y} + Q = S \frac{\partial h}{\partial t} \qquad (8)$$

where T = T(x,y).

The sensitivity equation can be found on the basis of definition (7). Thus, the sensitivity equation is obtained from the partial derivative of equation (8) with respect to T.

$$\frac{\partial}{\partial T}\left[\frac{\partial}{\partial x} T \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} T \frac{\partial h}{\partial y} + Q\right] = \frac{\partial}{\partial T}\left[S \frac{\partial h}{\partial t}\right] \qquad (9)$$

The aquifer parameters, T and S, and the discharge or the recharge terms are assumed independent of each other, i.e.,

$$\frac{\partial S}{\partial T} = 0,\ \frac{\partial Q}{\partial T} = 0,\ \frac{\partial T}{\partial S} = 0,\ \frac{\partial Q}{\partial S} = 0 \qquad (10)$$

Now we can expand equation (9)

$$\frac{\partial}{\partial T}\left[\frac{\partial T}{\partial x} \cdot \frac{\partial h}{\partial x} + T \frac{\partial^2 h}{\partial x^2} + \frac{\partial T}{\partial y} \cdot \frac{\partial h}{\partial y} + T \frac{\partial^2 h}{\partial y^2} + Q\right] = \frac{\partial}{\partial T}\left[S \frac{\partial h}{\partial t}\right]$$

and take partial derivatives with respect to T.

$$\frac{\partial^2 T}{\partial T \partial x} \cdot \frac{\partial h}{\partial x} + \frac{\partial T}{\partial x} \cdot \frac{\partial^2 h}{\partial T \partial x} + \frac{\partial^2 h}{\partial x^2} + T \frac{\partial^3 h}{\partial T \partial x^2} + \frac{\partial^2 T}{\partial T \partial y} \cdot \frac{\partial h}{\partial y} + \frac{\partial T}{\partial y} \cdot \frac{\partial^2 h}{\partial T \partial y} +$$

$$\frac{\partial^2 h}{\partial y^2} + T \frac{\partial^3 h}{\partial T \partial y^2} = S \frac{\partial^2 h}{\partial T \partial t} \qquad (11)$$

If we interchange the order of derivatives in equation (11)

$$\frac{\partial}{\partial x}\left[\frac{\partial T}{\partial T}\right] \cdot \frac{\partial h}{\partial x} + \frac{\partial T}{\partial x} \cdot \left[\frac{\partial}{\partial x} \frac{\partial h}{\partial T}\right] + \frac{\partial^2 h}{\partial x^2} + T\left[\frac{\partial^2}{\partial x^2} \frac{\partial h}{\partial T}\right] + \frac{\partial}{\partial y}\left[\frac{\partial T}{\partial T}\right] \cdot \frac{\partial h}{\partial y} +$$

$$\frac{\partial T}{\partial y} \cdot \left[\frac{\partial}{\partial y} \frac{\partial h}{\partial T}\right] + \frac{\partial^2 h}{\partial y^2} + T\left[\frac{\partial^2}{\partial y^2} \frac{\partial h}{\partial T}\right] = S\left[\frac{\partial}{\partial t} \frac{\partial h}{\partial T}\right] \qquad (12)$$

remembering that

$$\frac{\partial h}{\partial T} = u$$

the sensitivity coefficient given in expression (7), equation (12) can be written as

$$\frac{\partial T}{\partial x} \cdot \frac{\partial u}{\partial x} + T\,\frac{\partial^2 u}{\partial x^2} + \frac{\partial T}{\partial y} \cdot \frac{\partial u}{\partial y} + T\,\frac{\partial^2 u}{\partial y^2} = S\,\frac{\partial u}{\partial t} - \left[\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right]$$

or

$$\frac{\partial}{\partial x}\,T\,\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\,T\,\frac{\partial u}{\partial y} + \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = S\,\frac{\partial u}{\partial t} \qquad (13)$$

This is the sensitivity equation. We obtain the sensitivity of the confined flow system to variations in transmissivity values from the solution of equation (13). The sensitivity coefficient u is a function of space and time. The solution of equation (13) will therefore show the system's sensitivity spatially at any time. The general assumption which we made in deriving equation (13) was that the solution of equation (4) is analytically dependent on the parameters (Tomovic', 1963).

SOLUTION OF THE SENSITIVITY EQUATION

The solution of the sensitivity equation is obtained by finite-difference techniques. The application of the Crank Nicolson scheme to equation (13) gives (see Appendix E),

$$\frac{1}{\Delta x_i}\left[T_{i+\frac{1}{2},j}\,\frac{u^{n+\frac{1}{2}}_{i+1,j} - u^{n+\frac{1}{2}}_{i,j}}{\Delta x_{i+\frac{1}{2}}} - T_{i-\frac{1}{2},j}\,\frac{u^{n+\frac{1}{2}}_{i,j} - u^{n+\frac{1}{2}}_{i-1,j}}{\Delta x_{i-\frac{1}{2}}}\right] +$$

$$\frac{1}{\Delta y_j}\left[T_{i,j+\frac{1}{2}}\,\frac{u^{n+\frac{1}{2}}_{i,j+1} - u^{n+\frac{1}{2}}_{i,j}}{\Delta y_{j+\frac{1}{2}}} - T_{i,j-\frac{1}{2}}\,\frac{u^{n+\frac{1}{2}}_{i,j} - u^{n+\frac{1}{2}}_{i,j-1}}{\Delta y_{j-\frac{1}{2}}}\right] +$$

$$\frac{1}{\Delta x_i}\left[\frac{h^{n+\frac{1}{2}}_{i+1,j} - h^{n+\frac{1}{2}}_{i,j}}{\Delta x_{i+\frac{1}{2}}} - \frac{h^{n+\frac{1}{2}}_{i,j} - h^{n+\frac{1}{2}}_{i-1,j}}{\Delta x_{i-\frac{1}{2}}}\right] +$$

$$\frac{1}{\Delta y_j}\left[\frac{h^{n+\frac{1}{2}}_{i,j+1} - h^{n+\frac{1}{2}}_{i,j}}{\Delta y_{j+\frac{1}{2}}} - \frac{h^{n+\frac{1}{2}}_{i,j} - h^{n+\frac{1}{2}}_{i,j-1}}{\Delta y_{j-\frac{1}{2}}}\right] = S_{i,j}\,\frac{u^{n+1}_{i,j} - u^{n}_{i,j}}{\Delta t} \qquad (14)$$

The sensitivity coefficients, u, are obtained by simultaneous numerical solution of the groundwater flow equation (8) and the sensitivity equation (13).

ANALYSIS OF THE SENSITIVITY COEFFICIENTS

The sensitivity equation (13) is solved for two different cases. The first case is an aquifer with no flow boundaries, and the second case is an aquifer with constant head boundaries. The mathematical formulation of the problem is as follows.

STEP A.

The solution of equation (8) is found for the given initial and boundary conditions. We will express this solution by $h(x,y,t;\ T,S,Q)$.

STEP B.

The transmissivity values are perturbed by some amount, ΔT, then equation (8) is solved for these perturbed transmissivity values. We shall express this solution by $h^*(x,y,t;\ T+\Delta T,S,Q)$.

STEP C.

The sensitivity coefficients of the original case are found by solving the sensitivity equation (13) and using the results of the flow equation (8) for unperturbed T values.

STEP D.

The difference between $h(x,y,t;\ T,S,Q)$ and $h^*(x,y,t;\ T+\Delta T,S,Q)$ is determined. This is the error in head due to a disturbance, ΔT, in the T values. Since we have assumed that the solutions depend analytically on the parameters, and that ΔT is small, the function $h^*(x,y,t;\ T+\Delta T, S,Q)$ can be expanded into a Taylor series (Tomovic', 1963).

If the second and higher terms are neglected,

$$h^* (x,y,t; T\pm\Delta T,S,Q) = h (x,y,t; T,S,Q) \pm \Delta T \frac{\partial h}{\partial T} \qquad (15)$$

is obtained. The error in head can be expressed as,

$$\Delta h = h^* (x,y,t; T\pm\Delta T,S,Q) - h (x,y,t; T,S,Q)$$

$$= h^* - h \qquad (16)$$

Now, equation (15) becomes

$$\Delta h = \pm\Delta T \frac{\partial h}{\partial T} = \pm\Delta Tu \qquad (17)$$

We can easily calculate the error in head from equation (17). However, there is a small error in equation (17) due to omission of higher order terms. The magnitude of this error is represented by R in equation (18),

$$\Delta h = \pm\Delta Tu + R (x,y,t; T,S,Q) \qquad (18)$$

R(x,y,t; T,S,Q) is the sum of higher order terms. This truncation error can be expressed by

$$R (x,y,t; T,S,Q) = R = \Delta h - (\pm\Delta Tu) \qquad (19)$$

We have determined Δh by subtracting the head value in Step B from the one in Step A. ΔT is known and u is calculated in Step C. The truncation error can now be calculated.

The reduction of the calculation of parameter tolerances to a linear problem, by equation (17), is common in the literature. The validity of equation (17) is checked in this study by determining the magnitude of truncation errors from equation (19).

CASE I. NO FLOW BOUNDARIES

Consider a porous sandstone artesian aquifer as shown in Figure 5. The shape of the aquifer in the x-y plane is a square with the

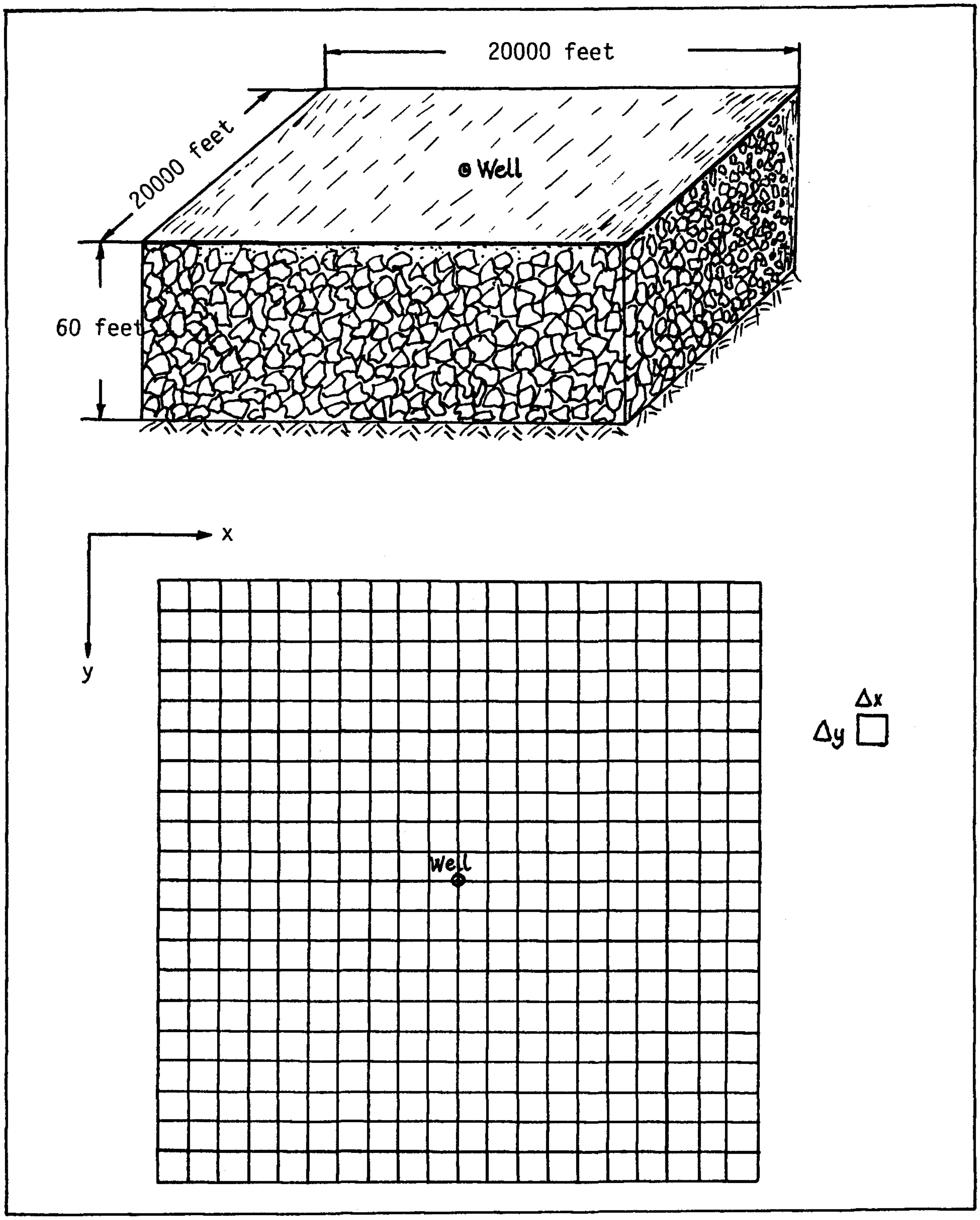


Figure 5. Sandstone artesian aquifer and the grid system.

sides of 20,000 feet. A discharging well is located at the center of the aquifer. The parameters of the system are as follows:

Hydraulic conductivity (K)	= 400 gal/day-ft^2
Initial head (h_o)	= 150 ft.
Saturated thickness (b)	= 60 ft.
Storage coefficient (S)	= 0.000948
Transmissivity (T = Kb)	= 24,000 gal/day-ft
Well discharge (Q)	= 240,000 gal/day

All the four boundaries are barrier boundaries, i.e.,

$$\frac{\partial h}{\partial x}\bigg|_{x=0} = \frac{\partial h}{\partial x}\bigg|_{x=20,000} = \frac{\partial h}{\partial y}\bigg|_{y=o} = \frac{\partial h}{\partial y}\bigg|_{y=20,000} = 0 \qquad (20)$$

Boundary conditions of the sensitivity equation are:

$$\frac{\partial}{\partial x}\left[\frac{\partial h}{\partial T}\right]\bigg|_{x=0} = \frac{\partial}{\partial x} u \bigg|_{x=0} = 0 \qquad (21a)$$

$$\frac{\partial}{\partial x}\left[\frac{\partial h}{\partial T}\right]\bigg|_{x=20,000} = \frac{\partial}{\partial x} u \bigg|_{x=20,000} = 0 \qquad (21b)$$

$$\frac{\partial}{\partial y}\left[\frac{\partial h}{\partial T}\right]\bigg|_{y=0} = \frac{\partial}{\partial y} u \bigg|_{y=0} = 0 \qquad (21c)$$

$$\frac{\partial}{\partial y}\left[\frac{\partial h}{\partial T}\right]\bigg|_{y=20,000} = \frac{\partial}{\partial y} u \bigg|_{y=20,000} = 0 \qquad (21d)$$

We shall assume uniform hydraulic conductivity, saturated thickness, and storage coefficient throughout the aquifer. These assumptions will simplify the illustration of parameter sensitivity in the groundwater flow system. However, the nonuniform case can be studied as easily as the uniform case.

The drawdown in the system is calculated for three different transmissivity values:

i. T_0 = 24,000 gal/day-ft.

ii. T_1 = (24,000 + 0.5 * 24,000) = 36,000 gal/day-ft.

iii. T_2 = (24,000 - 0.5 * 24,000) = 12,000 gal/day-ft.

Thus, the original transmissivity value, T_0, is perturbed by $\pm$ 50%. Since the saturated thickness is assumed uniform throughout the aquifer, perturbation of the transmissivity values by $\pm$ 50% means perturbation of the hydraulic conductivity values by $\pm$ 50%. Remembering that the hydraulic conductivity of a porous material is a characteristic property reflecting the ease of water transmission through its interstices, the T_1 value will show increased water transmission and T_2 will indicate a decreased water transmitting characteristic of the aquifer.

As soon as the pump begins discharging water from the well located at the center of the aquifer, a hydraulic gradient from all directions is established, resulting in a flow of water toward the well. Since barrier boundaries are assumed, the water being pumped is supplied by the water in storage. The rate of flow toward the well can be defined by

$$Q = V.A$$
$$= -K.A \frac{\partial h}{\partial r}$$

where

A = cross-sectional area (ft^2),

K = hydraulic conductivity (gal/day-ft^2),

Q = flow rate (gal/day),

and

$\frac{\partial h}{\partial r}$ = hydraulic gradient.

If the flow rate, Q, and cross-sectional area, A, are kept constant, or increase in hydraulic conductivity, K, will cause decrease in the hydraulic gradient, $\partial h/\partial r$, and vice versa.

Now we can analyze the effects of perturbating the transmissivity values, T_0. Figure 6 illustrates the variation in hydraulic head in time along the cross-section A-A' across the aquifer, with T_0 values. Figure 7 illustrates the change in hydraulic head in time if T_1 values are used. It is easily observed that the hydraulic gradient is less than the former case. Increased hydraulic conductivity results in a decreased hydraulic gradient. As a result, the radius of the cone of depression is greater for the case of higher hydraulic conductivity and the drawdown at the well is less. Figure 8 illustrates the case when the transmissivity is reduced by 50%. The hydraulic conductivity is smaller so the hydraulic gradient is steeper than for the unperturbed case. Notice that the drawdown at the well is greater and the radius of the cone of depression is smaller than for the unperturbed case.

We will assume that the unperturbed case represents the true transmissivity values. Thus, we can observe the error in hydraulic head when transmissivity values are overestimated by 50% and underestimated by 50%. An error of this magnitude is unlikely, but it is chosen to show the relationship between the error in hydraulic head and the sensitivity coefficients in a worst case situation.

The spatial distribution of errors in hydraulic head for over-estimated transmissivity values, T_1, and underestimated transmissivity values, T_2, are illustrated by Figures 9a, 10a, 11a, 12a, 9b, 10b, 11b, and 12b, at 0.03, 1.27, 10.23, and 40.95 hours, respectively. The corresponding sensitivity function values are shown by Figures 9c, 10c, 11c, and 12c.

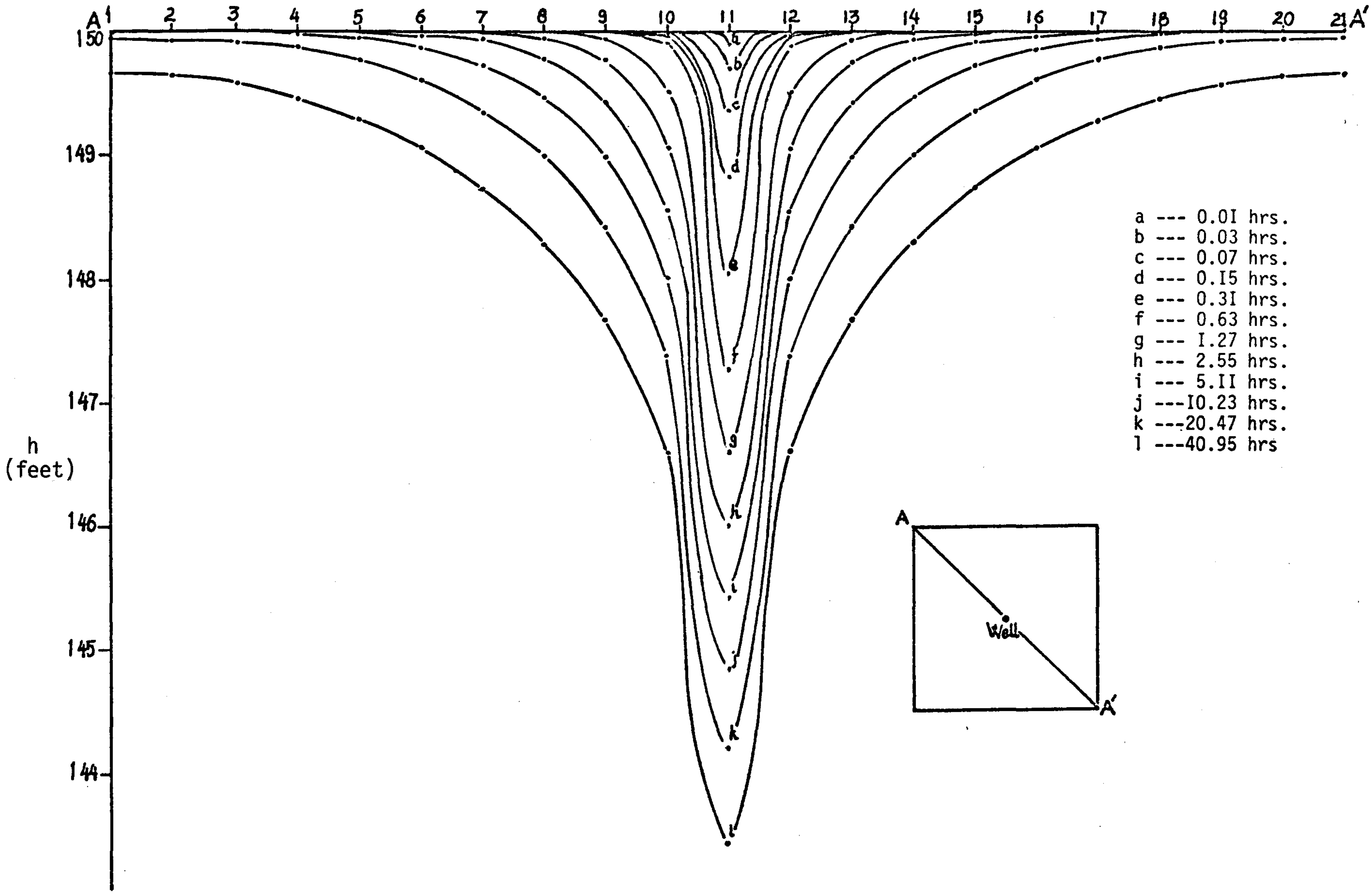


Figure 6. Variation of hydraulic head in time along the cross-section A-A' across the aquifer. Transmissivity is 24,000 gal./day-ft.

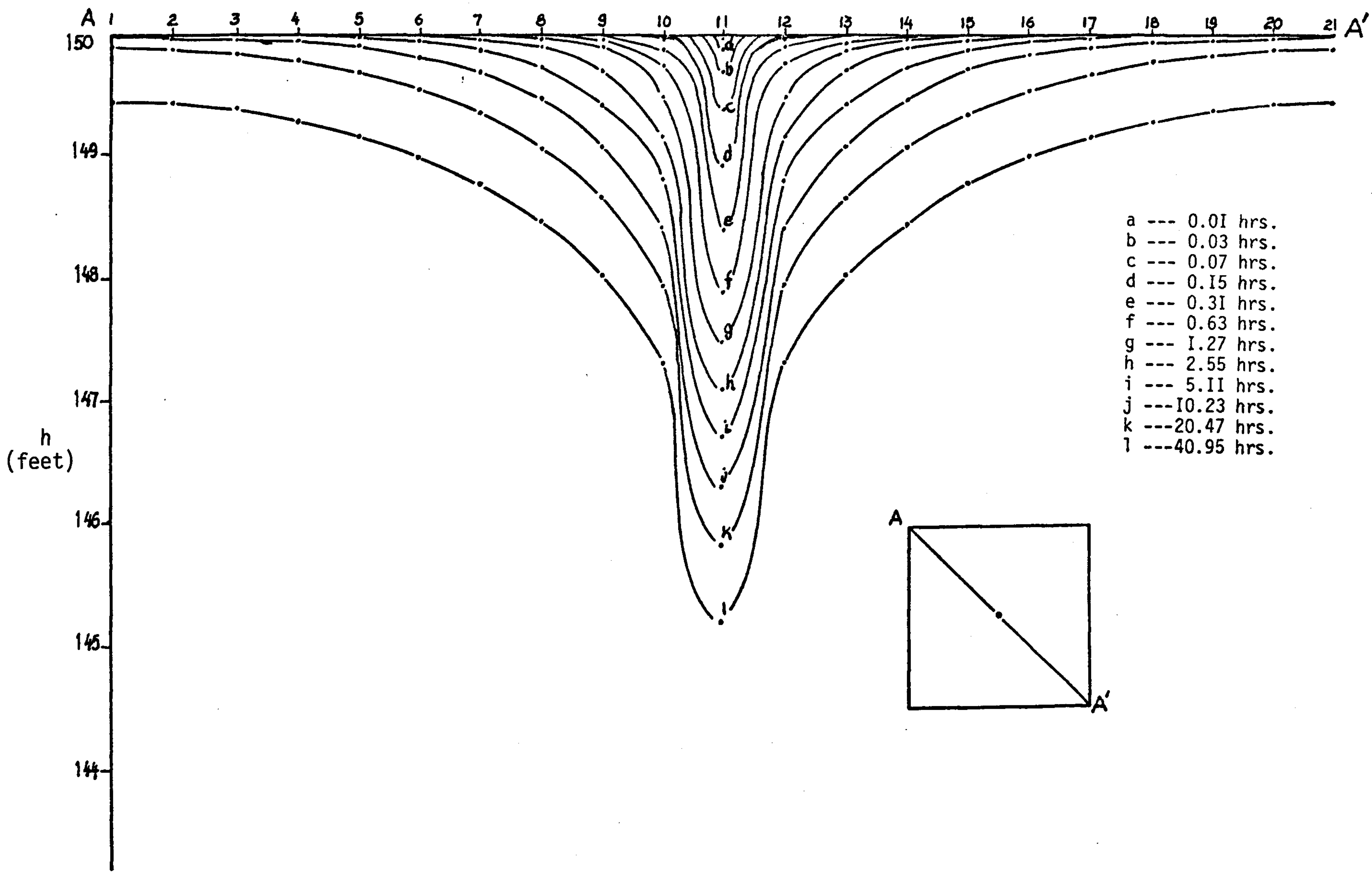


Figure 7. Variation of hydraulic head in time along the cross-section A-A' across the aquifer. Transmissivity is 36,000 gal./day-ft.

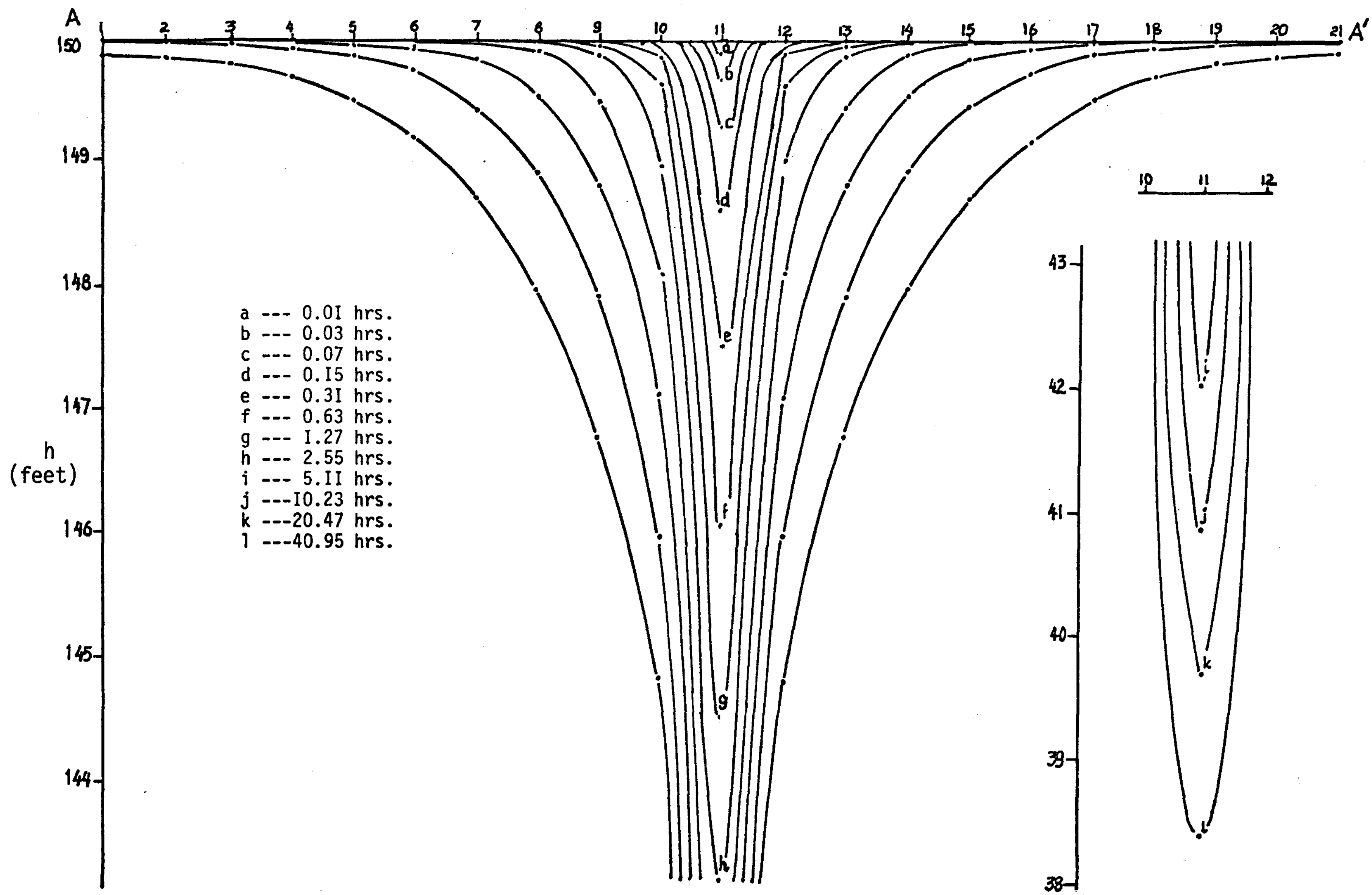


Figure 8. Variation of hydraulic head in time along the cross-section A-A' across the aquifer. Transmissivity is 12,000 gal./day-ft.

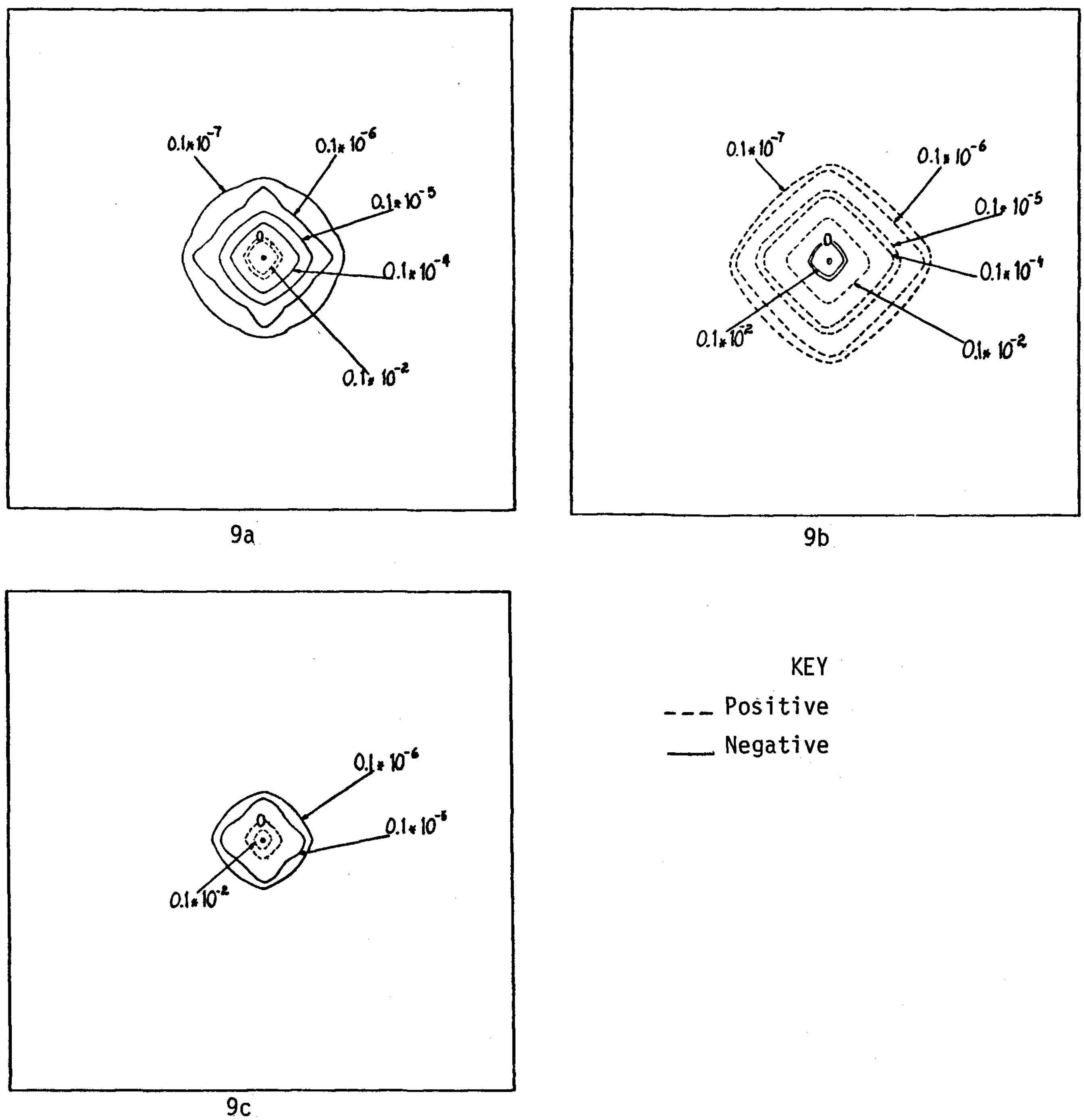


Figure 9. The spatial distribution of errors in hydraulic head for perturbed transmissivity values T_1 and T_2 are shown on (9a) and (9b) at 0.03 hours, respectively. The sensitivity coefficients are shown on (9c) at 0.03 hours.

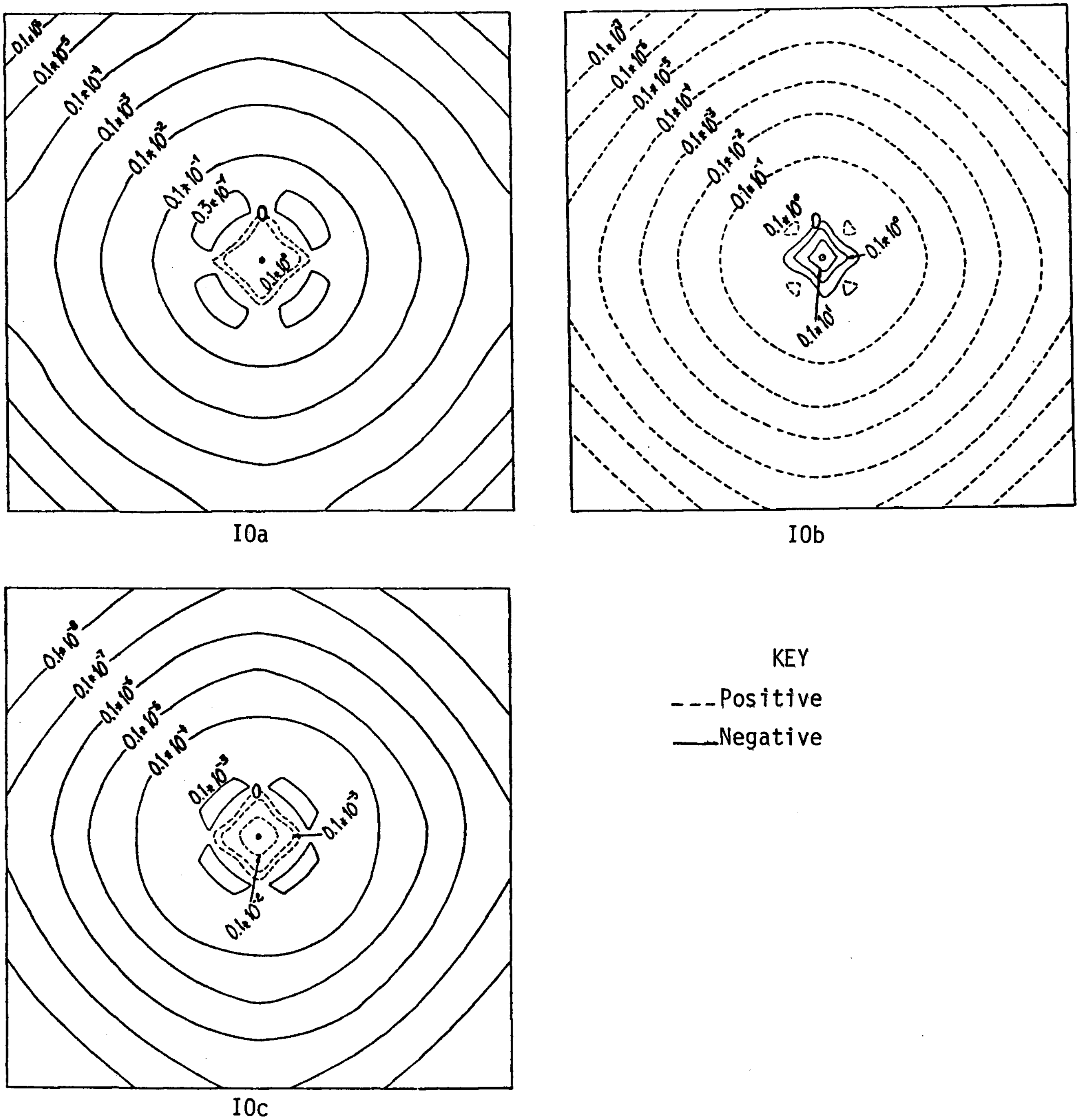


Figure IO. The spatial distribution of errors in hydraulic head for perturbed transmissivity values T_I and T_2 are shown on (IOa) and (IOb) at I.27 hours,respectively.The sensitivity coefficients are shown on (IOc) at I.27 hours.

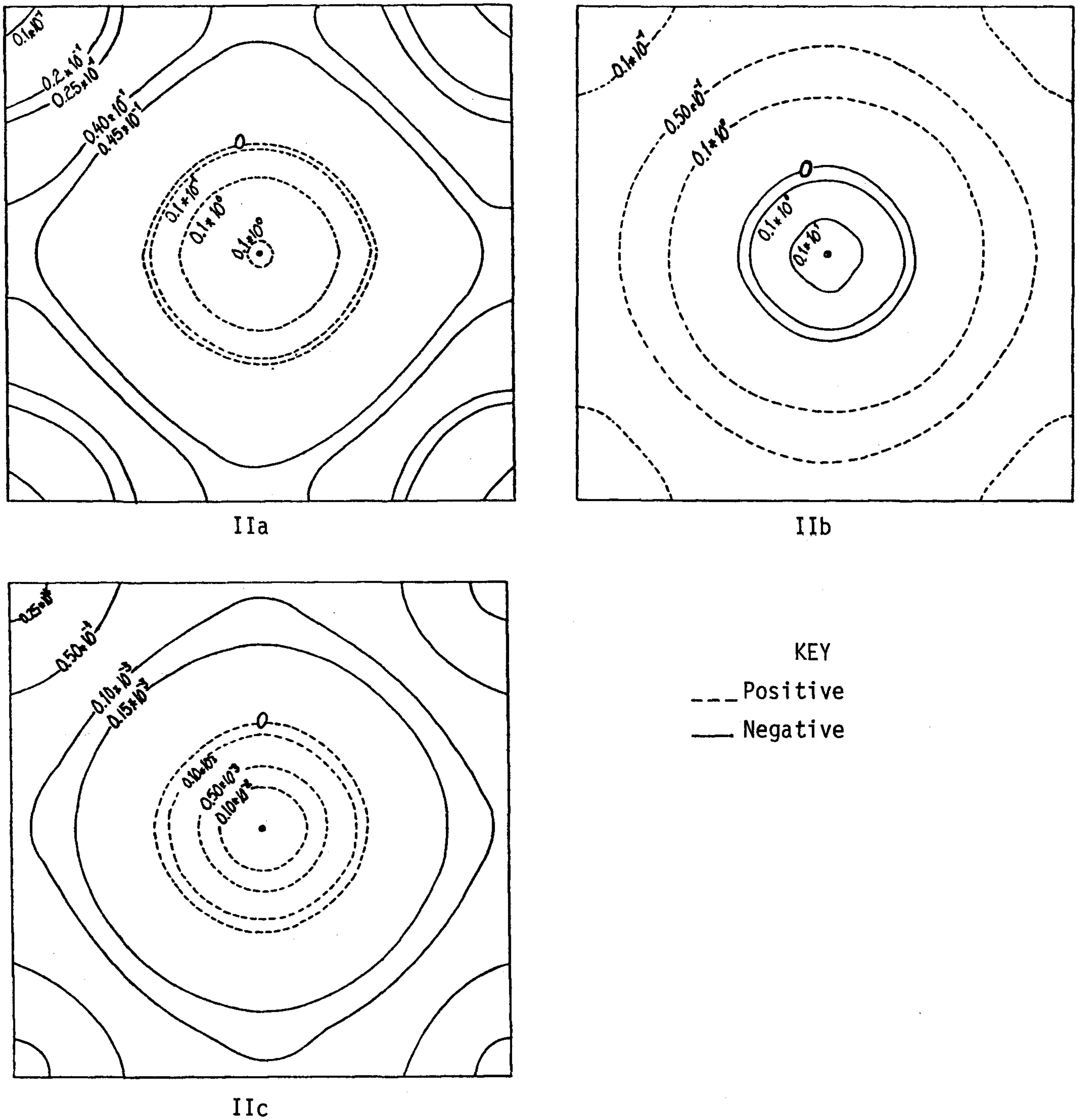


Figure II. The spatial variation of errors in hydraulic head for perturbed transmissivity values T_1 and T_2 are shown on (IIa) and (IIb) at 10.23 hours, respectively. The sensitivity coefficients are shown on (IIc) at 10.23 hours.

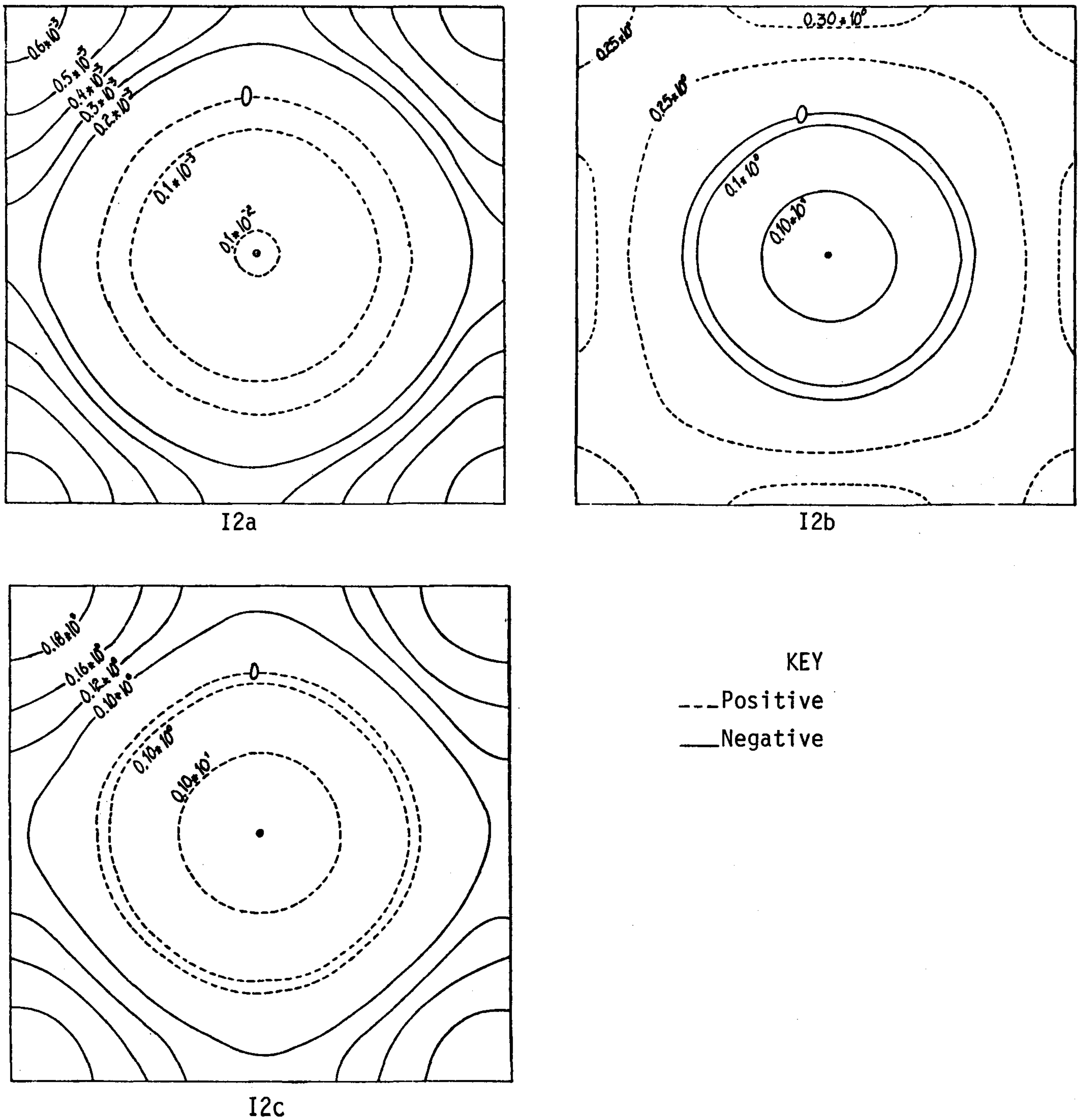


Figure I2. The spatial distribution of errors in hydraulic head for perturbed transmissivity values T_1 and T_2 are shown on (I2a) and (I2b) at 40.95 hours, respectively. The sensitivity coefficients are shown on (I2c) at 40.95 hours.

The product of the sensitivity coefficient and the disturbance in transmissivity, ΔT, gives the error made by perturbing the transmissivity values. The sign of the product indicates whether the system's response is overestimated (positive) or underestimated (negative) (Figures 9, 10, 11 and 12).

The zero contour lines on Figures 9c, 10c, 11c, and 12c show the lines along which the sensitivity coefficients are zero. The system is insensitive to parameter variations along these lines, and the error function values are zero along them.

The magnitude of the sensitivity coefficient shows the system's sensitivity to parameter variations at a particular location at a given time. Examination of the Figures 9c, 10c, 11c, and 12c shows that the values of the sensitivity coefficients are the greatest at the well and decrease away from it. Thus, the greatest error in hydraulic head when T is perturbed is at the well and the error decreases away from the well. However, the absolute values of sensitivity coefficients and the magnitude of errors increase beyond the zero line. This is due to different propagation rates of the cone of depression with different hydraulic conductivity values.

Figures 9 through 12 show that errors in hydraulic head can be easily determined qualitatively from the sensitivity coefficients. However, we are mostly interested in quantitative error analysis. We can determine the spatial distribution of error in time from equation (18). This equation has higher order terms which can not be solved readily. The problem is to find the conditions under which the higher order terms, R, can be neglected. Then we can determine the error in hydraulic head by equation (17).

The groundwater equation (8) is solved for disturbed transmissivity values, ± 50%, ± 20%, and ± 10%. The error in hydraulic head, h, due to disturbances in transmissivity values are calculated by subtracting the hydraulic head values of the perturbed cases from the unperturbed case. Then the magnitudes of the higher order terms are calculated from equation (19).

The changes in hydraulic head at the well determined from equation (8) and calculated by equation (17) for T_0, T_1, and T_2 values are plotted (Figures 13, 14, and 15). It is evident that equation (17) is not valid for ± 50% disturbance in transmissivity values (Figure 13). However, equation (17) yields better results for ± 20% disturbance (Figure 14) and the best results are obtained for ± 10% disturbance. The magnitudes of truncation errors, R, are given in Table 2. If truncation errors, R, less than 2.5 feet are acceptable, the equation (17) can be used to calculate errors in hydraulic head for ± 20% and ± 10% disturbances.

The drawdown halfway between the well and the boundary and 1,000 feet away from the boundary for ± 50%, ± 20% and ± 10% perturbation are illustrated by Figures 16, 17, and 18, respectively. The truncation errors, R, at these locations are given in Tables 3 and 4.

The magnitude of truncation errors in feet does not always give an insight into the system. Thus, the expression of these errors as percentage of drawdown may be more useful. The percentage truncation errors, R(%), at the well are given in Table 5. The percentage errors are less than or equal to 1% for ± 10% disturbances and less than or equal to 3.7% for ± 20% disturbances.

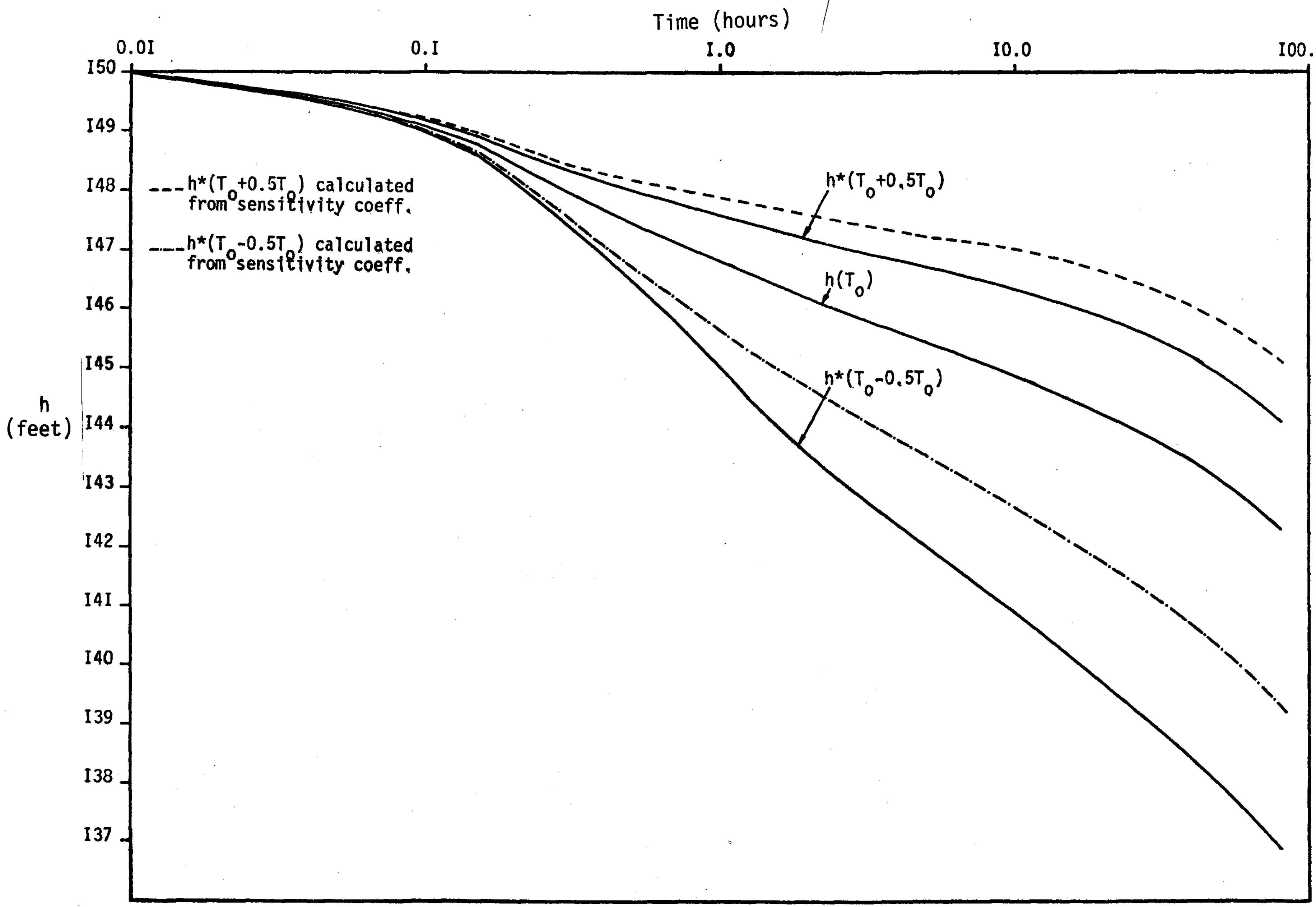


Figure 13. Comparison of hydraulic head values determined at the well by direct solution and calculated from sensitivity coefficients for ±50% disturbance in transmissivity values.

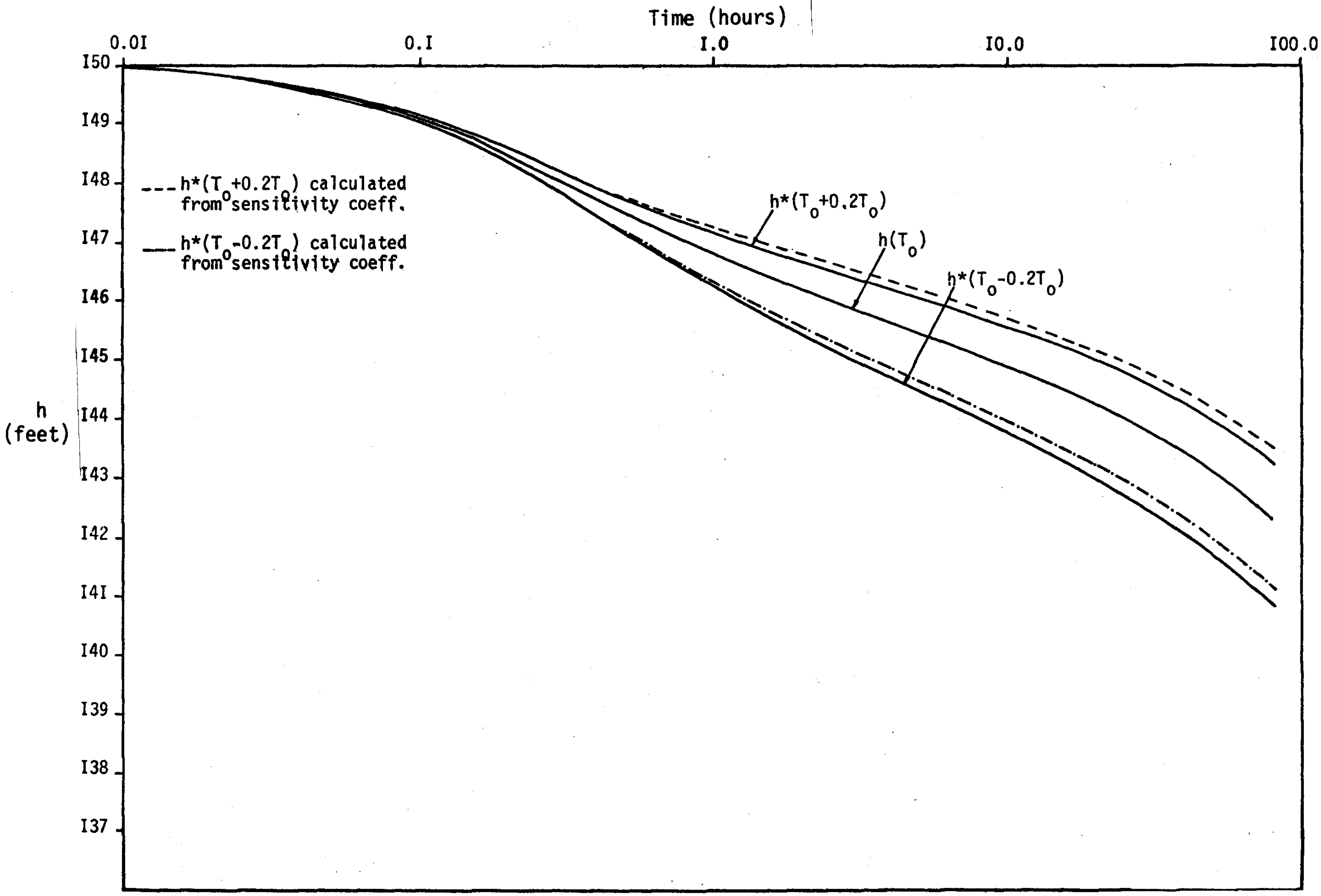


Figure I4. Comparison of hydraulic head values determined at the well by direct solution and calculated from sensitivity coefficients for ±20% disturbance in transmissivity values.

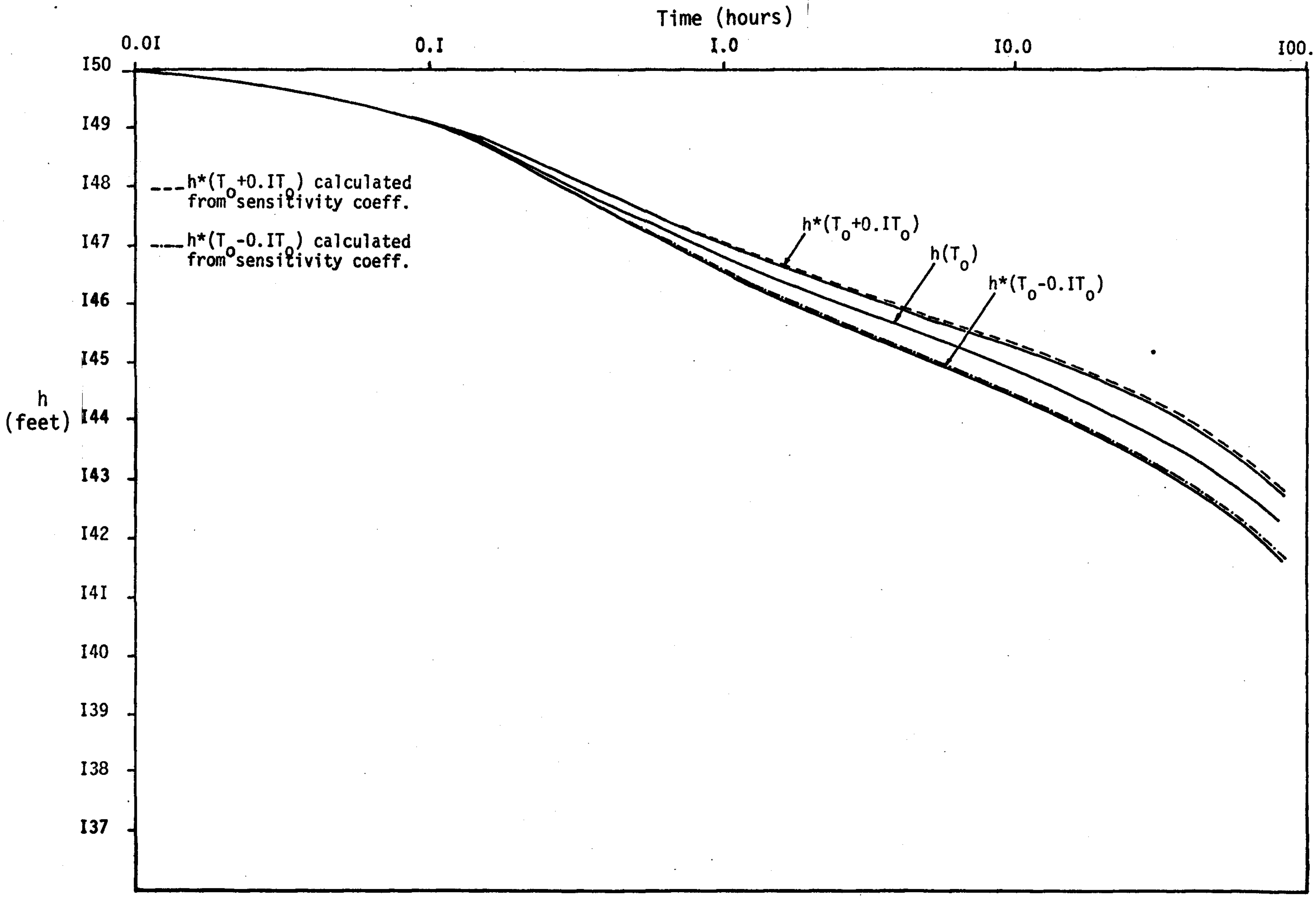


Figure I5. Comparison of hydraulic head values determined at the well by direct solution and calculated from sensitivity coefficients for ±I0% disturbance in transmissivity values.

TRUNCATION ERROR,R,AT THE WELL

Time (in hrs.)	$R(T_o+50\%T_o)$ (in feet)	$R(T_o+20\%T_o)$ (in feet)	$R(T_o+10\%T_o)$ (in feet)	$R(T_o)$	$R(T_o-10\%T_o)$ (in feet)	$R(T_o-20\%T_o)$ (in feet)	$R(T_o-50\%T_o)$ (in feet)
0.01	-0.000013	-0.000002	-0.000000	--	-0.000000	-0.000002	-0.000014
0.03	-0.000249	-0.000041	-0.000010	--	-0.000010	-0.000042	-0.000264
0.07	-0.002410	-0.000399	-0.000101	--	-0.000103	-0.000418	-0.002710
0.15	-0.015600	-0.002650	-0.000677	--	-0.000706	-0.002890	-0.019400
0.31	-0.068700	-0.012100	-0.003140	--	-0.003380	-0.014100	-0.099900
0.63	-0.196000	-0.036300	-0.009560	--	-0.010700	-0.045600	-0.352000
1.27	-0.357000	-0.068800	-0.018500	--	-0.021600	-0.094100	-0.804000
2.55	-0.473000	-0.092800	-0.025100	--	-0.029900	-0.133000	-1.240000
5.11	-0.577000	-0.114000	-0.031300	--	-0.035800	-0.161000	-1.540000
10.23	-0.689000	-0.141000	-0.040400	--	-0.038600	-0.184000	-1.820000
20.47	-0.814000	-0.174000	-0.052700	--	-0.036300	-0.193000	-2.050000
40.95	-0.949000	-0.209000	-0.066200	--	-0.034200	-0.204000	-2.270000
81.91	-1.020000	-0.226000	-0.072100	--	-0.037000	-0.222000	-2.510000

Table 2. Truncation errors,R,under varying disturbances in transmissivity values at the well.

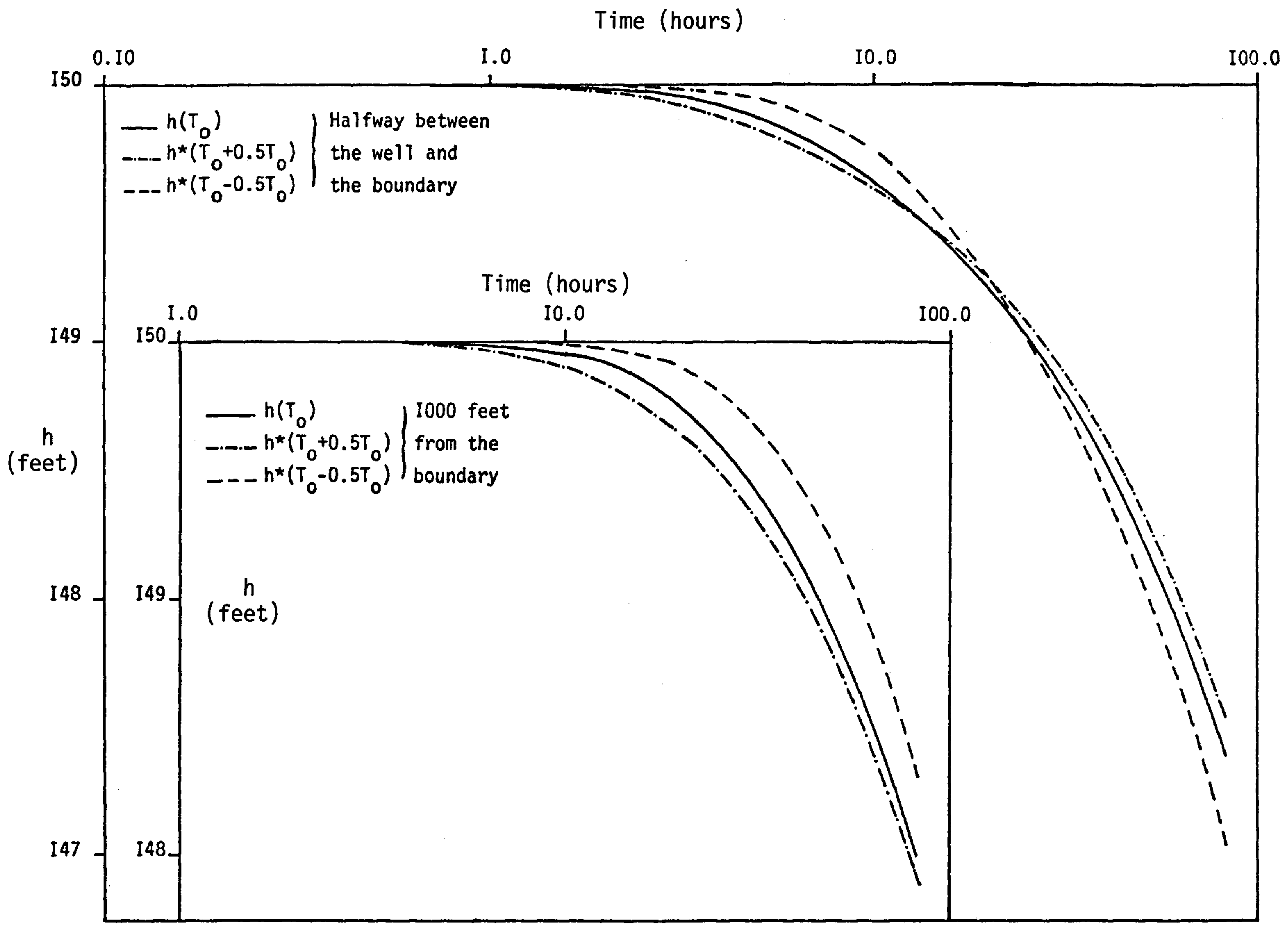


Figure 16. Hydraulic head values determined by direct solution for original and ±50% disturbance in transmissivity values, halfway between the boundary and 1000 feet from the boundary.

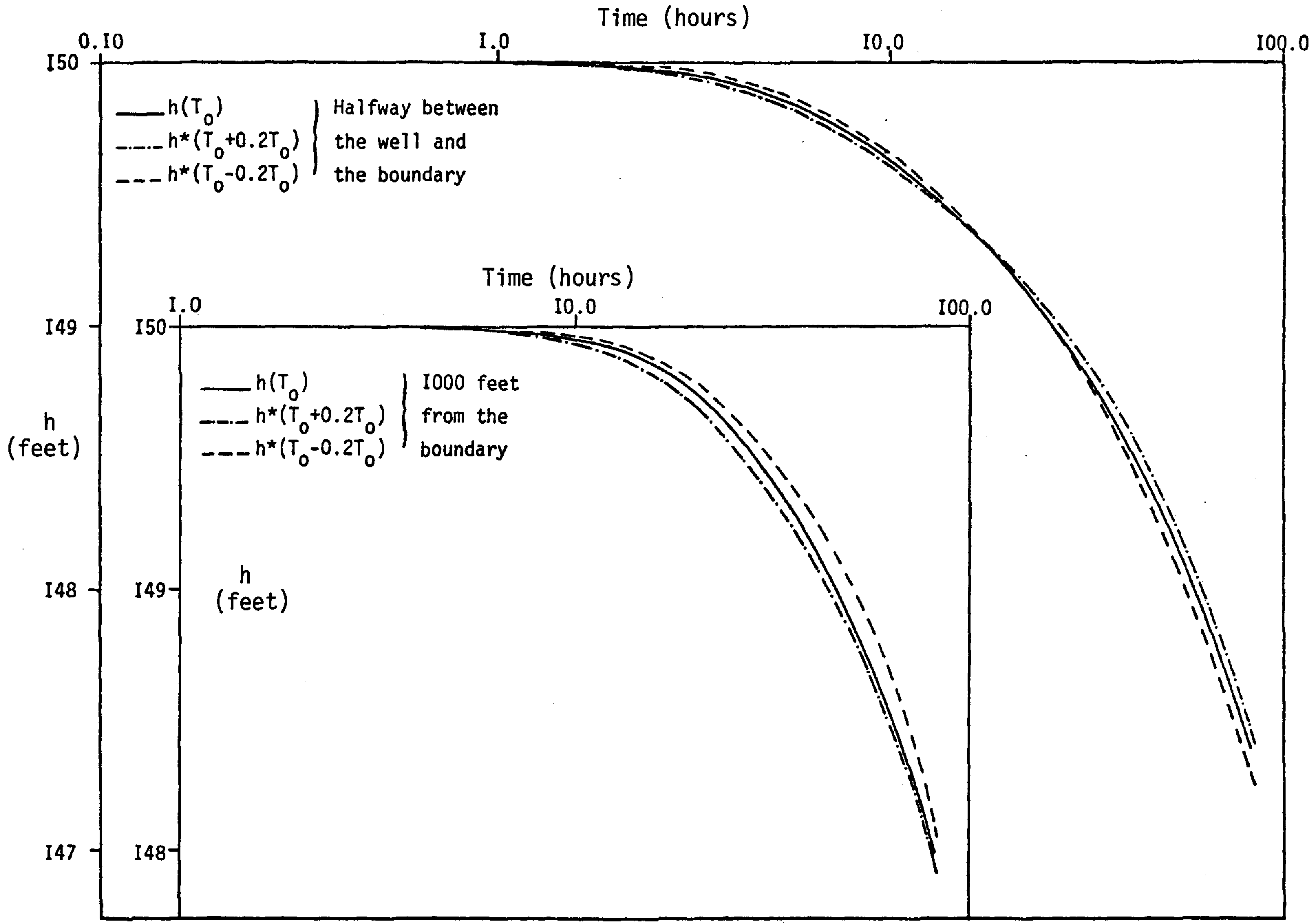


Figure 17. Hydraulic head values determined by direct solution for original and ±20% disturbance in transmissivity values, halfway between the well and the boundary and 1000 feet from the boundary.

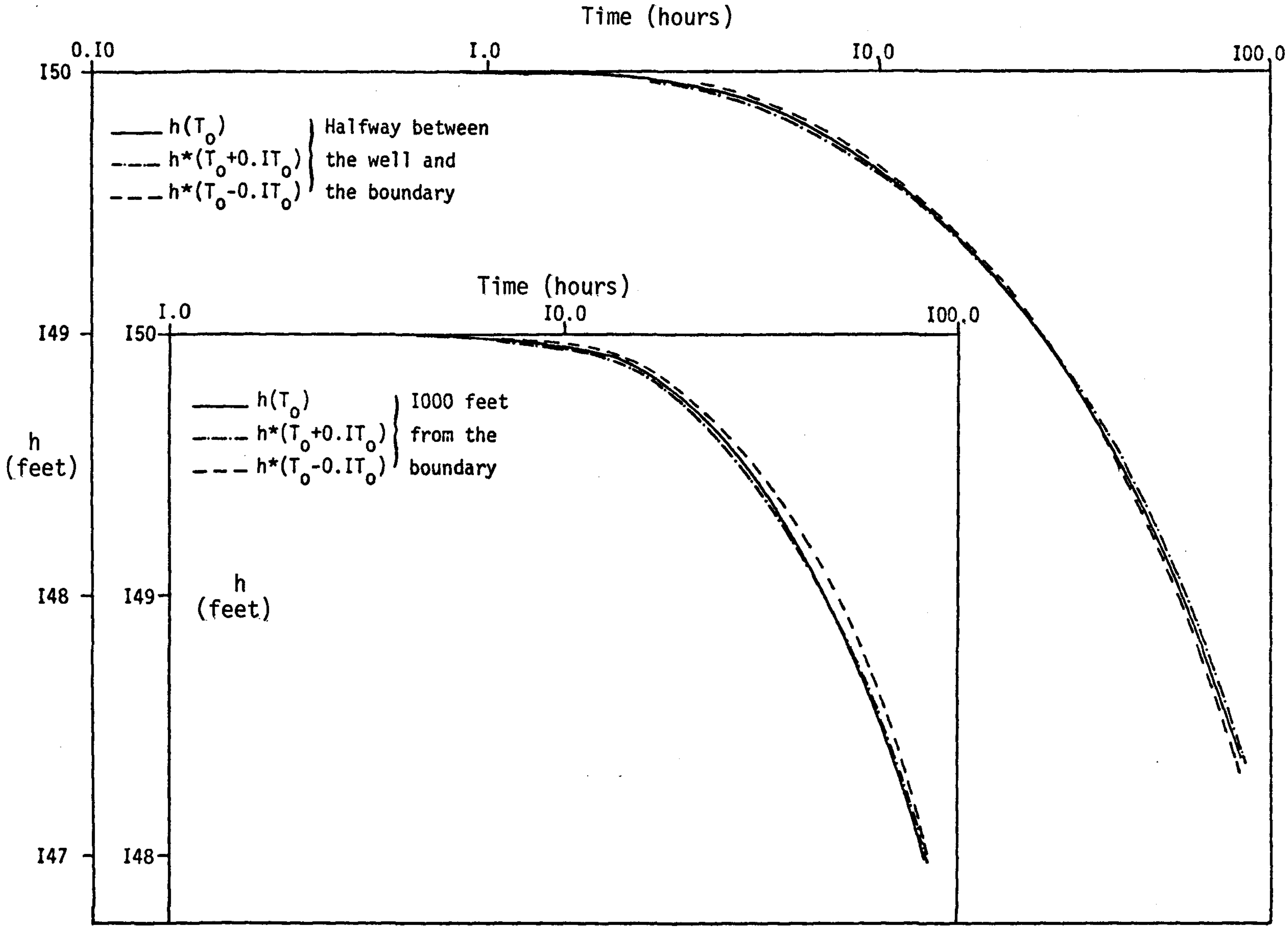


Figure 18. Hydraulic head values determined by direct solution for original and ±10% disturbance in transmissivity values, halfway between the well and the boundary and 1000 feet from the boundary.

TRUNCATION ERROR, R,							
Time (in hrs.)	R(T_o+50%T_o) (in feet)	R(T_o+20%T_o) (in feet)	R(T_o+I0%T_o) (in feet)	R(T_o)	R(T_o-I0%T_o) (in feet)	R(T_o-20%T_o) (in feet)	R(T_o-50%T_o) (in feet)
0.0I	---	---	---	---	---	---	---
0.03	---	---	---	---	---	---	---
0.07	---	---	---	---	---	---	---
0.I5	---	---	---	---	---	---	---
0.3I	0.0000I4	-0.000002	-0.00000I	---	-0.00000I	-0.000002	0.000009
0.63	0.000I7I	-0.000028	-0.000007	---	-0.000007	-0.000026	0.000I56
I.27	0.000764	-0.000I4I	-0.000035	---	-0.00004I	-0.000I73	0.00I200
2.55	-0.00I420	0.000I35	0.000020	---	0.0000I0	-0.000030	0.00I680
5.II	-0.0I5500	0.002580	0.000594	---	0.000930	0.003480	-0.020I00
I0.23	-0.02I000	0.002930	0.000I8I	---	0.003220	0.0II000	-0.0777
20.47	0.0I8300	-0.008300	-0.004570	---	0.006I80	0.0I5300	-0.09I700
40.95	0.078600	-0.026I00	-0.0I2I00	---	0.009990	0.0I7900	-0.039400
8I.9I	0.I40000	-0.04I600	-0.0I7800	---	0.0I0600	0.0I2500	0.052500

Table 3. Truncation errors, R, under varying disturbances in transmissivity values at halfway between the well and the boundary.

TRUNCATION ERROR, R

Time (in hrs.)	$R(T_o+50\%T_o)$ (in feet)	$R(T_o+20\%T_o)$ (in feet)	$R(T_o+10\%T_o)$ (in feet)	$R(T_o)$	$R(T_o-10\%T_o)$ (in feet)	$R(T_o-20\%T_o)$ (in feet)	$R(T_o-50\%T_o)$ (in feet)
0.01	---	---	---	---	---	---	---
0.03	---	---	---	---	---	---	---
0.07	---	---	---	---	---	---	---
0.15	---	---	---	---	---	---	---
0.31	---	---	---	---	---	---	---
0.63	---	---	---	---	---	---	---
1.27	0.000021	-0.000003	---	---	---	-0.000002	0.000010
2.55	0.000353	-0.000049	-0.000010	---	-0.000019	-0.000063	0.000316
5.11	0.001080	-0.000056	0.000049	---	-0.000234	-0.000680	0.00346
10.23	-0.007820	0.002300	0.001060	---	-0.001000	-0.002130	0.008980
20.47	-0.036500	0.008200	0.002860	---	-0.000112	0.002720	-0.028000
40.95	-0.039300	0.003920	-0.000634	---	0.007190	0.022800	-0.148000
81.91	0.009190	-0.013900	-0.009370	---	0.016300	0.042400	-0.241000

Table 4. Truncation errors, R, under varying disturbances in transmissivity values at 1000 feet from the boundary.

PERCENTAGE TRUNCATION ERROR,R(%),AT THE WELL

Time (in hours)	R(%) ($T_o+0.5T_o$)	R(%) ($T_o+0.2T_o$)	R(%) ($T_o+0.1T_o$)	R(%) (T_o)	R(%) ($T_o-0.1T_o$)	R(%) ($T_o-0.2T_o$)	R(%) ($T_o-0.5T_o$)
0.01	0.012700	0.001940	0.000485	---	0.000483	0.001930	0.013400
0.03	0.068200	0.013900	0.003380	---	0.003340	0.013900	0.086400
0.07	0.400000	0.063900	0.015900	---	0.015800	0.064500	0.395000
0.15	1.470000	0.212000	0.058100	---	0.057600	0.230000	1.420000
0.31	4.300000	0.680000	0.169000	---	0.167000	0.666000	4.080000
0.63	9.400000	1.490000	0.373000	---	0.370000	1.470000	9.010000
1.27	14.300000	2.320000	0.583000	---	0.587000	2.360000	14.700000
2.55	16.400000	2.690000	0.679000	---	0.691000	2.800000	16.500000
5.11	17.700000	2.910000	0.741000	---	0.720000	2.950000	19.300000
10.23	18.800000	3.190000	0.849000	---	0.687000	2.960000	19.900000
20.47	19.600000	3.480000	0.987000	---	0.575000	2.770000	19.900000
40.95	19.800000	3.680000	1.090000	---	0.478000	2.600000	19.600000
81.91	17.600000	3.350000	1.000000	---	0.444000	2.450000	19.100000

Table 5. Percentage truncation errors,R(%),under varying transmissivity values at the well.

Using equation (17), we can define the sensitivity of the groundwater flow system in yet another way. We assumed earlier that a disturbance in the transmissivity value, ΔT, is constant. Now let us assume that we have fixed the permissible error in hydraulic head, Δh. In this case we must change ΔT at every point so that the condition Δh = constant will be satisfied. Solving equation (17) with respect to ΔT, we obtain

$$|\Delta T| \leq \left|\frac{\Delta h}{u}\right| \quad (22)$$

The function $|\Delta T|$ shows directly the maximum value of ΔT for which the error in hydraulic head is equal to a certain value at a particular location at a given time. This function is a direct measure of the sensitivity of the groundwater flow system at a given location and time. Locations where the system is the most sensitive require the smallest tolerance in $|\Delta T|$.

Equation (22) can also be written in terms of drawdown

$$\begin{aligned} |\Delta T| &\leq \left|\frac{h^*-h}{u}\right| \\ &\leq \left|\frac{(H_0-s^*)-(H_0-s)}{u}\right| \\ &\leq \left|\frac{\Delta s}{u}\right| \end{aligned} \quad (23)$$

where

H_0 is the initial head value (feet),

S^* is the drawdown of the perturbed case (feet), and

S is the drawdown of the unperturbed case (feet).

The system's sensitivity to variations in transmissivity for various unperturbed transmissivity values has been calculated. The unperturbed transmissivity values range from 36,000 to 12,000 gal/day-ft, that is from moderately high to low transmissivity. It is found that the sensitivity of the groundwater flow system increases as the transmissivity value decreases. The changes in sensitivity coefficients at the well are illustrated by figure 19. We can conclude that a more accurate determination of transmissivity values is required as the transmissivity of the system decreases.

Using equation (23) and the sensitivity coefficients given by figure 19 maximum values of ΔT are computed for the systems with transmissivity values 36,000, 24,000, and 12,000 gal/day-ft. (Figure 20). The permissible error in Δs is assumed to be 2% of the drawdown of the unperturbed case. Notice that permissible errors (ΔT) in T are larger for the system of higher transmissivity since it is less sensitive as shown by the sensitivity coefficients in figure 19.

Examination of figure 20 shows that permissible ΔT values decrease up to a certain time and then show an increase again. This increase is due to boundary conditions. The cone of depression reaches the barrier boundaries earlier in the system of higher transmissivity and the boundary effects are seen earlier. Thus, the increase in ΔT is detected after 10.23, 20.47, and 40.95 hours in the systems with transmissivity values 36,000, 24,000 and 12,000 gal/day-ft, respectively. This phenomenon is due to rapid increase in drawdown values. Magnitudes of the sensitivity coefficients also show a great increase when the boundary effects are sensed in the system. This results in larger magnitude of errors in the system. Hence, these conditions have to be avoided.

CASE II. CONSTANT HEAD BOUNDARIES

The same aquifer described in Case I is considered with the same aquifer parameters. Constant head boundaries are considered. The head values on four boundaries are maintained at 150 feet. Since the sensitivity

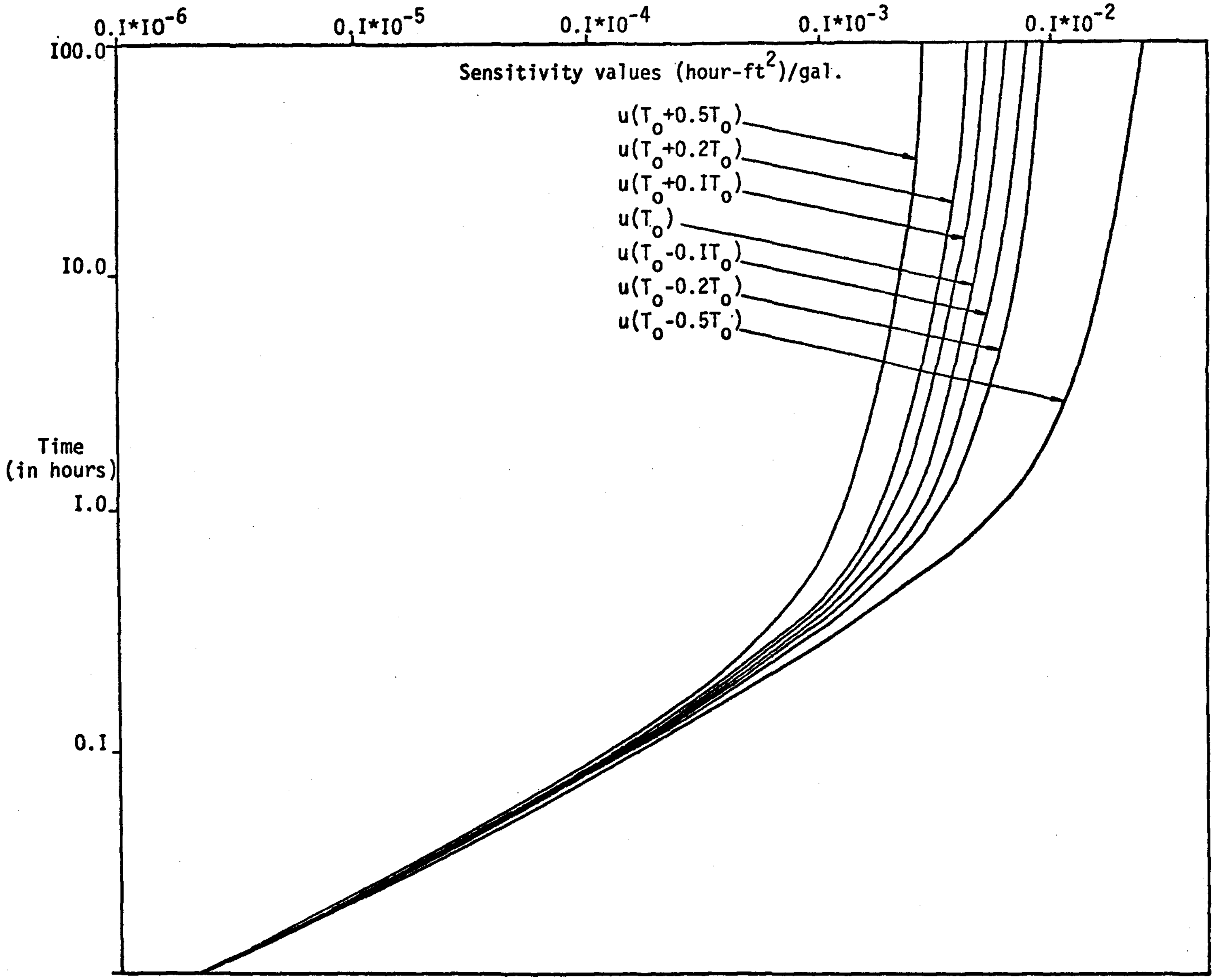


Figure I9. Variations in the sensitivity coefficients at the well with a change in transmissivity value.

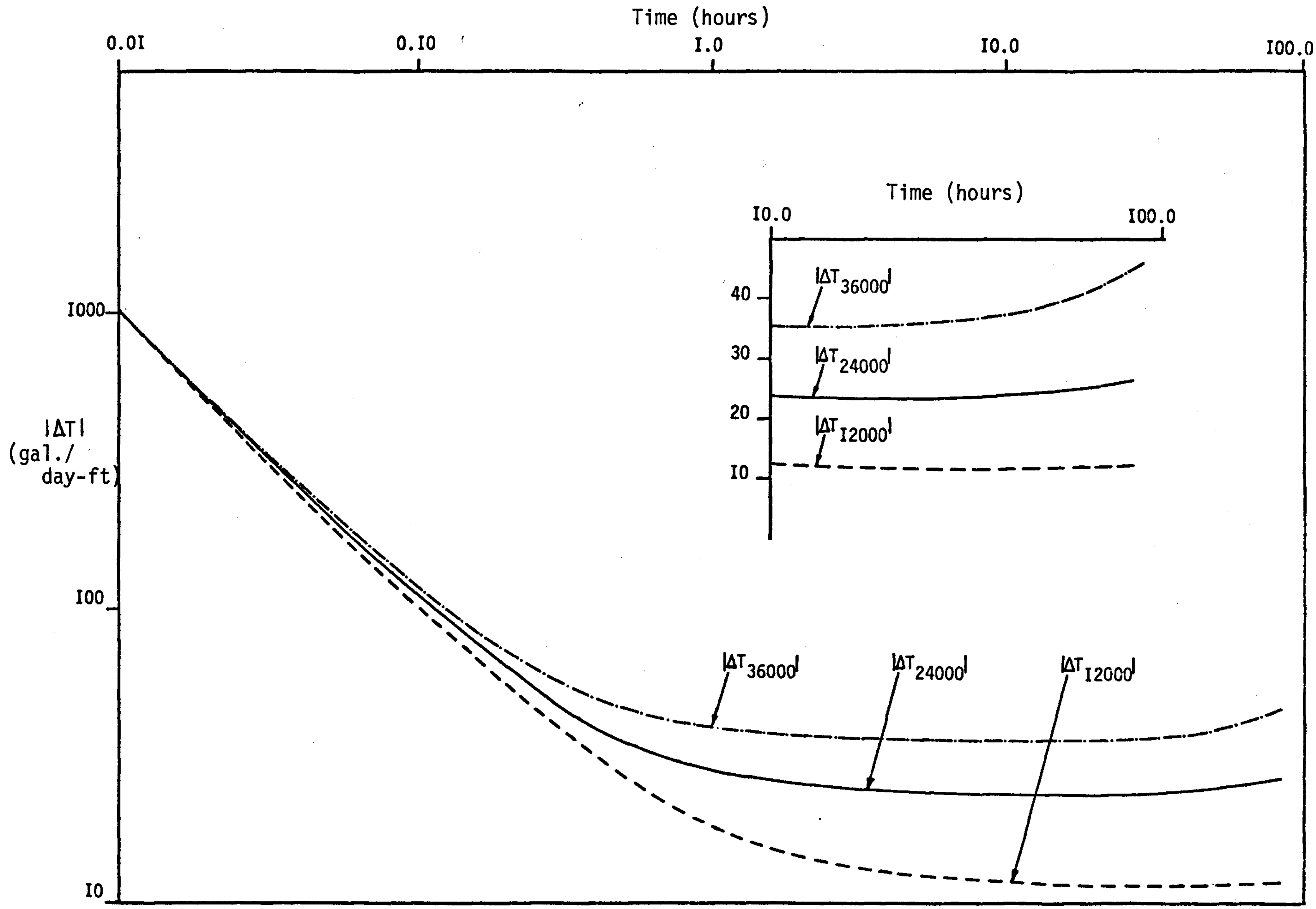


Figure 20. Maximum allowable errors in the transmissivity values at the well when a 2% error in the drawdown value is permissible.

coefficient is defined as, $u = \frac{\partial h}{\partial T}$, the sensitivity coefficients on the boundaries are.

$$u = \frac{\partial h}{\partial T} = \frac{\partial 150}{\partial T} = 0 \qquad (24)$$

Thus the system is insensitive to parameter variations on the boundaries.

The drawdown in the system is calculated for the three different transmissivity values, T_0, T_1, and T_2 (Figures 21, 22, and 23). Notice that the radius of the cone of depression is greater for the case of higher transmissivity (Figure 22) and it is smaller for the case of lower transmissivity (Figure 23). This behavior is similar to what was observed in Case I. However, the drawdown values are different on the boundaries. As soon as the cone of depression reaches the boundaries in Case II, there is a flow of water into the system to maintain the head constant, 150 feet in this case along the boundaries. In Case I no water can flow in or out through the boundaries and the head values show a drop along the boundaries with time. Therefore, the systems in Case I and Case II behave differently after the cone of depression reaches the boundaries.

The groundwater equation (8) has been solved for disturbed transmissivity values, $\pm$ 50%, $\pm$ 20%, and $\pm$ 10%. The maximum error occurs at the well and the variation in error away from the well followed the same pattern observed in Case I, $\pm$ 50% (Figures 9, 10, 11, and 12).

Hydraulic head values at the well for disturbed transmissivity values, are computed separately from equation (8) and from equation (17), to illustrate the time variation of the truncation errors, R (Figure 24). The truncation errors are small for $\pm$ 20% and $\pm$ 10% perturbations as in Case I and are not shown. Notice that the slope of the hydraulic

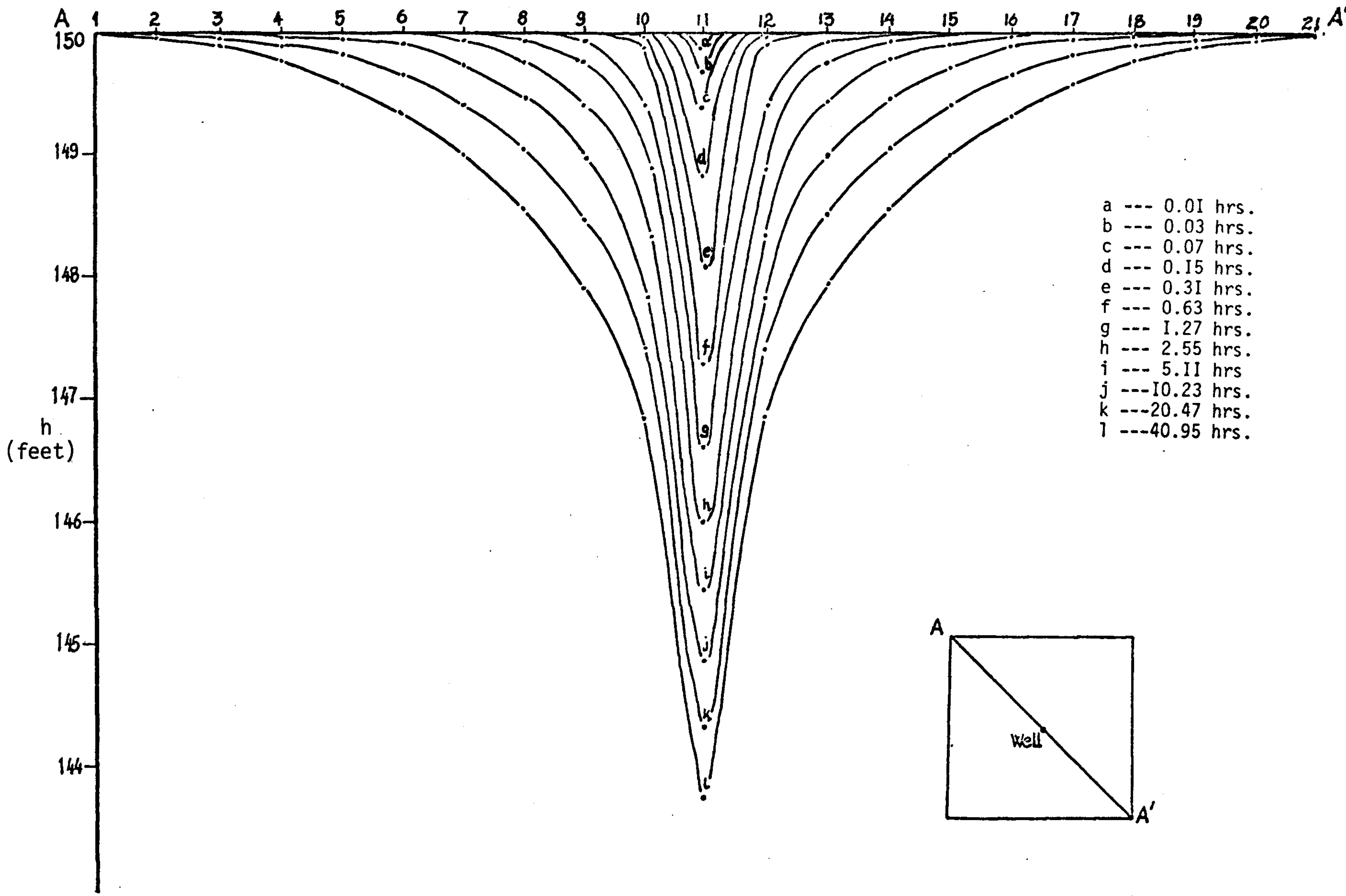


Figure 2I. Variation of hydraulic head in time along the cross-section A-A' across the aquifer. Transmissivity is 24,000 gal./day-ft.

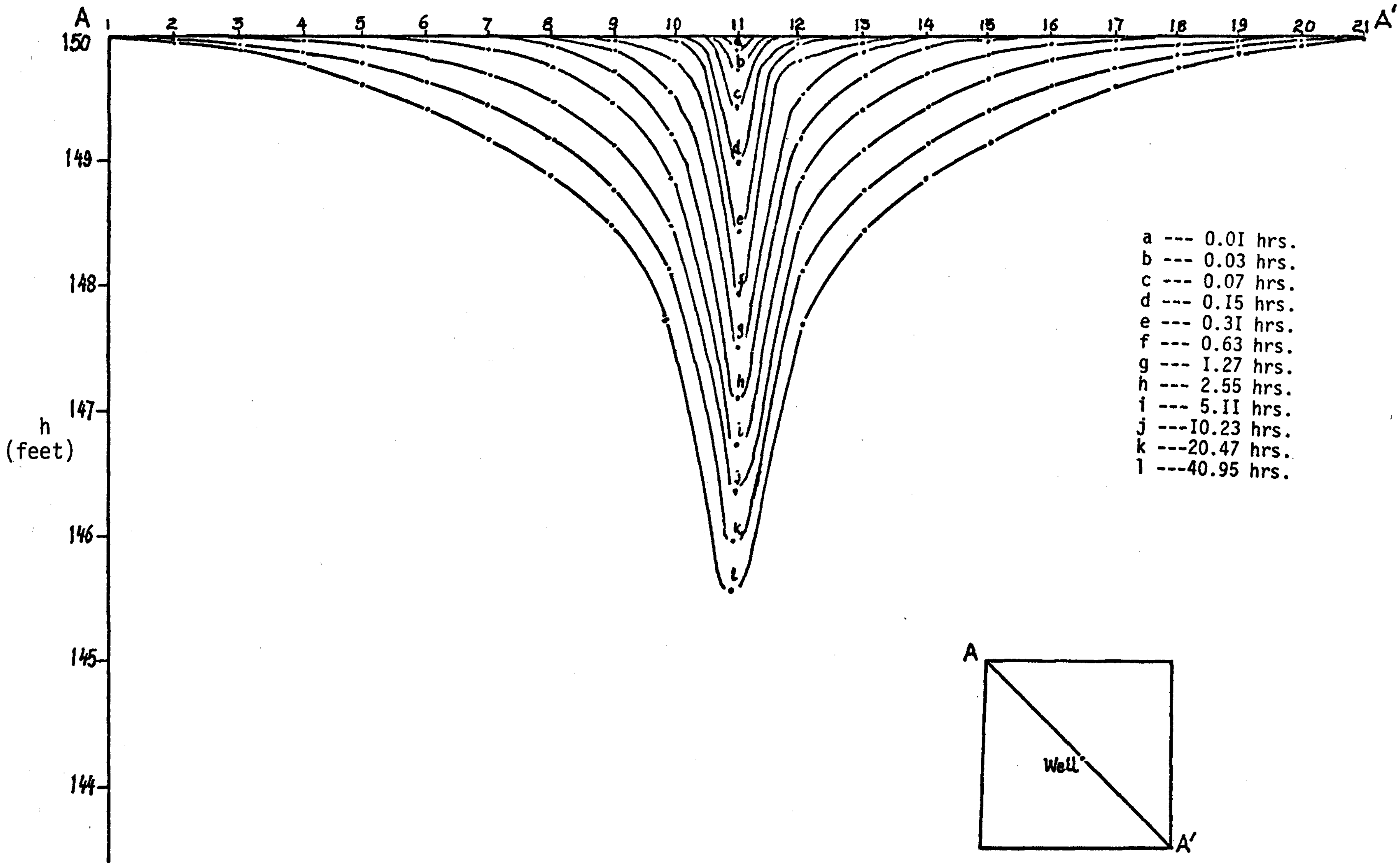


Figure 22. Variation of hydraulic head in time along the cross-section A-A' across the aquifer. Transmissivity is 36,000 gal./day-ft.

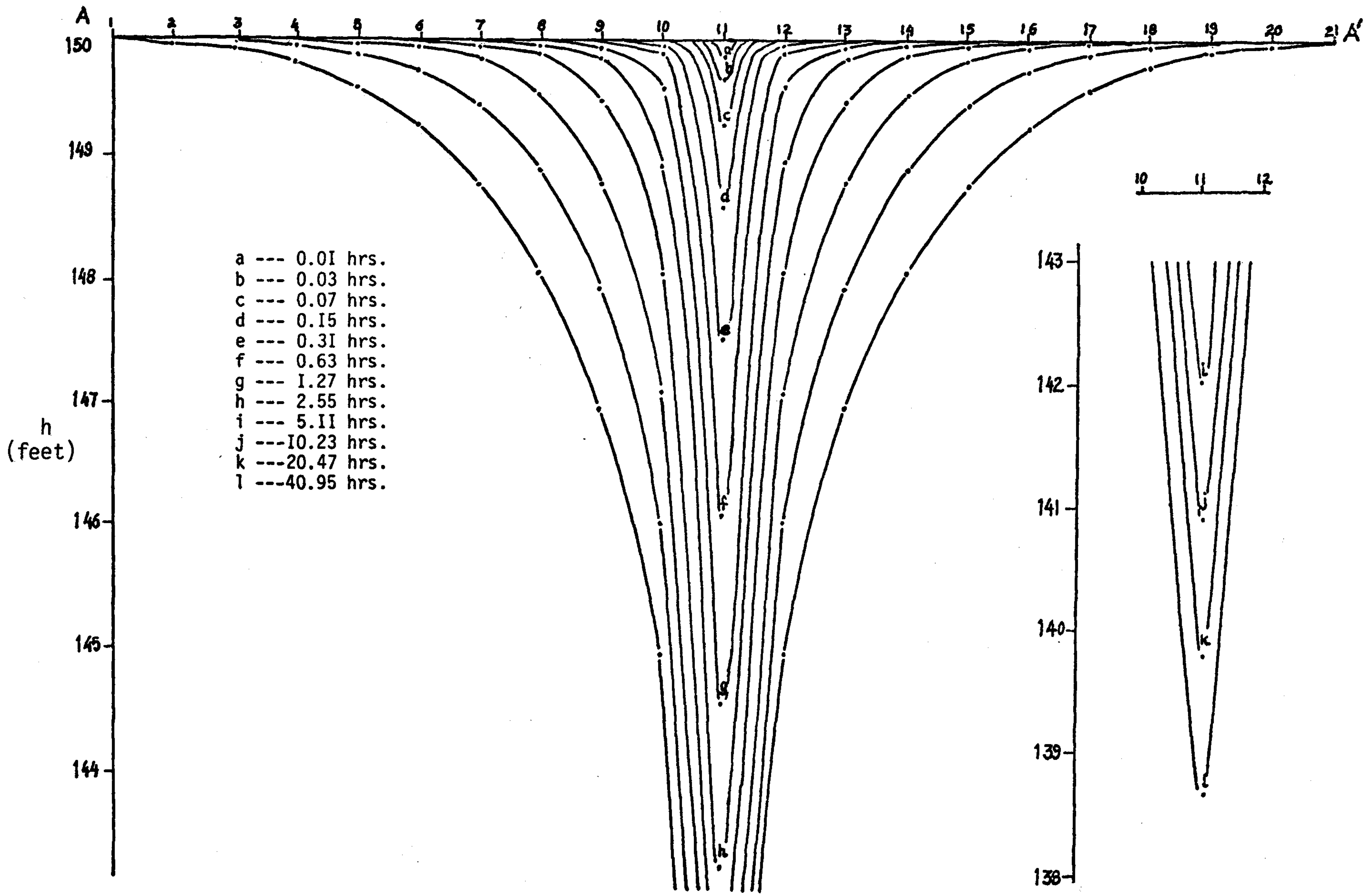


Figure 23. Variation of hydraulic head in time along the cross-section A-A' across the aquifer. Transmissivity is I2,000 gal./day-ft.

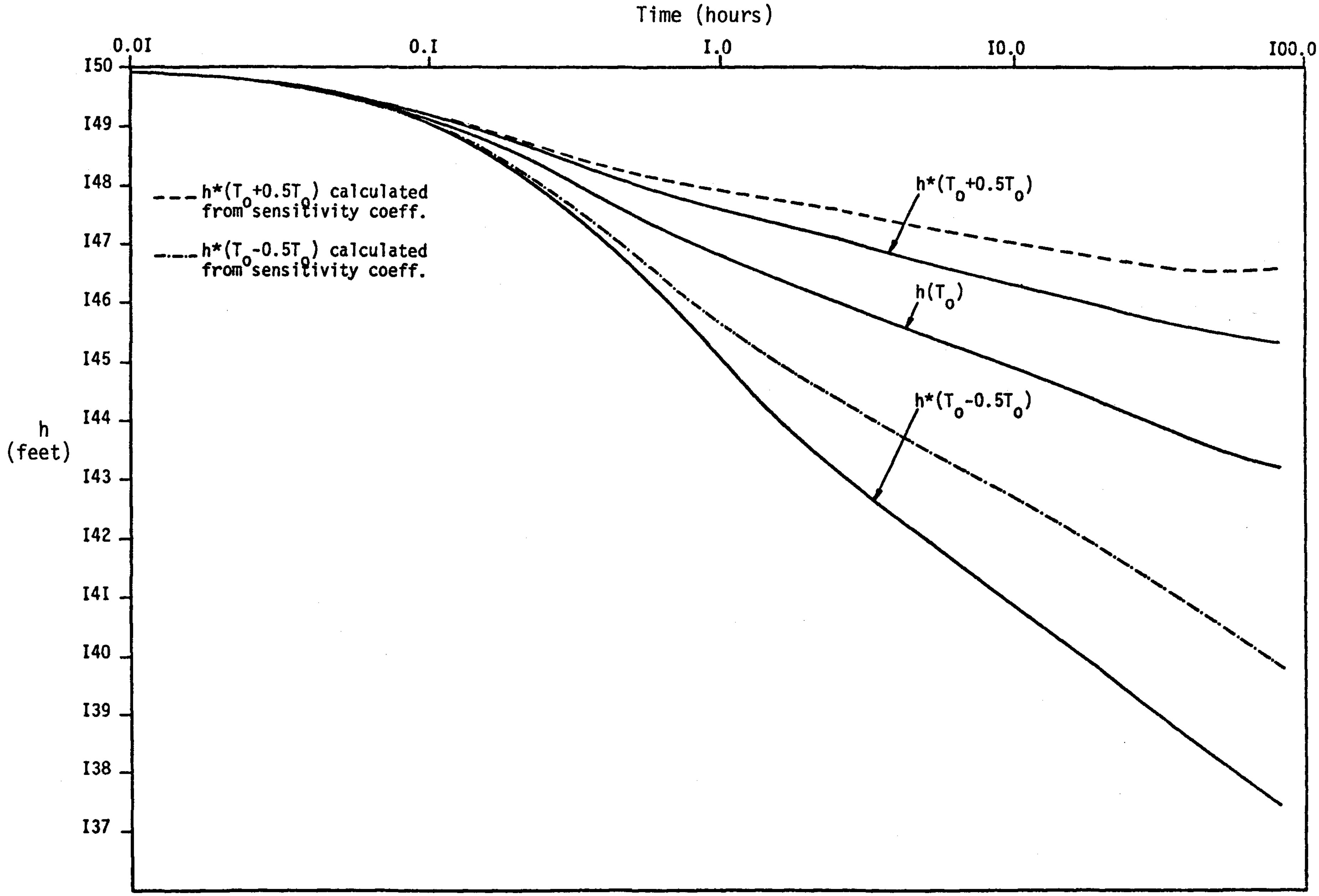


Figure 24. Comparison of hydraulic head values determined at the well by direct solution and calculated from sensitivity coefficients for ±50% disturbance in transmissivity values.

head decreases with time in Case II (Figure 24) whereas the slope increases in Case I (Figure 13). This behavior is more obvious at the points halfway between the well and the boundary and very close to the boundary (Figure 25). The decrease in the slope of the head function is due to a flow of water into the system along the boundaries when the cone of depression reaches the boundaries. Eventually an equilibrium state would be reached where the drawdown would not change. This phenomenon is detected earlier in the system of higher transmissivity where the radius of the cone of depression increases faster (Figures 24 and 25).

The sensitivity coefficients of the system are calculated by equation (13) for ± 50%, ± 20%, and ± 10% disturbances in transmissivity values (Figure 26). Using equation (23) and the sensitivity coefficients given by figure 26, the maximum allowable variations in transmissivity, ΔT, are computed for the systems with transmissivity values 36,000, 24,000, and 12,000 gal/day-ft, respectively (Figure 27). The permissible error in Δs is assumed to be 2% of the drawdown of the unperturbed case. It is observed that the ΔT values are larger for the system of higher transmissivity similar to what is seen in Case I. However, the boundary conditions have a different effect in Case II. The ΔT values decrease up to a certain time and then level off. When the boundary effects are sensed in the system the ΔT values show a decrease in magnitude. This behavior is exhibited first in the system of highest transmissivity and last in the system of lowest transmissivity. Similar to Case I the cone of depression reaches the boundaries earlier in the system of higher transmissivity and flow into the system through the boundaries begins to maintain the head constant along the boundaries. This causes a slowing of the drawdown. On the other hand, the sensitivity coefficients continue

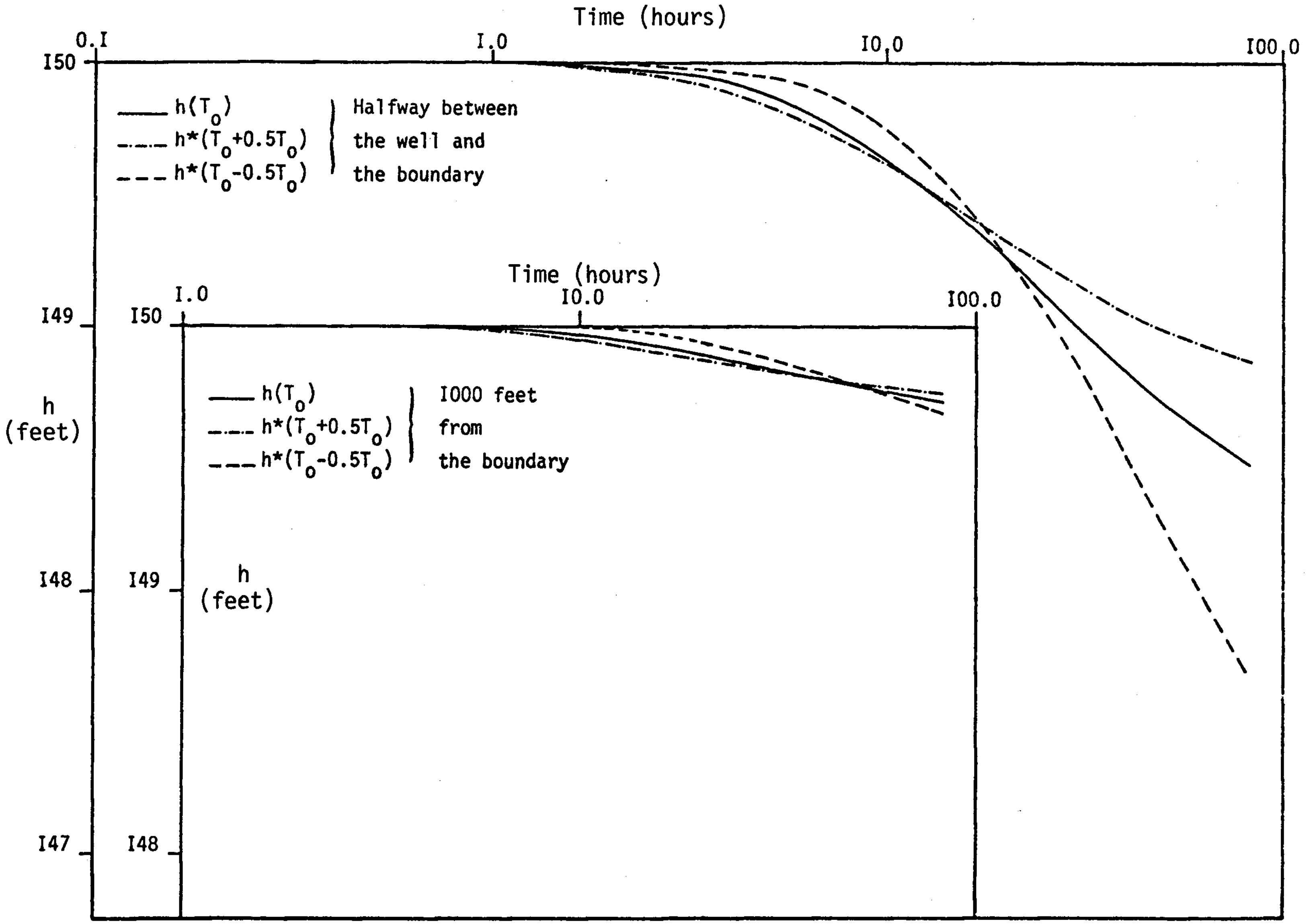


Figure 25. Hydraulic head values determined by direct solution for original and ±50% disturbance in transmissivity values, halfway between the well and the boundary and 1000 feet from the boundary.

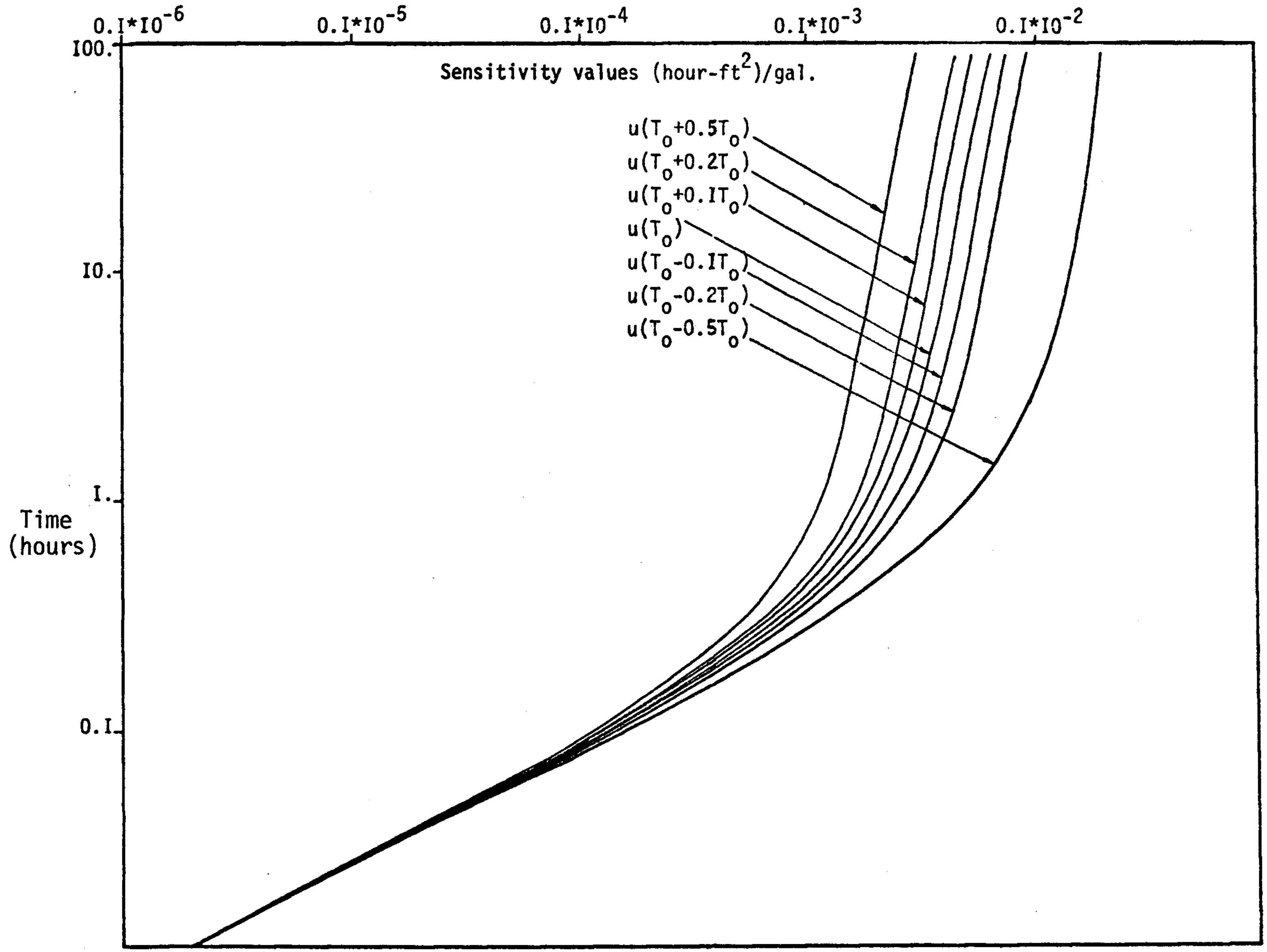


Figure 26. Variations in sensitivity coefficient values with changes in transmissivity values, at the well.

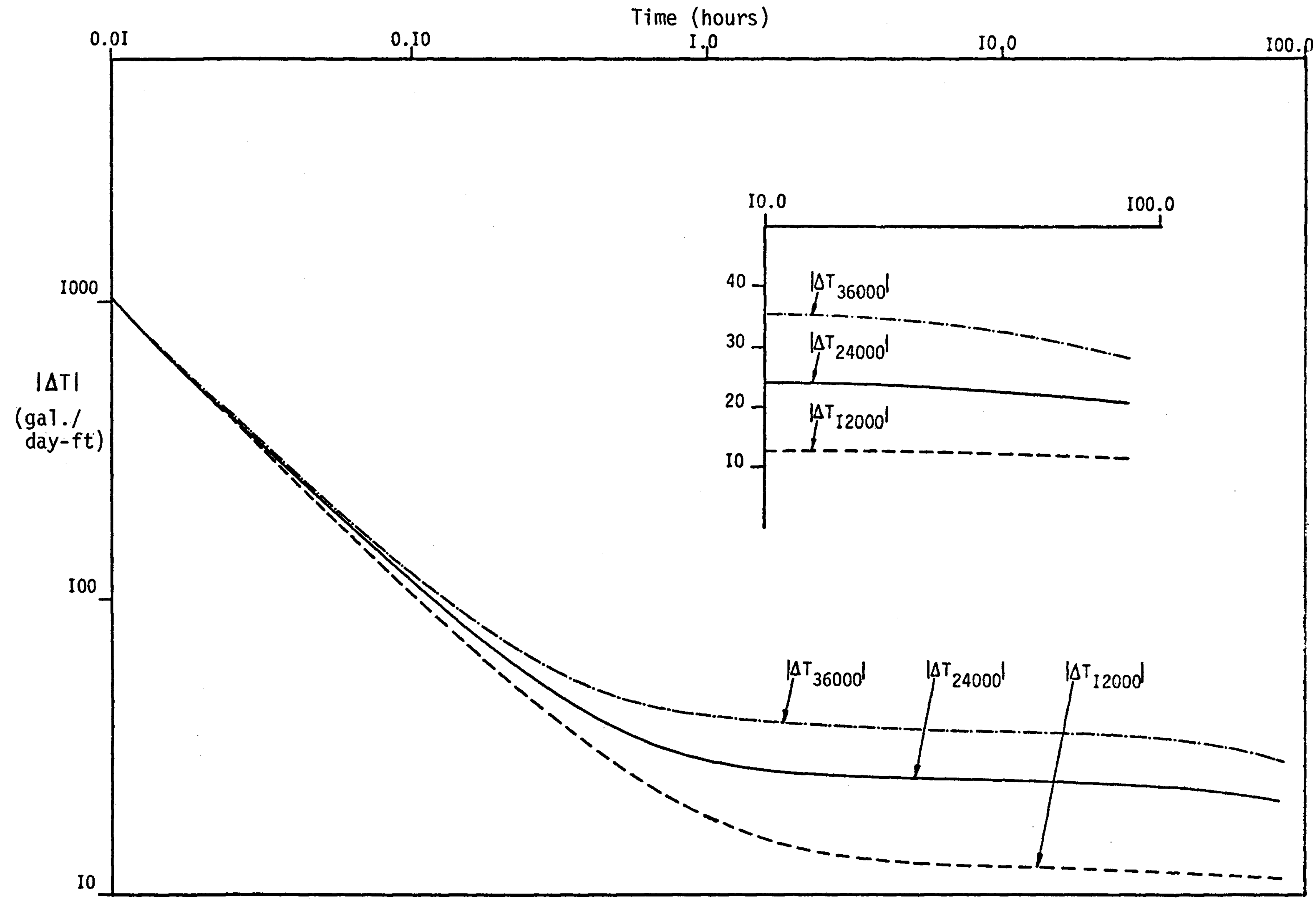


Figure 27. Maximum allowable errors in transmissivity values at the well when 2% error in drawdown value is permissible.

to increase resulting in a decrease in the ΔT values. Since increasing sensitivity values cause larger magnitude of errors in the system, these conditions must be avoided.

CASE III. NONUNIFORM TRANSMISSIVITY DISTRIBUTIONS

Uniform transmissivity and storage coefficient distributions are assumed in Case I and Case II. These assumptions are made to simplify the illustration of error distribution in the system. Since a discharging well is located at the center of the aquifer, hydraulic head, sensitivity coefficient, and error distributions are radially symmetric with respect to the well. However, nonuniform distributions of aquifer parameters, transmissivity and storage coefficient, destroy this symmetry.

Three computer runs are made to show that the nonuniform case can be studied as easily as the uniform case. The same aquifer as in Case I is considered with no flow boundaries and nonuniform transmissivity distributions:

i. T = 24,000 gal/day-ft close to the boundaries, A and B, and gradually drops to 12,000 gal/day-ft at the middle of the aquifer, c-c', (Figure 28);

ii. T = 24,000 gal/day-ft close to the boundary A and drops to 12,000 gal/day-ft close to the boundary B (Figure 29);

iii. T = 12,000 gal/day-ft near the boundaries, A and B, and gradually increases to 24,000 gal/day-ft at the middle of the aquifer, c-c', (Figure 30).

The variations with time in the head, h, and the sensitivity coefficient, u, are plotted (Figures 28, 29, and 30). Since the maximum

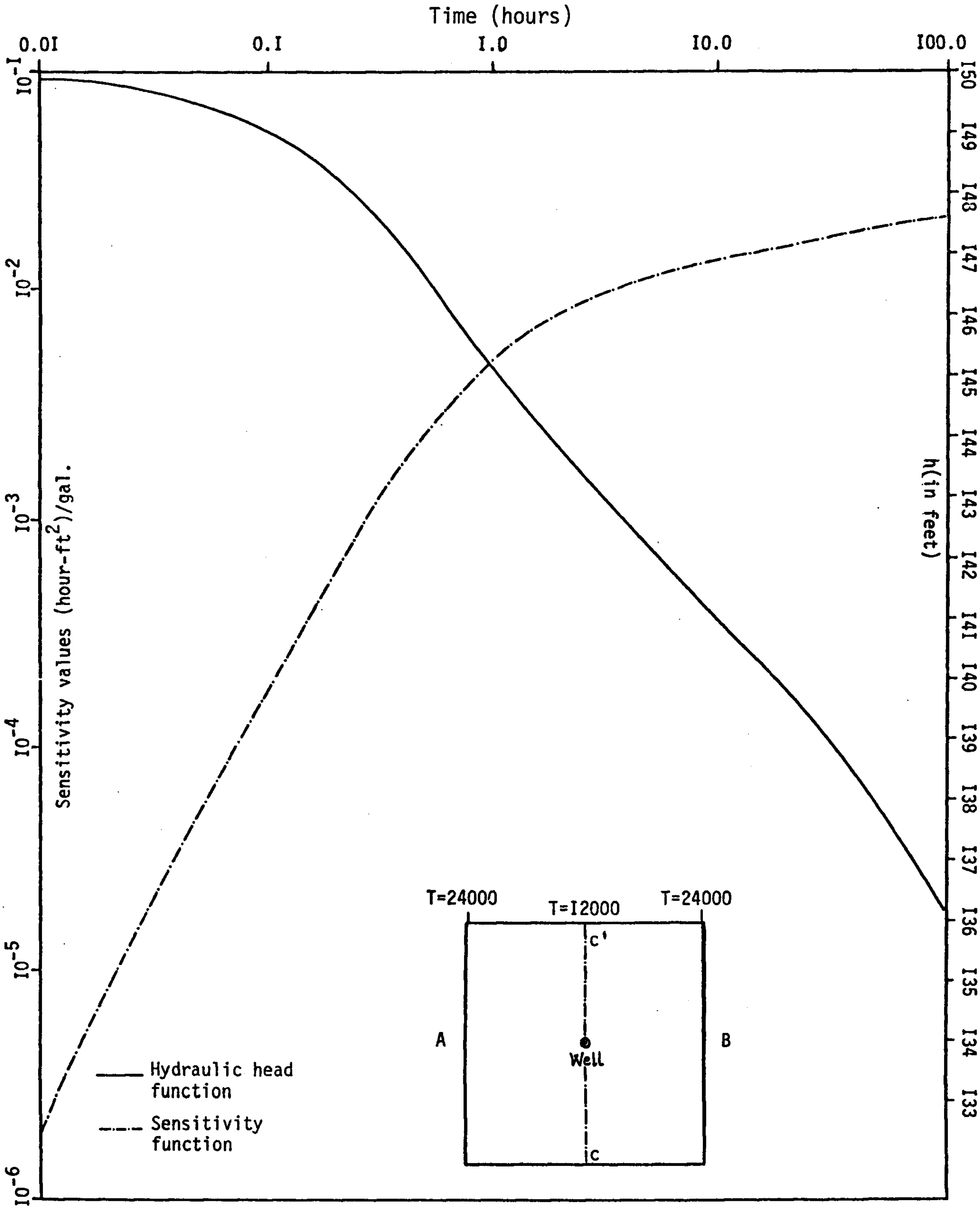


Figure 28. Change in hydraulic head and the sensitivity coefficient at the well (T=24,000 ..... T=I2,000 ..... T=24,000).

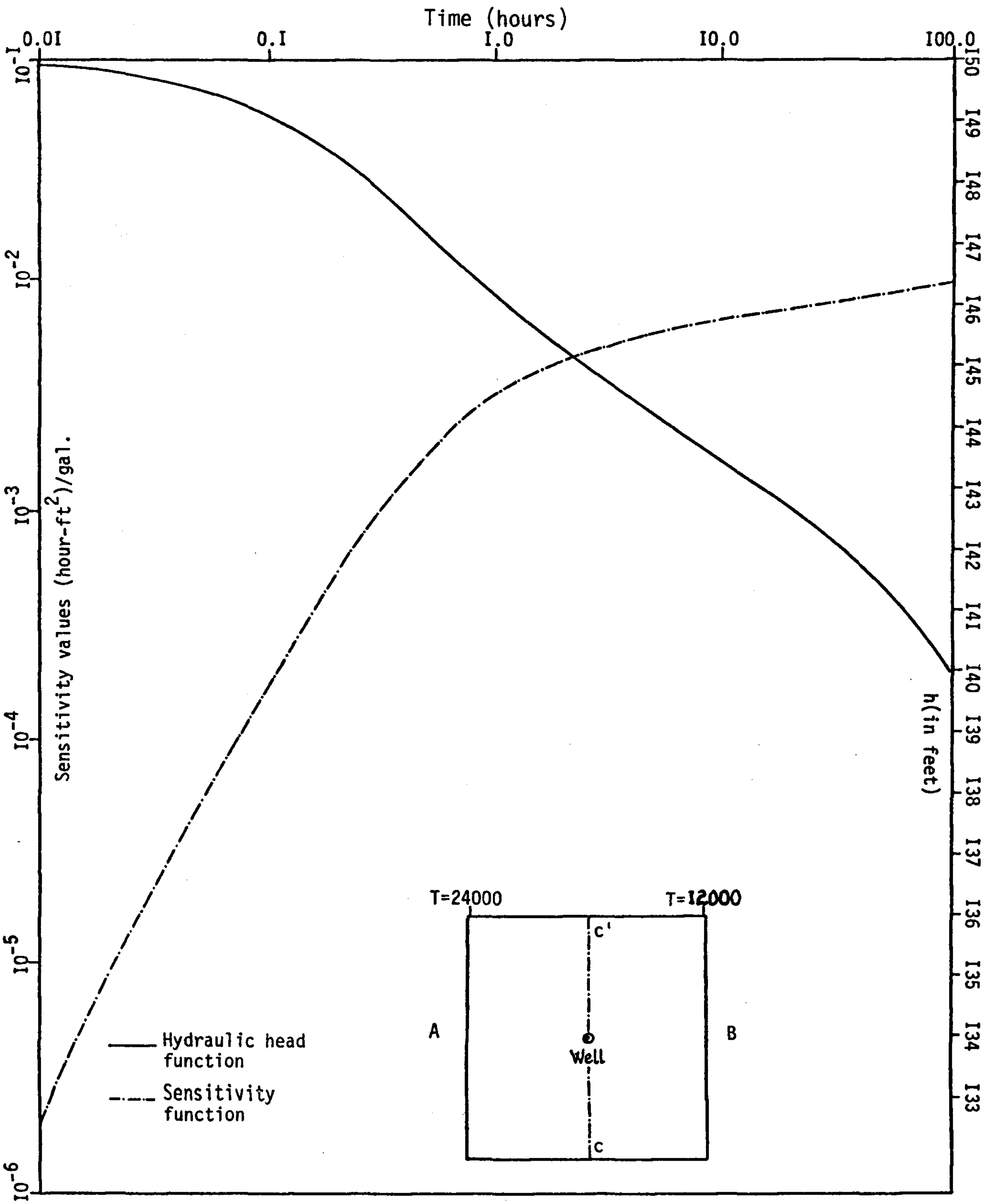


Figure 29. Change in hydraulic head and the sensitivity coefficients at the well (T=24,000 .... T=I2,000).

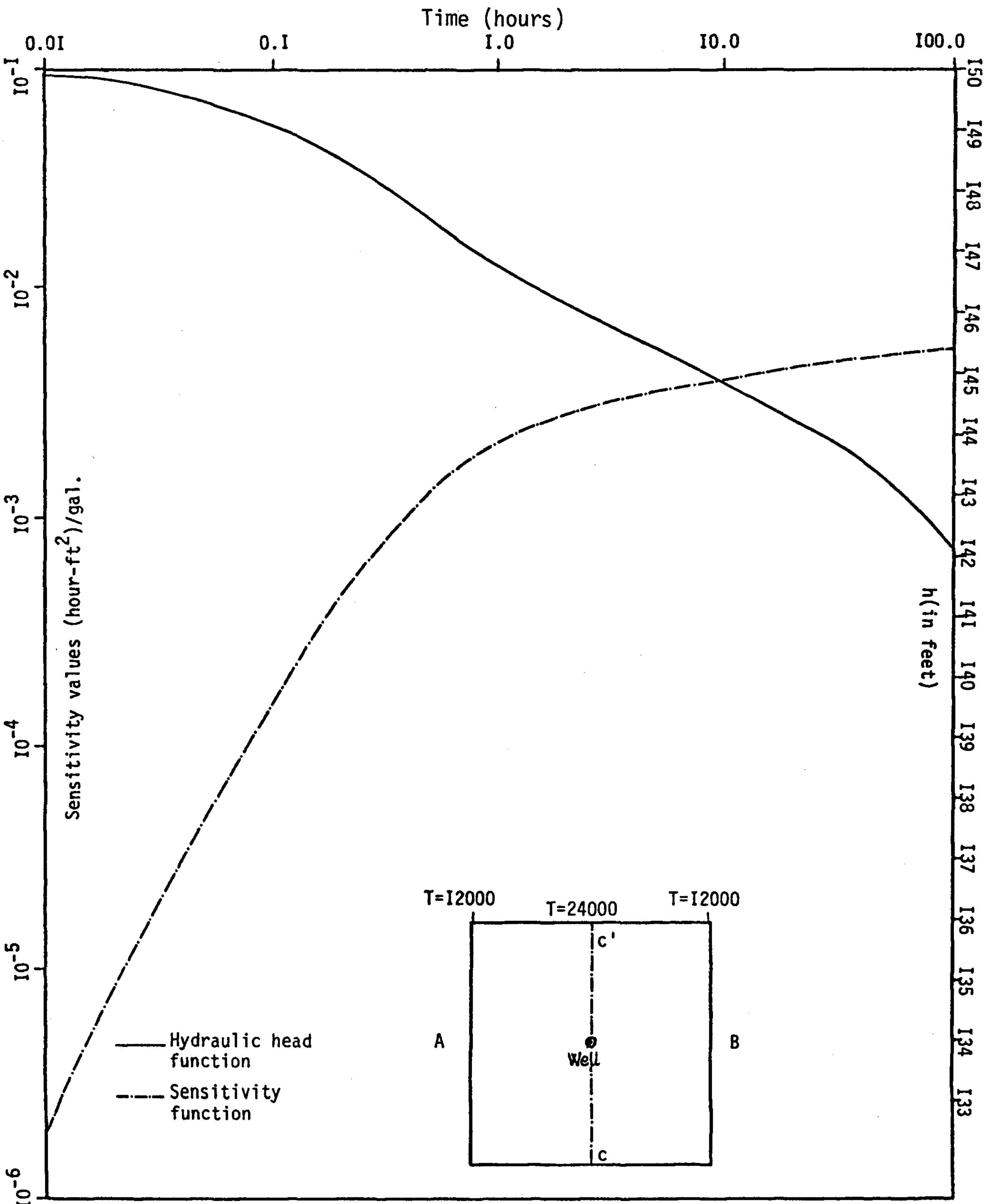


Figure 30. Change in hydraulic head and the sensitivity coefficients at the well (T=I2,000 ..... T=24,000 ..... T=I2,000).

variations in h and u, and the maximum errors, Δh, in h occur at the well only the grid point at the well is considered. The rate of drawdown is the fastest with condition (i) and the slowest with condition (iii). The cone of depression travels faster away from the well in a region of high hydraulic conductivity and a larger portion of the aquifer responds to the pumpage resulting in less drawdown. Since the hydraulic conductivity values around the well are the lowest in (i) and the highest in (iii), the magnitude of the drawdown is the greatest in (i) and the smallest in (iii).

The sensitivity coefficient values are the highest in (i) and the lowest in (iii). This is in accord with what is observed in Case I and Case II. We can conclude that the lower the hydraulic conductivity, the higher is the rate of drawdown; thus, the system is more sensitive to variations in hydraulic conductivity values.

THE SYSTEM'S SENSITIVITY WITH RESPECT TO CHANGES IN THE STORAGE COEFFICIENT

The sensitivity equation for the storage coefficient can be found on the basis of an expression analogous to (7). The procedure is similar to that used in determining the system's sensitivity with respect to transmissivity. We take the partial derivative of the flow equation (8) with respect to the storage coefficient:

$$\frac{\partial}{\partial S}\left[\frac{\partial}{\partial x} T \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} T \frac{\partial h}{\partial y} + Q\right] = \frac{\partial}{\partial S}\left[S \frac{\partial h}{\partial t}\right] \qquad (24)$$

where

$$\frac{\partial T}{\partial S} = 0 \text{ and } \frac{\partial Q}{\partial S} = 0 \qquad (25)$$

Now we can expand equation (24)

$$\frac{\partial}{\partial S}\left[\frac{\partial T}{\partial x}\cdot\frac{\partial h}{\partial x} + T\frac{\partial^2 h}{\partial x^2} + \frac{\partial T}{\partial y}\cdot\frac{\partial h}{\partial y} + T\frac{\partial^2 h}{\partial y^2} + Q\right] = \frac{\partial}{\partial S}\left[S\frac{\partial h}{\partial t}\right]$$

and take partial derivatives with respect to S

$$\frac{\partial^2 T}{\partial S\partial x}\cdot\frac{\partial h}{\partial x} + \frac{\partial T}{\partial x}\cdot\frac{\partial^2 h}{\partial S\partial x} + \frac{\partial T}{\partial S}\cdot\frac{\partial^2 h}{\partial x^2} + T\frac{\partial^3 h}{\partial S\partial x^2} + \frac{\partial^2 T}{\partial S\partial y}\cdot\frac{\partial h}{\partial y} +$$

$$\frac{\partial T}{\partial y}\cdot\frac{\partial^2 h}{\partial S\partial y} + \frac{\partial T}{\partial S}\cdot\frac{\partial^2 h}{\partial y^2} + T\frac{\partial^3 h}{\partial S\partial y^2} + \frac{\partial Q}{\partial S} = \frac{\partial h}{\partial t} + S\frac{\partial^2 h}{\partial S\partial t} \qquad (26)$$

Remembering expressions (25) and interchanging the derivatives in equation (26)

$$\frac{\partial T}{\partial x}\left\{\frac{\partial}{\partial x}\frac{\partial h}{\partial S}\right\} + T\left\{\frac{\partial^2}{\partial x^2}\frac{\partial h}{\partial S}\right\} + \frac{\partial T}{\partial y}\left\{\frac{\partial}{\partial y}\frac{\partial h}{\partial S}\right\} + T\left\{\frac{\partial^2}{\partial y^2}\frac{\partial h}{\partial S}\right\} = \frac{\partial h}{\partial t} + S\frac{\partial}{\partial t}\left\{\frac{\partial h}{\partial S}\right\} \qquad (27)$$

where $\frac{\partial h}{\partial S} = u'$ is the sensitivity coefficient of the system similar to the one given in expression (7).

Now equation (26) can be written

$$\frac{\partial T}{\partial x}\cdot\frac{\partial}{\partial x}u' + T\frac{\partial^2}{\partial x^2}u' + \frac{\partial T}{\partial y}\cdot\frac{\partial}{\partial y}u' + T\frac{\partial^2}{\partial y^2}u' = S\frac{\partial}{\partial t}u' + \frac{\partial h}{\partial t}$$

or

$$\frac{\partial}{\partial x}T\frac{\partial}{\partial x}u' + \frac{\partial}{\partial y}T\frac{\partial}{\partial y}u' = S\frac{\partial}{\partial t}u' + \frac{\partial h}{\partial t} \qquad (28)$$

This is the sensitivity equation from which the system's sensitivity with respect to the storage coefficient can be determined.

The sensitivity equation (28) is solved for two different cases: no flow boundaries and constant head boundaries. The mathematical formulation of the analysis of these sensitivity coefficients is similar to what is presented in the section on transmissivity.

SOLUTION OF THE SENSITIVITY EQUATION

The application of the Crank-Nicolson scheme to equation (28) gives

(Appendix E),

$$\frac{1}{\Delta x_i}\left[T_{i+\frac{1}{2},j}\ \frac{u_{i+1,j}^{n+\frac{1}{2}} - u_{i,j}^{n+\frac{1}{2}}}{\Delta x_{i+\frac{1}{2}}} - T_{i-\frac{1}{2},j}\ \frac{u_{i,j}^{n+\frac{1}{2}} - u_{i-1,j}^{n+\frac{1}{2}}}{\Delta x_{i-\frac{1}{2}}}\right] +$$

$$\frac{1}{\Delta y_i}\left[T_{i,j+\frac{1}{2}}\ \frac{u_{i,j+1}^{n+\frac{1}{2}} - u_{i,j}^{n+\frac{1}{2}}}{\Delta y_{j+\frac{1}{2}}} - T_{i,j-\frac{1}{2}}\ \frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j-1}^{n+\frac{1}{2}}}{\Delta y_{i-\frac{1}{2}}}\right] =$$

$$S_{i,j}\left[\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t}\right] + \frac{h_{i,j}^{n+1} - h_{i,j}^{n}}{\Delta t}$$

The unknowns in the above equation are: $u_{i,j}^{n+1}$ and $h_{i,j}^{n+1}$ but $h_{i,j}^{n+1}$ is calculated from the flow equation (8) and then the sensitivity equation is solved.

CASE I NO FLOW BOUNDARIES

The same artesian aquifer (Figure 5) is studied. In this case the effects of the disturbances in storage coefficient values are analyzed. The boundaries of the system are taken as no flow boundaries (equations 20 and 21). Since barrier boundaries are assumed, the water being pumped is supplied by the water in storage. Thus, variations in storage coefficient will directly affect the volume of water available for pumpage. Remembering that the storage coefficient is equal to the volume of water that a unit decline in head releases from storage in a vertical column of aquifer of unit cross-sectional area, the drawdown will be bigger in a system of lower storage coefficient compared to a system of higher storage coefficient provided that both systems are tapped by wells of equal discharge. Comparison of Figures 31, 32, and 33 shows that the drawdown is bigger when the storage coefficient is underestimated by 50% and lower when storage coefficient is overestimated by 50%.

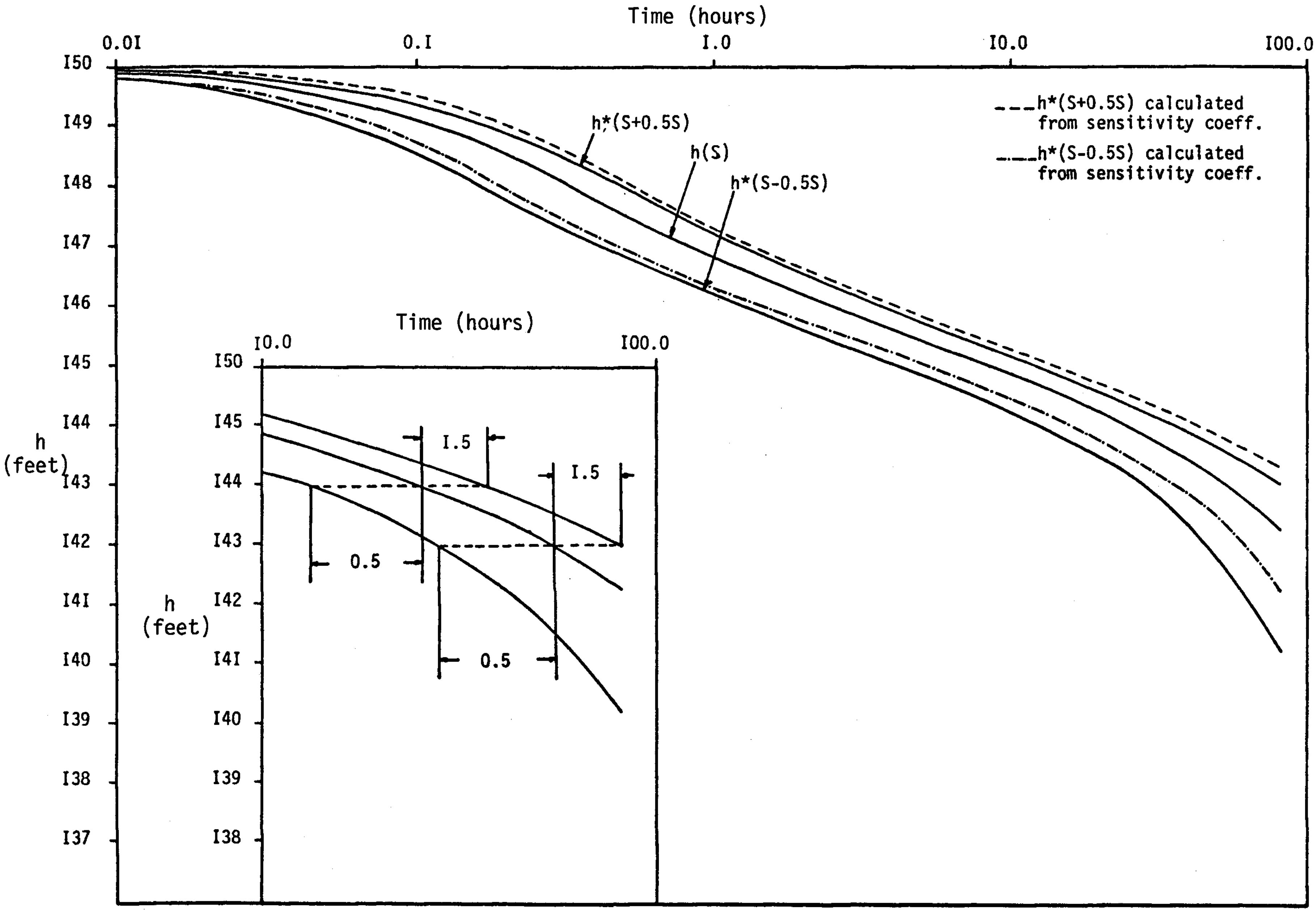


Figure 3I. Comparison of hydraulic head values determined at the well by direct solution and calculated from sensitivity coefficients for ±50% disturbance in storage coefficient values.

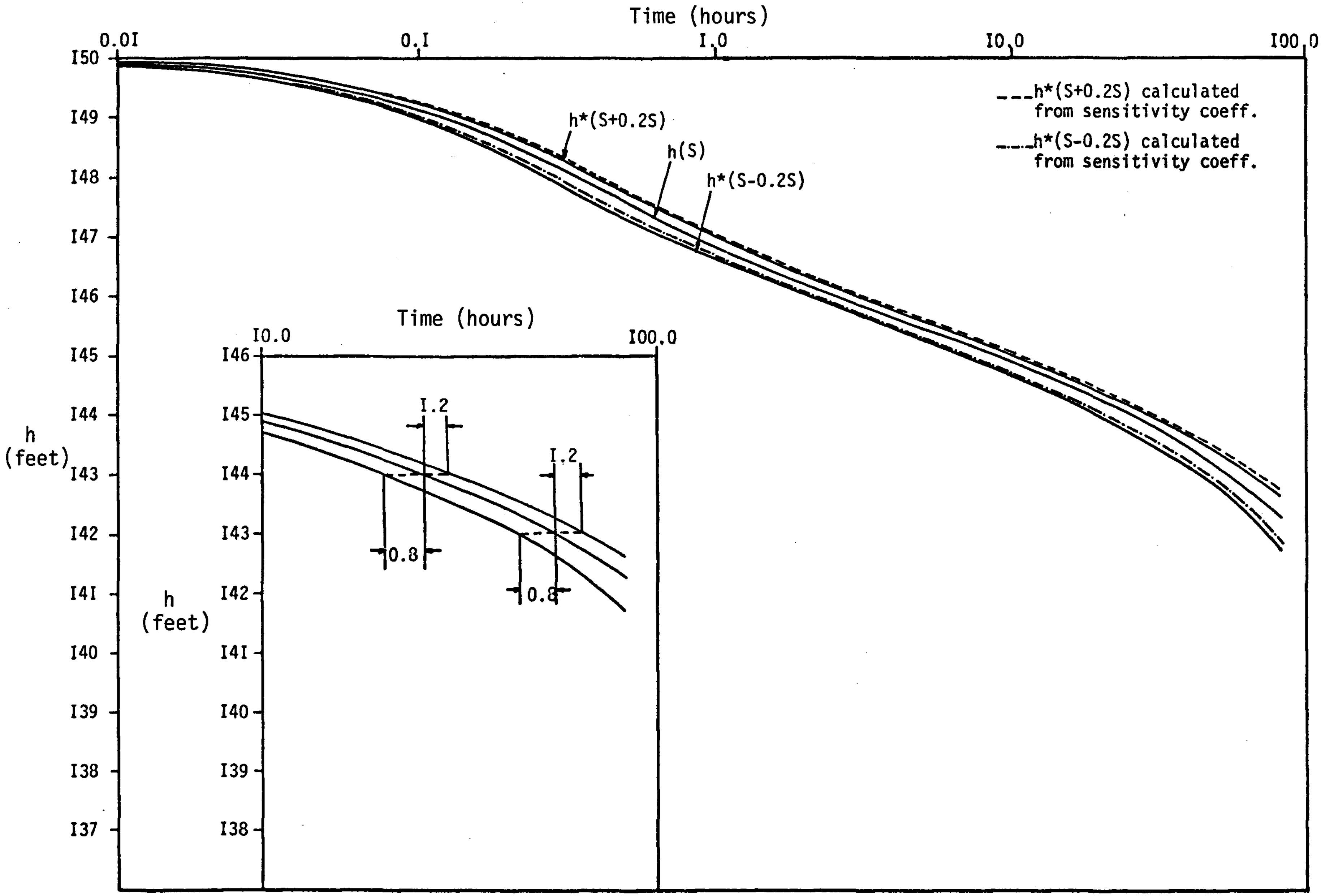


Figure 32. Comparison of hydraulic head values determined at the well by direct solution and calculated from sensitivity coefficients for ±20% disturbance in storage coefficient values.

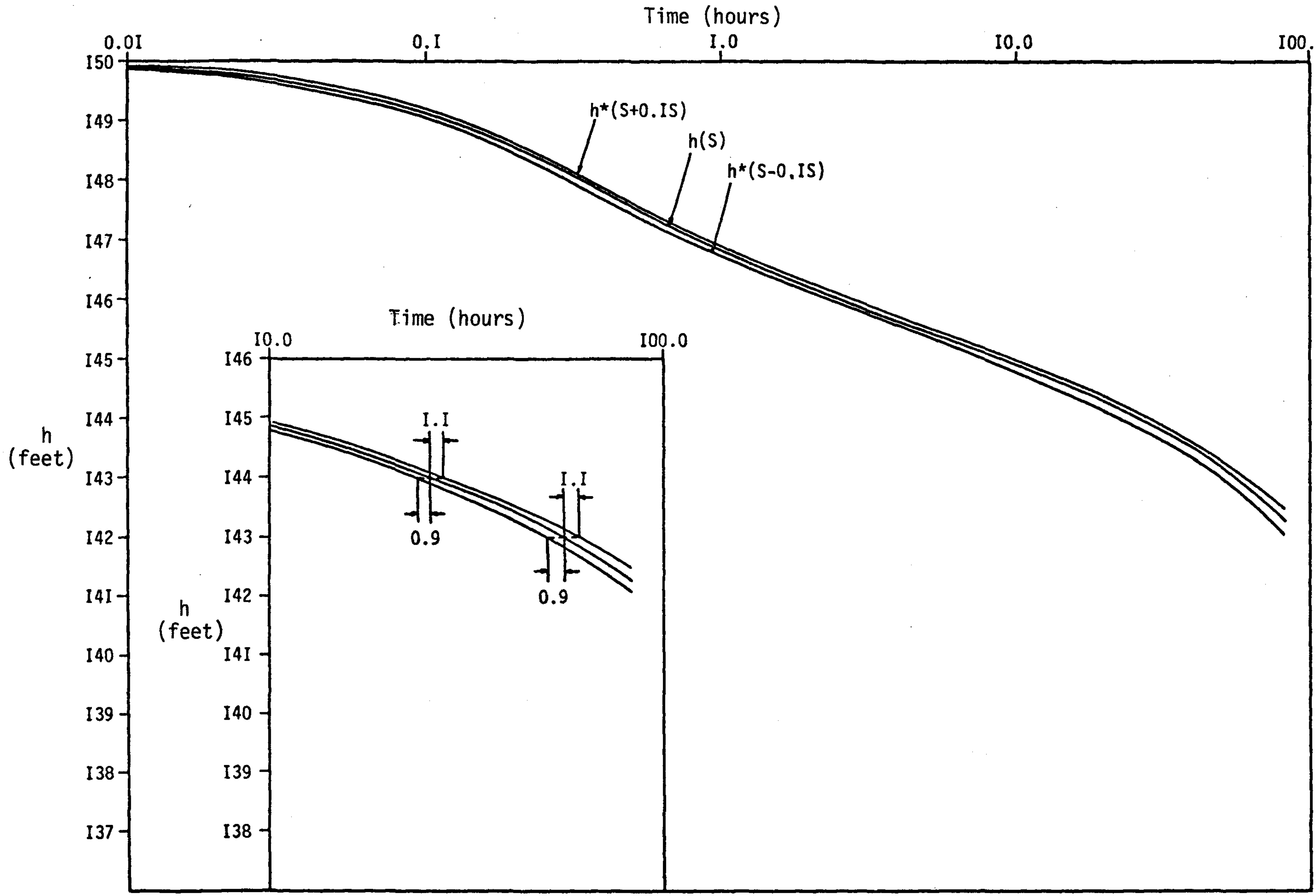


Figure 33. Comparison of hydraulic head values determined at the well by direct solution and calculated from sensitivity coefficients,these values were so close that they can not be shown separately, for ±I0% disturbance in storage coefficient values.

An interesting result is obtained from the comparison of head values for the original and perturbed cases. The change in storage coefficient merely caused phase differences in these curves with time. That is, the unperturbed head value curve is either delayed or pushed ahead in time. The time scale for the perturbed head is equal to the old time scale times the ratio of the perturbed storage coefficient to the unperturbed storage coefficient. The ratios of storage coefficients for different disturbances are;

for +50%, ratio = 0.001422/0.000948 = 1.50

+20%, ratio = 0.001138/0.000948 = 1.20

+10%, ratio = 0.001043/0.000948 = 1.10

-10%, ratio = 0.000853/0.000948 = 0.90

-20%, ratio = 0.000758/0.000948 = 0.80

-50%, ratio = 0.000474/0.000948 = 0.50

These same ratios in time are shown by inserts in Figures 31, 32, and 33. Therefore, the variation in storage coefficient merely causes a shift of the head curve in time.

The groundwater equation (8) is solved for disturbed storage coefficient values, $\pm$ 50%, $\pm$ 20%, $\pm$ 10%. The error in hydraulic head, h, is found as described in the previous section. Then the magnitudes of the truncation errors are calculated by equation (19). The percentage truncation errors, expressed as percentage of drawdown, R(%), at the well are given in table 6. Percentage errors are greater at the well at the early time steps and gradually decrease with time until the cone of depression reaches the boundary; then an increase in the truncation error is observed (Figures 31, 32, and 33). This is due to relatively

PERCENTAGE TRUNCATION ERROR,R(%),AT THE WELL

Time (hours)	R(%) (S+0.5S)	R(%) (S+0.2S)	R(%) (S+0.IS)	R(%) (S)	R(%) (S-0.IS)	R(%) (S-0.2S)	R(%) (S-0.5S)
0.0I	I6.00000	3.2I000	0.87500	---	I.06000	4.77000	47.00000
0.03	I5.00000	2.98000	0.80800	---	0.97600	4.35000	44.I7000
0.07	I3.I0000	2.57000	0.69300	---	0.82400	3.64000	33.I0000
0.I5	I0.I0000	I.94000	0.5I800	---	0.60000	2.60000	2I.80000
0.3I	6.33000	I.I6000	0.30500	---	0.34000	I.44000	II.00000
0.63	3.05000	0.54600	0.I4200	---	0.I5500	0.65600	5.I3000
I.27	I.96000	0.36400	0.09720	---	0.II300	0.49500	4.44000
2.55	I.86000	0.36200	0.09580	---	0.III00	0.47900	3.95000
5.II	I.59000	0.28I00	0.06700	---	0.II300	0.45400	3.73000
I0.23	I.43000	0.I8600	0.0I4I0	---	0.I8300	0.6I800	4.40000
20.47	I.23000	0.07550	0.04840	---	0.26500	0.8I200	5.58000
40.95	I.86000	0.22800	0.003I3	---	0.3I200	I.05000	8.I2000
8I.9I	4.05000	0.74500	0.I8200	---	0.32600	I.34000	I3.30000

Table 6. Percentage truncation error,R(%),under varying storage coefficient values at the well.

large errors in hydraulic head values at the early times when the drawdown values are small so greater percentage errors are observed at the early times (Figure 34). As time increases, drawdown values increase faster than the error in hydraulic head resulting in smaller percentage truncation errors. When the boundary effects are seen in the system, the error in hydraulic head increases suddenly which leads to increased truncation errors and increased percentage truncation errors. The percentage truncation errors are less than or equal to 1% for $\pm$ 10 disturbances and less than 5% for $\pm$ 20% disturbances.

The system's sensitivity to variations in the storage coefficient under different storage coefficient values has been calculated (Figure 35). The storage coefficient values range from 0.0014 to 0.00047, which is from high to average values. It is observed that the sensitivity coefficient values increase with a decrease in the storage coefficients; that is, the system becomes more sensitive to variations in this parameter. Thus, more accurate determination of storage coefficient values are required as these values get smaller.

Maximum values of allowable errors in the storage coefficients, ΔS, are computed using equation (22), ΔT is replaced by ΔS, and the sensitivity coefficients given by figure 35. The ΔS is computed for the systems with storage coefficients 0.0014, 0.000948, and 0.00047, respectively (Figure 36). The permissible error in Δs is assumed to be 2% of the drawdown of the unperturbed case. Examination of Figure 36 shows that permissible ΔS values increase up to a certain time and then show a rapid decrease. This rapid decrease is due to a rapid increase in the system's sensitivity when the boundary effects in the system become important (Figure 35). Since the cone of depression

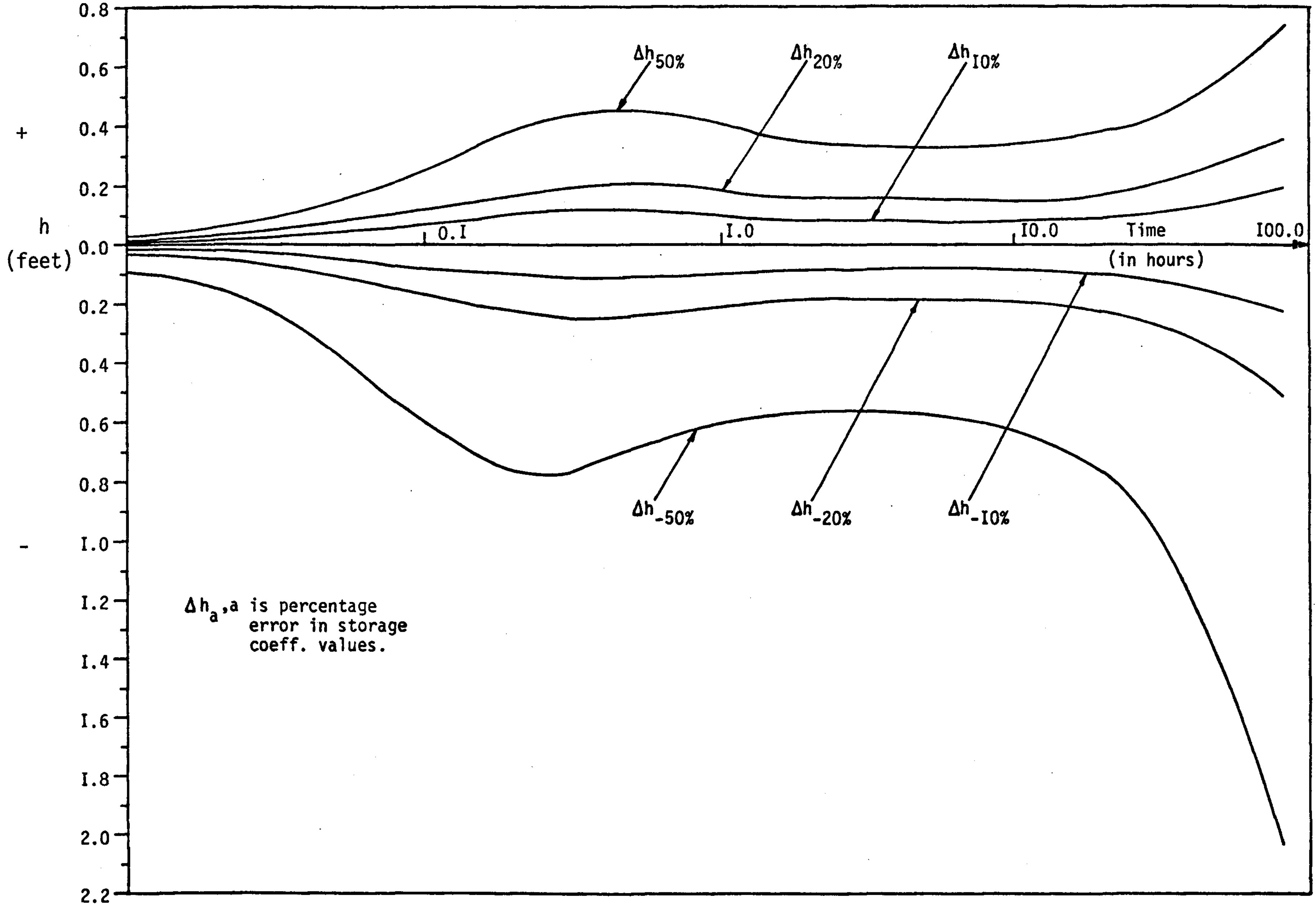


Figure 34. Distribution of error in hydraulic head, h,with time at the well.

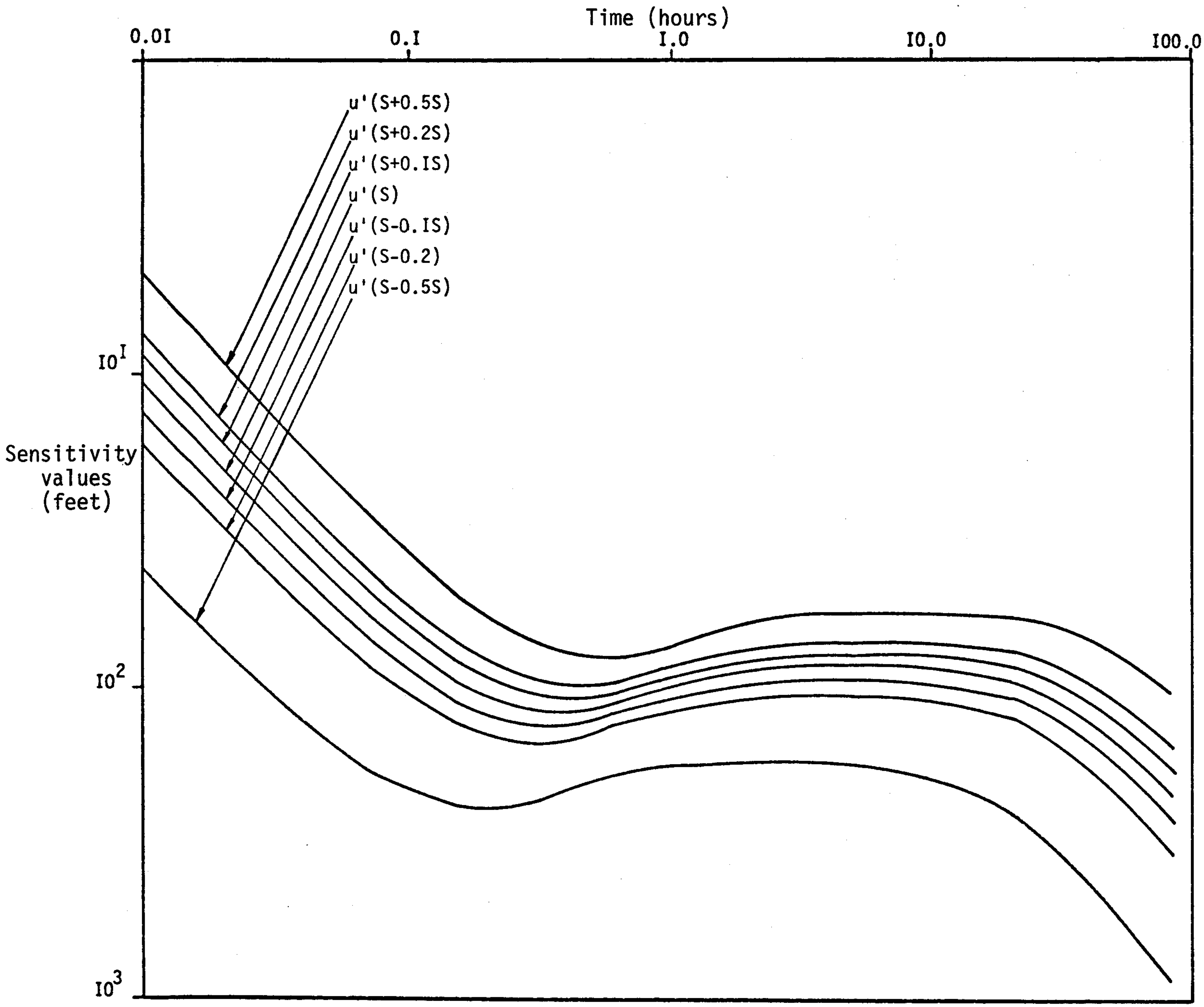


Figure 35. Variations in the sensitivity coefficients with change in storage coefficient values, at the well.

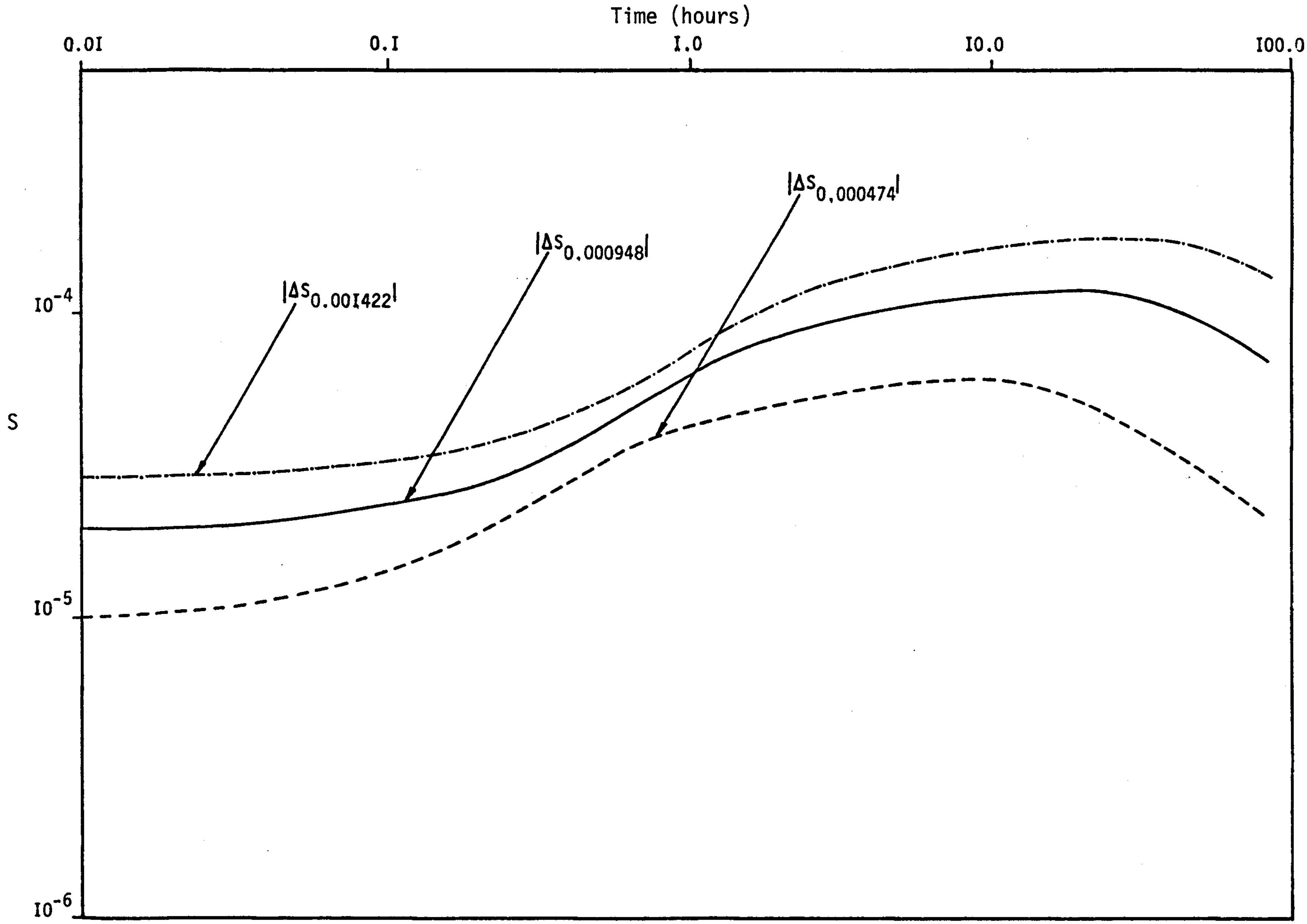


Figure 36. Maximum allowable errors in storage coefficient values at the well when 2% error in drawdown value is permissible.

reaches the boundary earlier in the lower storage coefficient case, the decrease in S value is first sensed in this case. As the system's sensitivity increases the storage coefficients must be known with great accuracy.

CASE II CONSTANT HEAD BOUNDARIES

The same system is studied under constant head boundary conditions. Initial hydraulic head values are constant, 150 feet, and sensitivity coefficients are zero along the boundaries. At the early time steps the water being pumped is supplied by the water in storage. Thus, variations in the storage coefficient will directly affect the volume of water available for pumpage and cause changes in the observed hydraulic head. However, when the cone of depression reaches the boundaries there will be an inflow of water through the boundaries to maintain the head values constant. Then the amount of water discharged is not only coming from a reduction of storage within the aquifer (i.e. from the expansion of the water and the compaction of sand) but also from the volume of water entering the system through the boundaries. The influence of the discharge progresses radially outward from the well in such a manner that the product of the rate of decline of head and the storage coefficient integrated over the area of influence equals the discharge (Jacob, 1950). Since the discharge is kept constant and some of it is supplied by water entering into the system the rate of decline of head decreases. The decrease in the rate of decline of head continues until the rate of change in hydraulic head with time is zero. This means that the system approaches a steady state case. In a steady state the sensitivity with respect to storage coefficient is zero.

At early time steps, before the boundary effects are sensed in the systems, the drawdown will be higher in the system of lower storage coefficient compared to the system of higher storage coefficient provided that the discharge is equal in these systems (Figure 37). Examination of figure 37 shows that the change in storage coefficient causes phase differences in the hydraulic head curves with time similar to what is observed in Case I.

R(%), at the well are determined as explained in Case I (Table 7). At early time steps percentage truncation errors are larger and gradually decrease as in Case I. The rate of decrease in these errors is larger than the no flow boundary case. When the boundary effects are seen in the system both the errors in head and the sensitivity coefficient values show a rapid decrease (Figures 38 and 39). The system approaches a steady state. Thus, the truncation errors become smaller. The percentage truncation errors are less than or equal to 1% for $\pm$ 10% disturbances and less than 5% for $\pm$ 20% disturbances.

The system's sensitivity to variations in the storage coefficient with different storage coefficient values, ranging from 0.0014 to 0.00047, has been calculated (Figure 39). The system is found to be more sensitive to variations in the storage coefficient when the system has low storage values. As soon as the boundary effects are seen in the system, the sensitivity coefficients show rapid declines. This is seen first in the system of lowest storage where the cone of depression reaches the boundaries first. This decline in sensitivity is due to the fact that the system is approaching a steady state. The right hand side of the flow equation (8) becomes zero. Therefore,

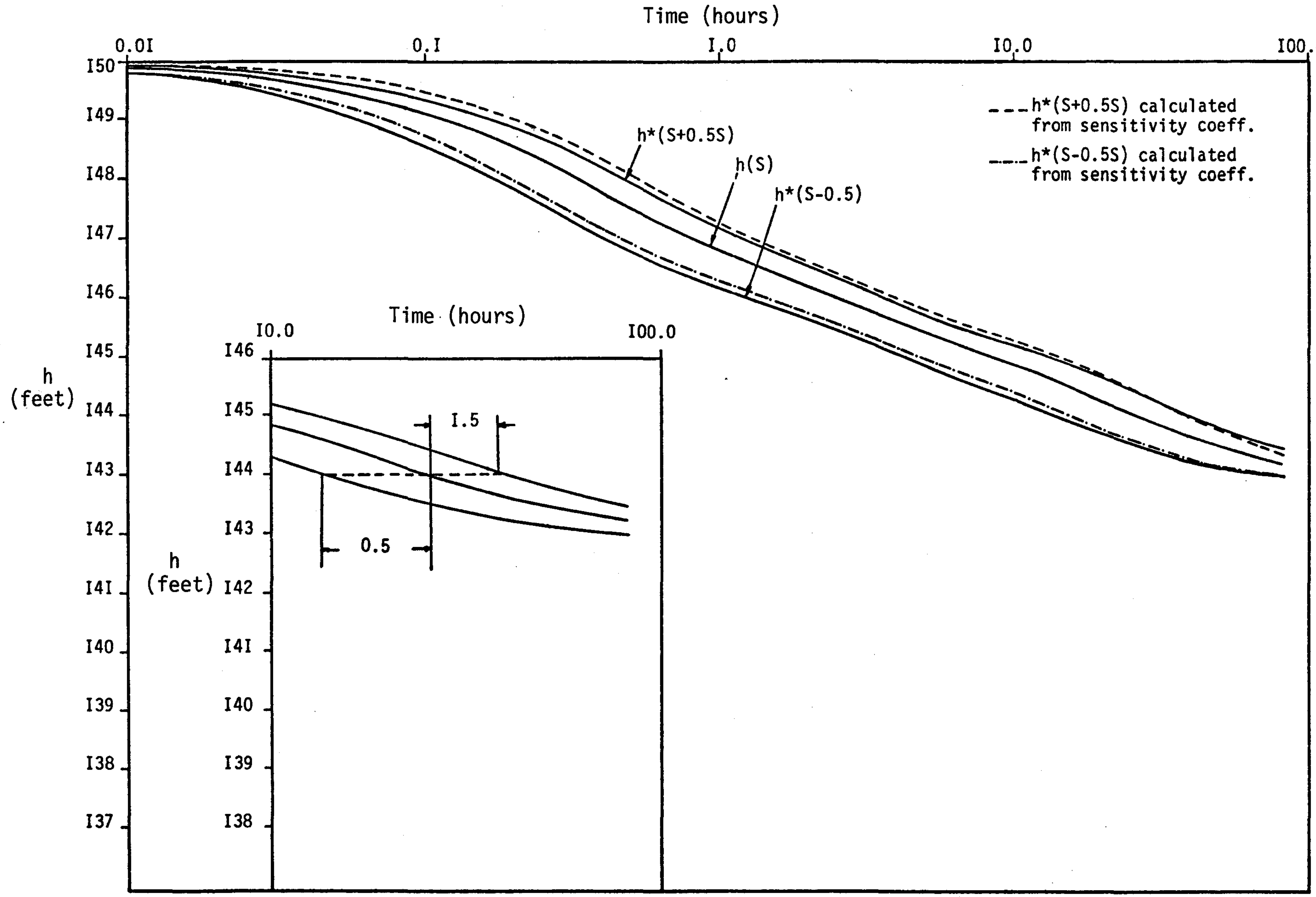


Figure 37. Comparison of hydraulic head values at the well determined by direct solution and calculated from sensitivity coefficients for ±50% disturbance in the storage coefficient values.

Time (hours)	R(%) (S+0.5S)	R(%) (S+0.2S)	R(%) (S+0.IS)	R(%) (S)	R(%) (S-0.IS)	R(%) (S-0.2S)	R(%) (S-0.5S)
			PERCENTAGE TRUNCATION ERROR,R(%),AT THE WELL				
0.0I	I6.I0000	3.2I000	0.87400	---	I.06000	4.77000	47.00000
0.03	I5.00000	2.98000	0.80800	---	0.97600	4.35000	4I.70000
0.07	I3.I0000	2.57000	0.69300	---	0.82500	3.64000	33.I0000
0.I5	I0.20000	I.94000	0.5I800	---	0.59900	2.60000	2I.80000
0.3I	6.33000	I.I6000	0.30500	---	0.34000	I.44000	II.00000
0.63	3.07000	0.54600	0.I4200	---	0.I5500	0.65600	5.I5000
I.27	I.93000	0.36400	0.09720	---	0.II300	0.49500	4.33000
2.55	I.90000	0.36300	0.09640	---	0.II000	0.47500	3.89000
5.II	I.6I000	0.30000	0.07920	---	0.09I90	0.39700	3.37000
I0.23	I.47000	0.26400	0.064I0	---	0.I0300	0.4I500	3.35000
20.47	I.09000	0.I2300	0.03390	---	0.I6000	0.5I300	3.28000
40.95	0.I7I00	0.I8I00	0.I3800	---	0.24300	0.59700	2.I7000
8I.9I	0.I5300	0.63400	0.3I200	---	0.28000	0.49000	0.05I90

Table 7. Percentage truncation error,R(%),with varying storage coefficient values at the well.

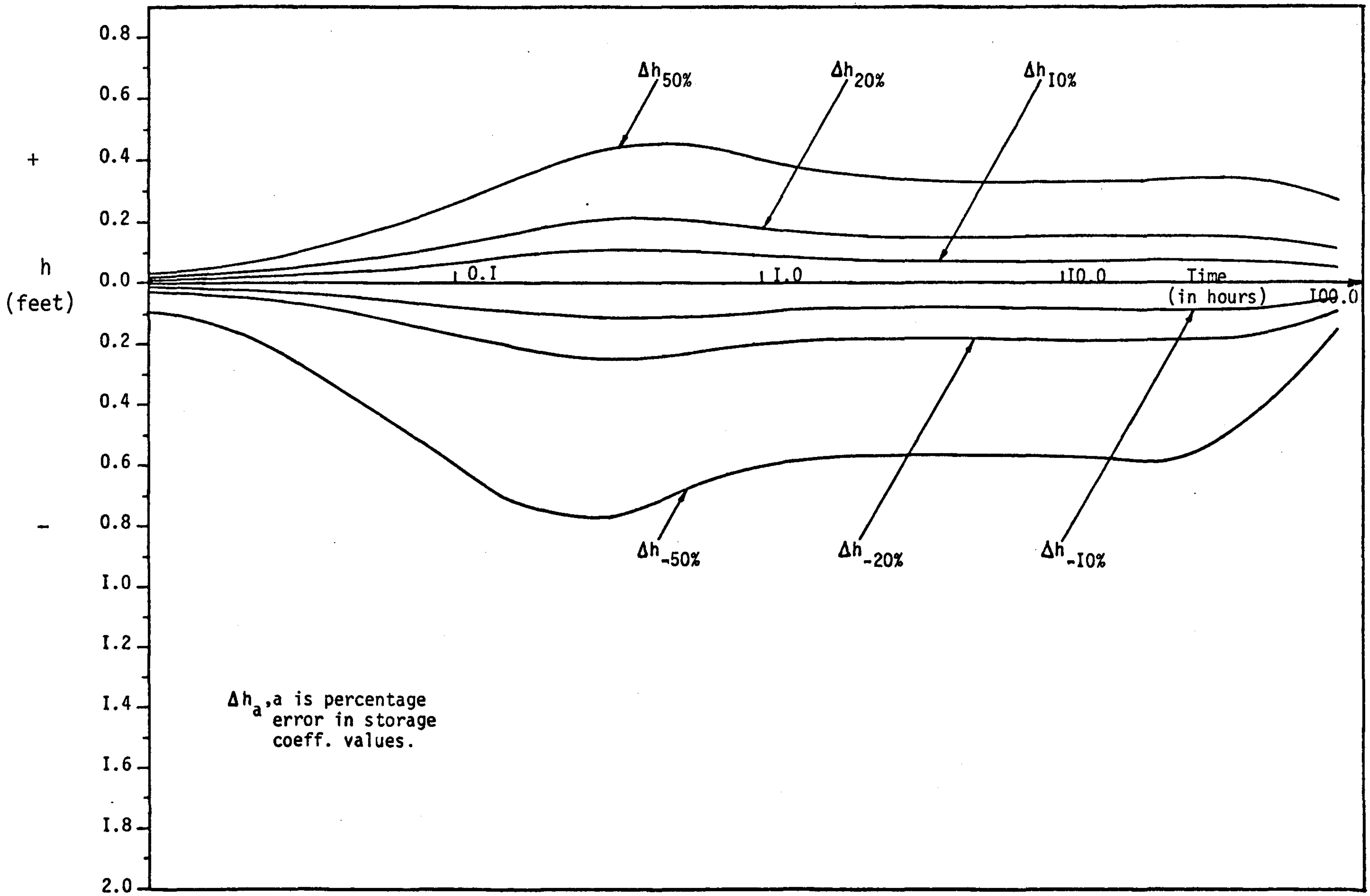


Figure 38. Distribution of errors in hydraulic head, h, with time at the well.

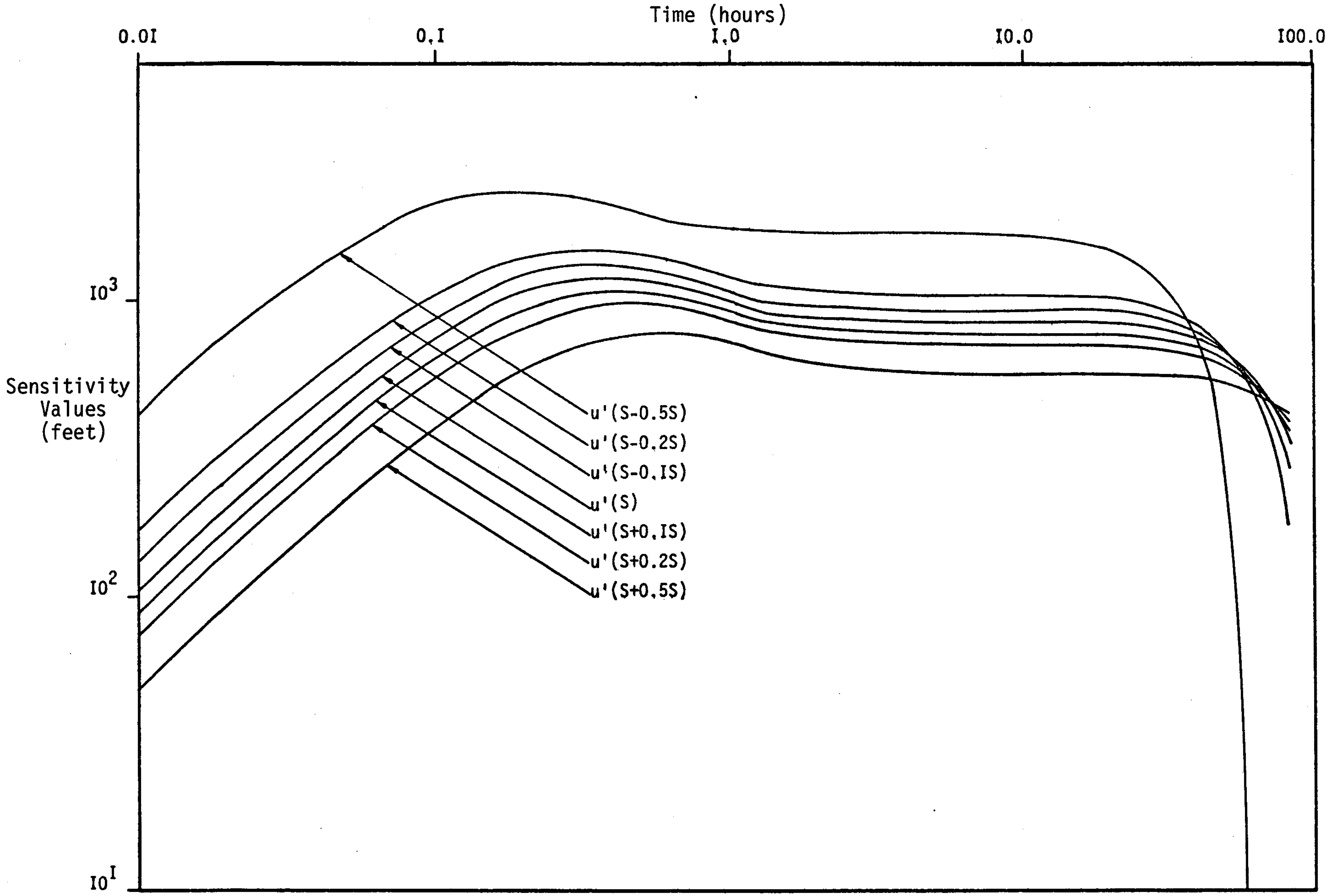


Figure 39. Variations in the sensitivity coefficient values at the well with a change in the storage coefficient.

in a steady state condition the system is indifferent to variations in storage coefficient; thus the sensitivity coefficients are zero.

Allowable errors in the storage coefficient (ΔS), when 2% error in drawdown values is permitted, are computed using equation (23) and the sensitivity coefficients given by figure 39 (Figure 40). These allowable errors, ΔS, show a rapid increase when the boundary effects are seen in the system. The first rapid response is in the system of smallest storage coefficient. This rapid increase in ΔS is due to a rapid decrease in the sensitivity of the system to the storage coefficient. Since the ΔS value can be used as a direct measure of the sensitivity of the flow system, the rapid increase in it shows that the system is becoming insensitive to variations in the storage coefficient.

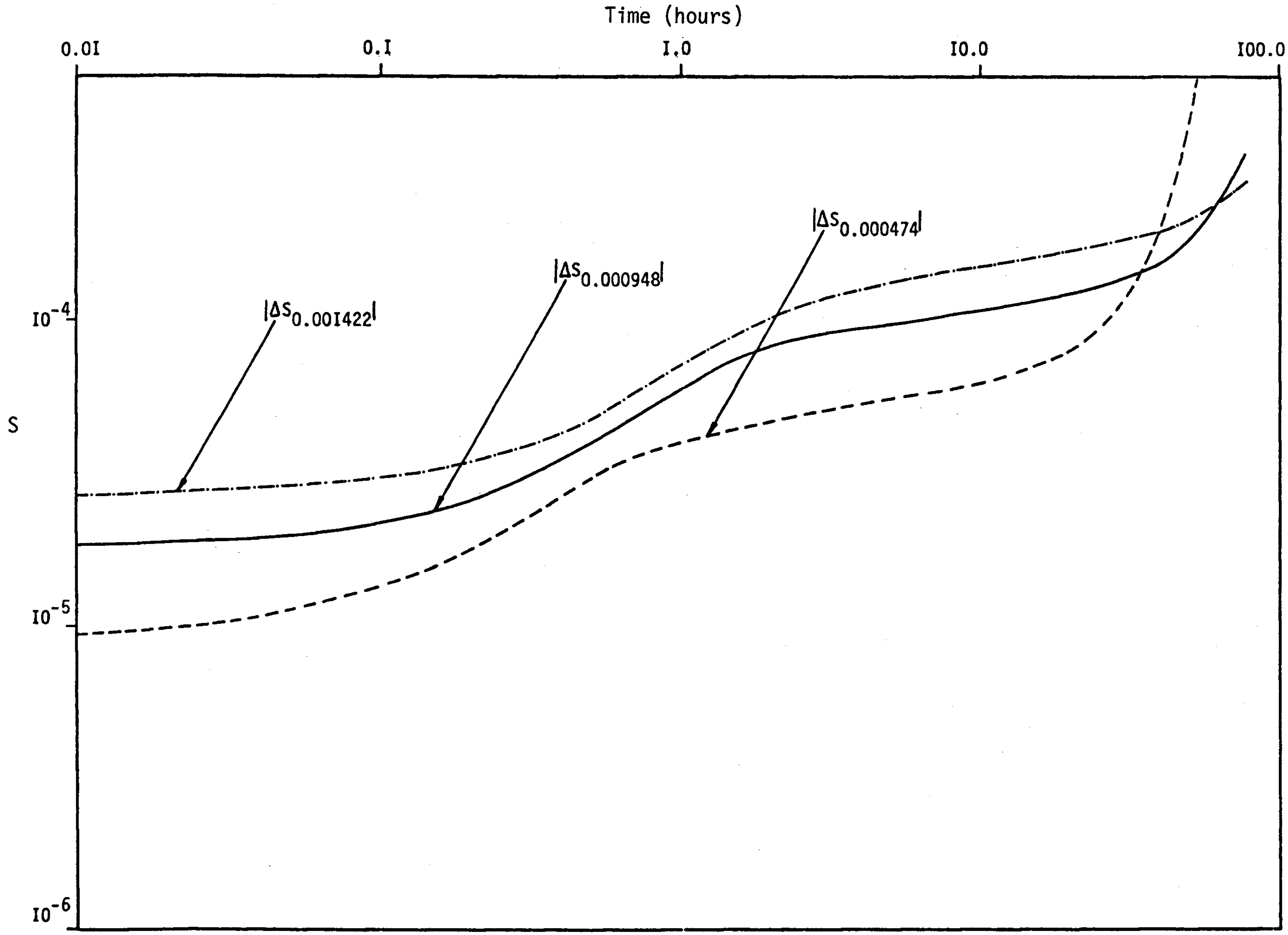


Figure 40. Maximum allowable errors in the storage coefficient values at the well when a 2% error in drawdown value is permissible.

SENSITIVITY ANALYSIS OF UNCONFINED FLOW SYSTEMS

SENSITIVITY WITH RESPECT TO HYDRAULIC CONDUCTIVITY VARIATIONS

The equation that describes the unconfined groundwater flow system is

$$\frac{\partial}{\partial x} Kh \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Kh \frac{\partial h}{\partial y} + Q = Sy \frac{\partial h}{\partial t} \qquad (2)$$

where $K = K(x,y)$.

The sensitivity equation is obtained by taking the partial derivative of the above equation with respect to K.

$$\frac{\partial}{\partial K}\left[\frac{\partial}{\partial x} Kh \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Kh \frac{\partial h}{\partial y} + Q\right] = \frac{\partial}{\partial K}\left[Sy \frac{\partial h}{\partial t}\right] \qquad (29)$$

Remembering that Q, S, and T are assumed independent of each other, i.e.,

$$\frac{\partial Sy}{\partial K} = 0 \quad \text{and} \quad \frac{\partial Q}{\partial K} = 0$$

we can expand equation (29)

$$\frac{\partial}{\partial K}\left[K \frac{\partial h}{\partial x} \cdot \frac{\partial h}{\partial x} + h \frac{\partial K}{\partial x} \cdot \frac{\partial h}{\partial x} + Kh \frac{\partial^2 h}{\partial x^2} + K \frac{\partial h}{\partial y} \cdot \frac{\partial h}{\partial y} + h \frac{\partial K}{\partial y} \cdot \frac{\partial h}{\partial y} + Kh \frac{\partial^2 h}{\partial y^2} + Q\right] = \frac{\partial}{\partial K}\left[Sy \frac{\partial h}{\partial t}\right]$$

take the partial derivatives with respect to K, and rearrange equation (29)

$$\frac{\partial}{\partial x} Kh \frac{\partial}{\partial x} u_c + \frac{\partial}{\partial x} Ku_c \frac{\partial h}{\partial x} + \frac{\partial}{\partial x} h \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Kh \frac{\partial}{\partial y} u_c + \frac{\partial}{\partial y} Ku_c \frac{\partial h}{\partial y} + \frac{\partial}{\partial y} h \frac{\partial h}{\partial y} = Sy \frac{\partial}{\partial t} u_c \qquad (30)$$

This is the sensitivity equation of the unconfined flow system with respect to variations in hydraulic conductivity. The detailed derivation of the equation is given in Appendix D. We shall study the sensitivity of the unconfined system with no flow boundaries and constant head boundaries.

SOLUTION OF THE SENSITIVITY EQUATION

The sensitivity equation (30) is solved with Crank-Nicolson scheme and the time extrapolation routine (Appendix E).

CASE I. NO FLOW BOUNDARIES

The same aquifer described by figure 5 is considered with the aquifer parameters as follows:

Hydraulic conductivity (K_1) = 9600 gal/day-ft^2

Hydraulic conductivity (K_2) = 355.2 gal/day-ft^2

Initial water level (h_o) = 150 feet

Specific yield (Sy_1) = 0.15

Specific yield (Sy_2) = 0.24

Well discharge (Q) = 240,000 gal/day

All the four boundaries are barrier boundaries. The boundary conditions for the head function and the sensitivity equation are as given by equations (16), and (17), respectively. Uniform hydraulic conductivity and specific yield are assumed throughout the aquifer. These assumptions are made to simplify the discussion of error distribution in the flow system. The more general case can easily be handled by the method developed here.

Two different sets of hydraulic conductivity and specific yield values are used to study the performance of the system with different aquifer parameters;

i. K_1, h_o, Sy_1, Q

K_1 and Sy_1 values are common in good aquifers composed of clean sands and gravels (Figures 41 and 42). The K_1 values are perturbed by $\pm$ 50%, $\pm$ 20%, and $\pm$ 10% and the change in the performance of the system is determined. Since K_1 values are quite high the drawdown at the well is small. Nevertheless, the effects of disturbance in K_1 values are easily observed from figures 43, 44, and 45. The declines in water levels with time are calculated from equation (2) and from the sensitivity coefficients determined by equation (30). The differences between these two sets of water levels are due to truncation errors, R. The truncation errors are less than 23% of the drawdown for ± 50%, less than 4% **for ± 20% and less than** 1.3% for ± 10% disturbances in K_1.

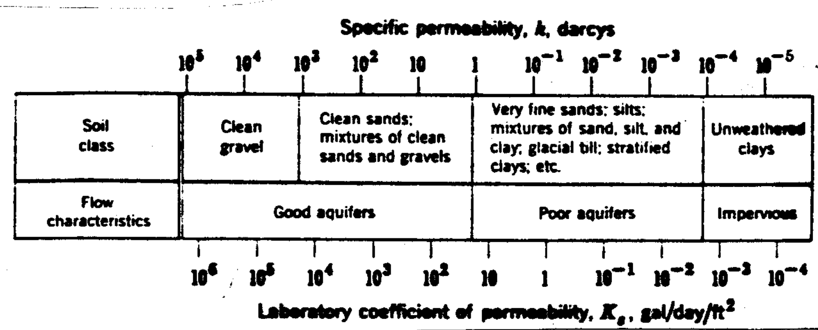


Figure 4I. Hydraulic conductivity for different soil types (Todd,I959).

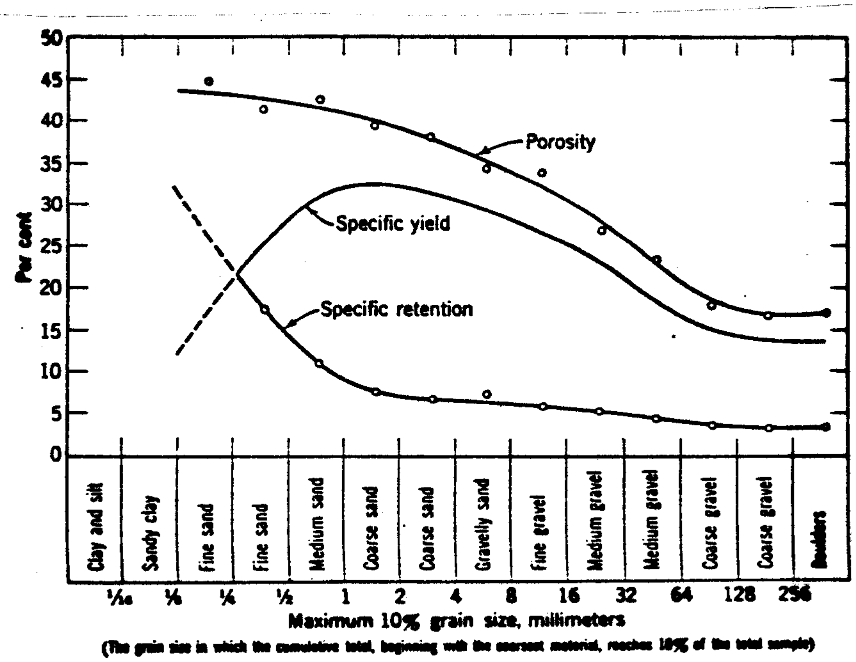


Figure 42. Porosity,specific yield,and specific retention variations with grain size (Todd,I959).

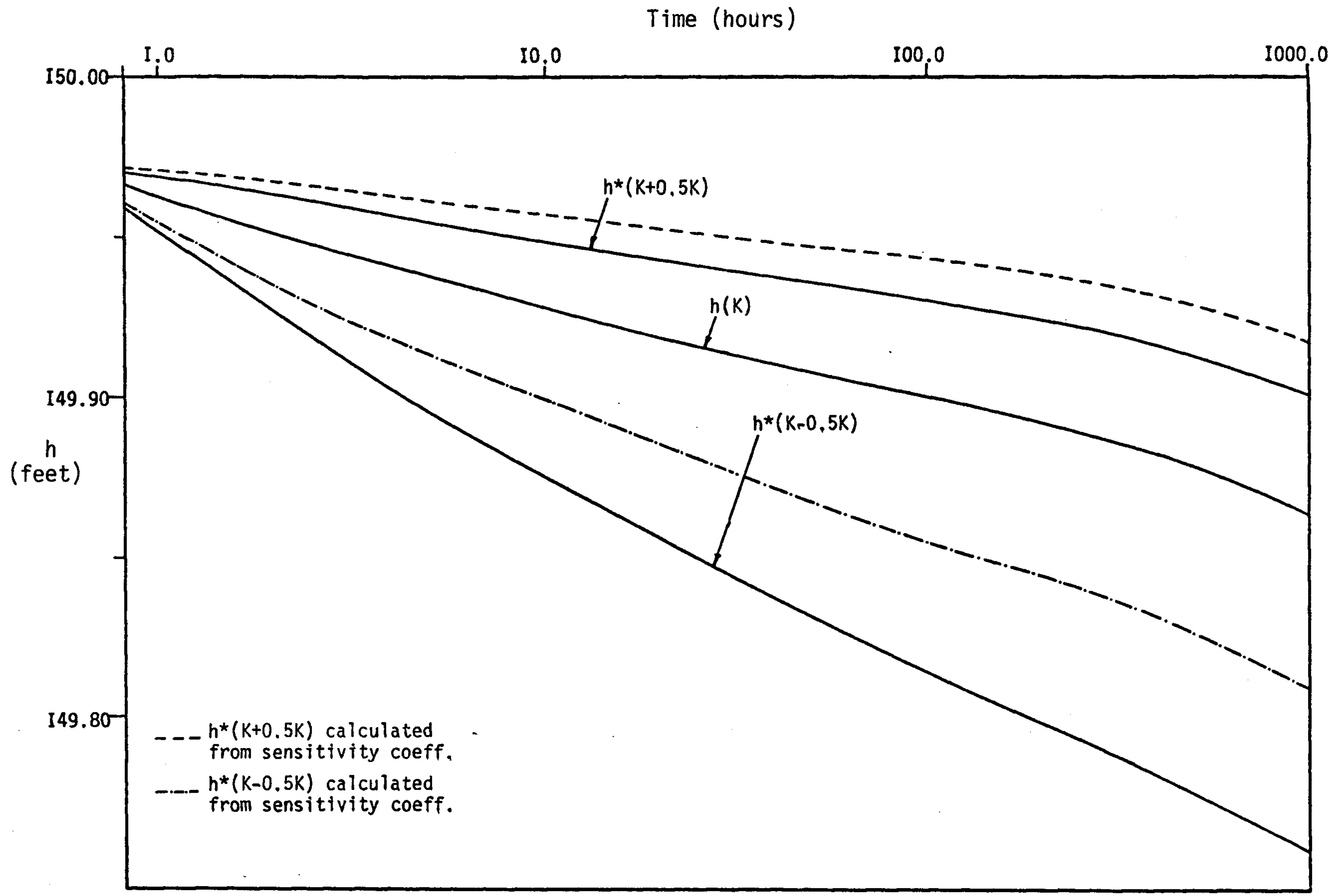


Figure 43. Comparison of the hydraulic head values determined at the well by direct solution and calculated from the sensitivity coefficients for ±50% disturbance in the hydraulic conductivity (K=9600 gal./day-ft^2, S_y=0.I5).

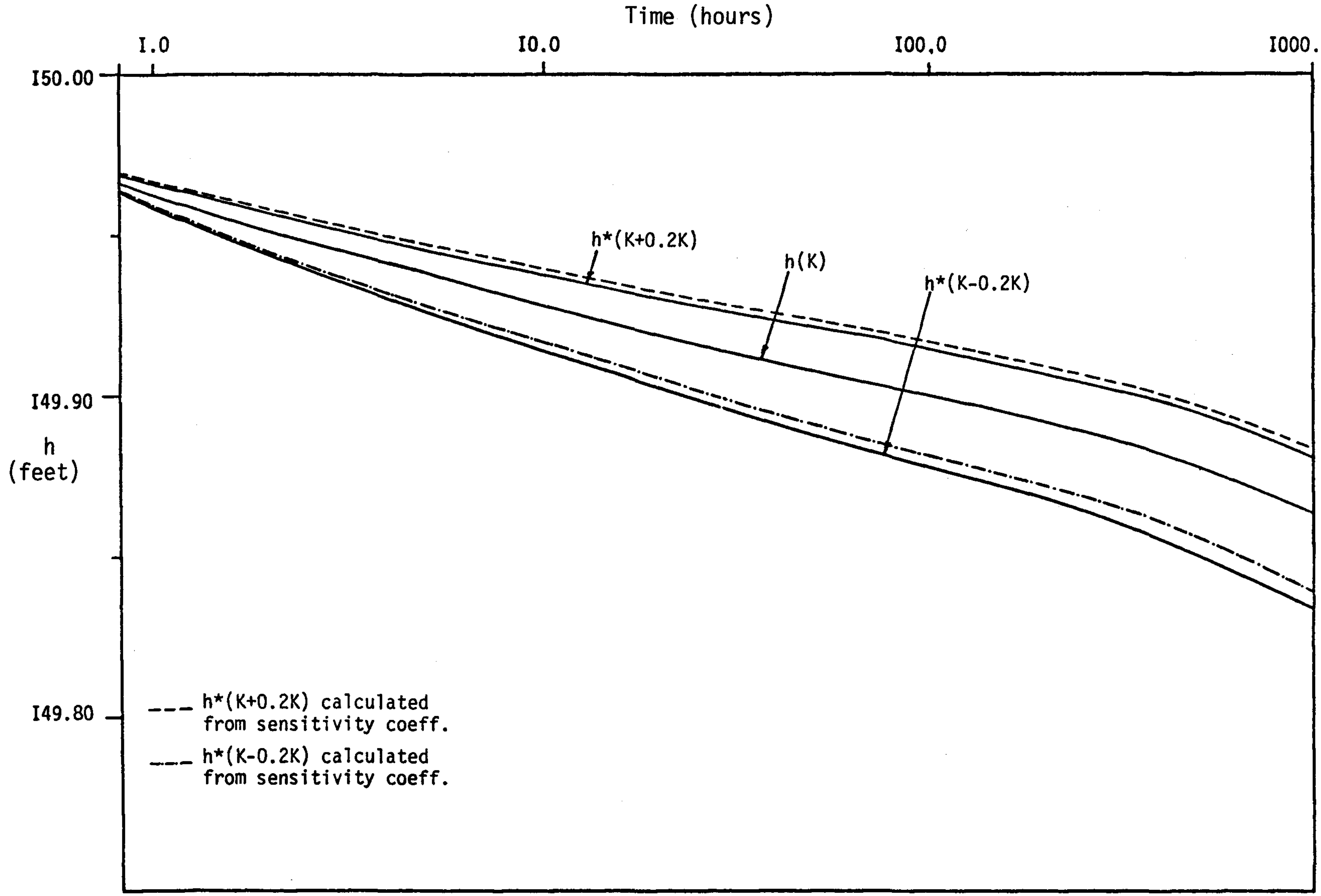


Figure 44. Comparison of the hydraulic head values determined at the well by direct solution and calculated from the sensitivity coefficients for ±20% disturbance in the hydraulic conductivity (K=9600 gal./day-ft^2, S_y=0.I5).

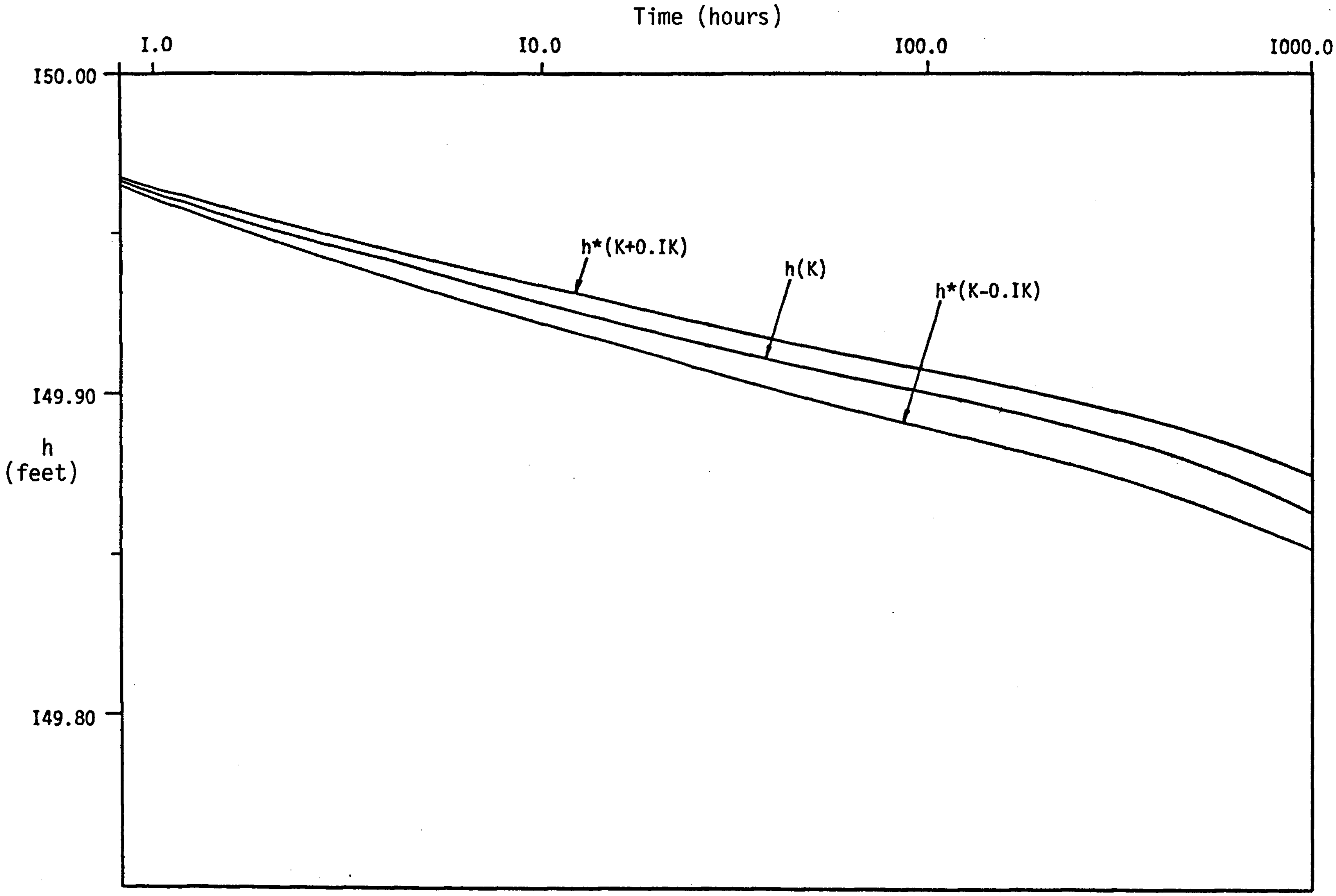


Figure 45. Comparison of the hydraulic head values determined at the well by direct solution and calculated from the sensitivity coefficients for ±I0% disturbance in the hydraulic conductivity (K=9600 gal./day-ft^2,S_y=0.I5).

ii. K_2, h_0, Sy_2, Q

K_2 and Sy_2 values are common in good aquifers composed of clean sands and gravel with some fine material. The K_2 values are considerably smaller than K_1 and the drawdown values are quite large compared to the previous case. The water levels are calculated from equation 2 and determined by equation 17 for the original and perturbed K_2 values (Figures 46 and 47). The percentage truncation errors, percent of drawdown, are less than 21% for $\pm$ 50%, less than 3.3% for $\pm$ 20%, and less than 1% for $\pm$10% disturbances in K_2 values.

Although the percentage truncation errors are larger in the aquifer with K_1 and Sy_1 values the truncation errors in feet are quite small. These errors appear quite large since they are divided by the drawdown values which are very small.

The comparison of the sensitivity coefficient values for K_1, perturbed K_1, K_2, and perturbed K_2 illustrates that the system of high hydraulic conductivity is not very sensitive to variations in hydraulic conductivity (Figure 48). Hence, the errors in water levels are less, remembering that error in water level is equal to the product of error in hydraulic conductivity times the sensitivity coefficient.

CASE II. CONSTANT HEAD BOUNDARIES

The same aquifer as in Case I is considered with the aquifer parameters K_2, h_0, Sy_2, and Q. The water levels on the four boundaries are maintained at 150 feet. Since the water levels are constant on the boundaries, the sensitivity coefficients are zero on the boundaries.

The drawdown and the sensitivity coefficients in the system are calculated for the original and perturbed, $\pm$ 50%, $\pm$ 20% and $\pm$ 10%,

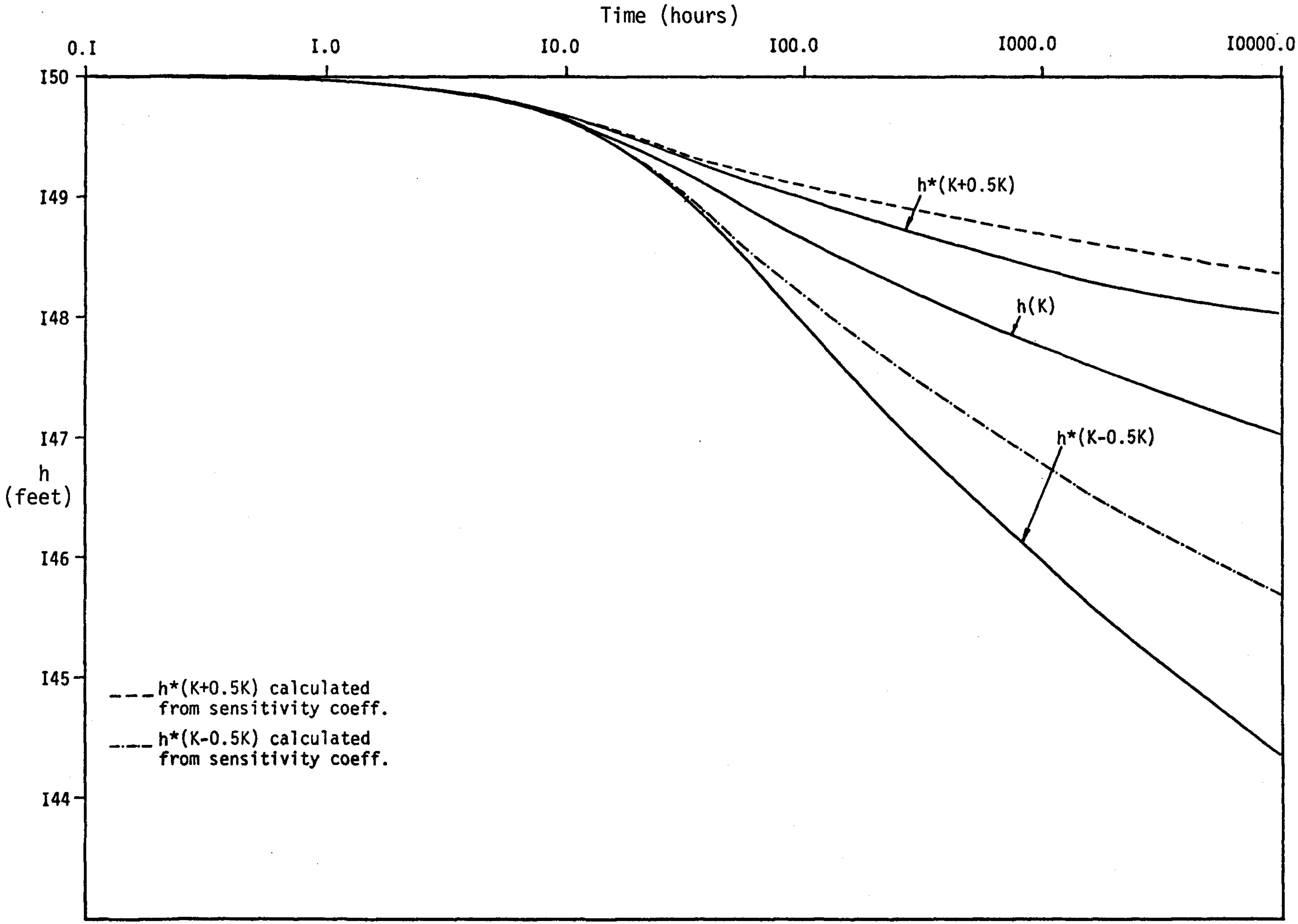


Figure 46. Comparison of the hydraulic head values determined at the well by direct solution and calculated from the sensitivity coefficients for ±50% disturbance in the hydraulic conductivity (K=355.2 gal./day-ft^2,S_y=0.24).

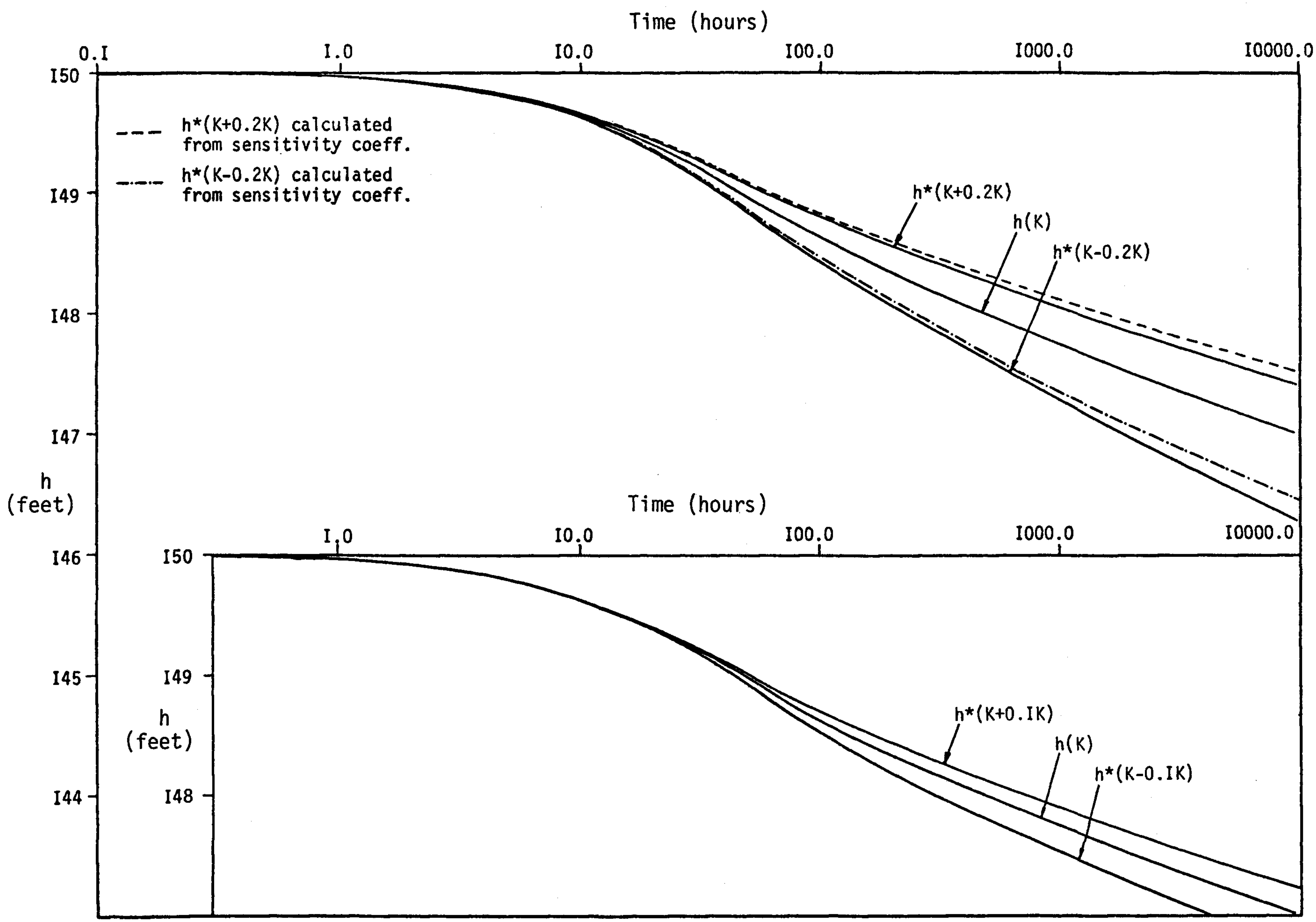


Figure 47. Comparison of the hydraulic head values determined at the well by direct solution and calculated from the sensitivity coefficients for ±20% and ±10% disturbance in the hydraulic conductivity ($K=355.2 gal/day-ft^2, S_y=0.24$).

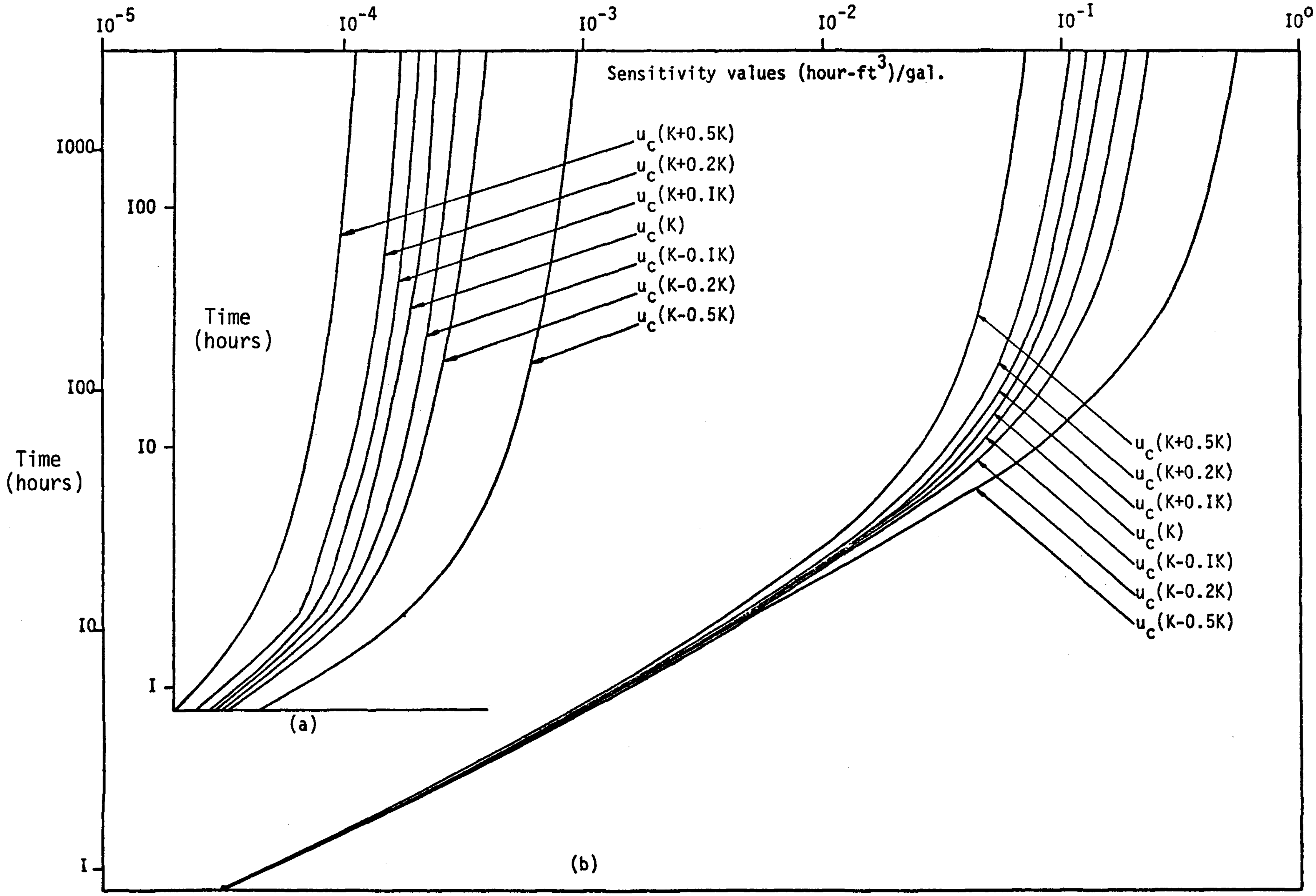


Figure 48. Variations in the sensitivity coefficient values at the well with changes in the hydraulic conductivity;(a) K=9600 gal./day-ft^2,S_y=0.I5,(b) K=355.2 gal./day-ft^2,S_y=0.24.

hydraulic conductivity values. Since the calculations were stopped before the boundary effects became important, the results of the constant head boundary case are expected to yield similar results to the no flow boundary case. Analysis of the results of Case I and Case II proves that the water levels and the sensitivity coefficient values are very close except for the system where the hydraulic conductivity values are perturbed by + 50% (Figures 43 and 49). The cone of depression travels faster in the system of higher hydraulic conductivity so that the boundary effects are sensed earlier in the system.

SENSITIVITY WITH RESPECT TO SPECIFIC YIELD

The sensitivity equation for the specific yield is derived by taking the partial derivative of the unconfined groundwater equation (2) with respect to specific yield.

$$\frac{\partial}{\partial Sy}\left[\frac{\partial}{\partial x} Kh \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Kh \frac{\partial h}{\partial y} + Q\right] = \frac{\partial}{\partial Sy}\left[Sy \frac{\partial h}{\partial t}\right] \tag{31}$$

Remembering that Q, Sy, and K are assumed independent of each other and rearranging equation (31) we obtain

$$\frac{\partial}{\partial x} Kh \frac{\partial}{\partial x} u_c' + \frac{\partial}{\partial x} Ku_c' \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Kh \frac{\partial}{\partial y} u_c' + \frac{\partial}{\partial y} Ku_c' \frac{\partial h}{\partial y}$$

$$= Sy \frac{\partial}{\partial t} u_c' + \frac{\partial h}{\partial t} \tag{32}$$

This is the sensitivity equation for the specific yield where

$$u_c' = \frac{\partial h}{\partial Sy}$$

The derivation of this equation is given in detail in Appendix D. Equation (32) is solved for both no flow boundaries and constant head boundaries.

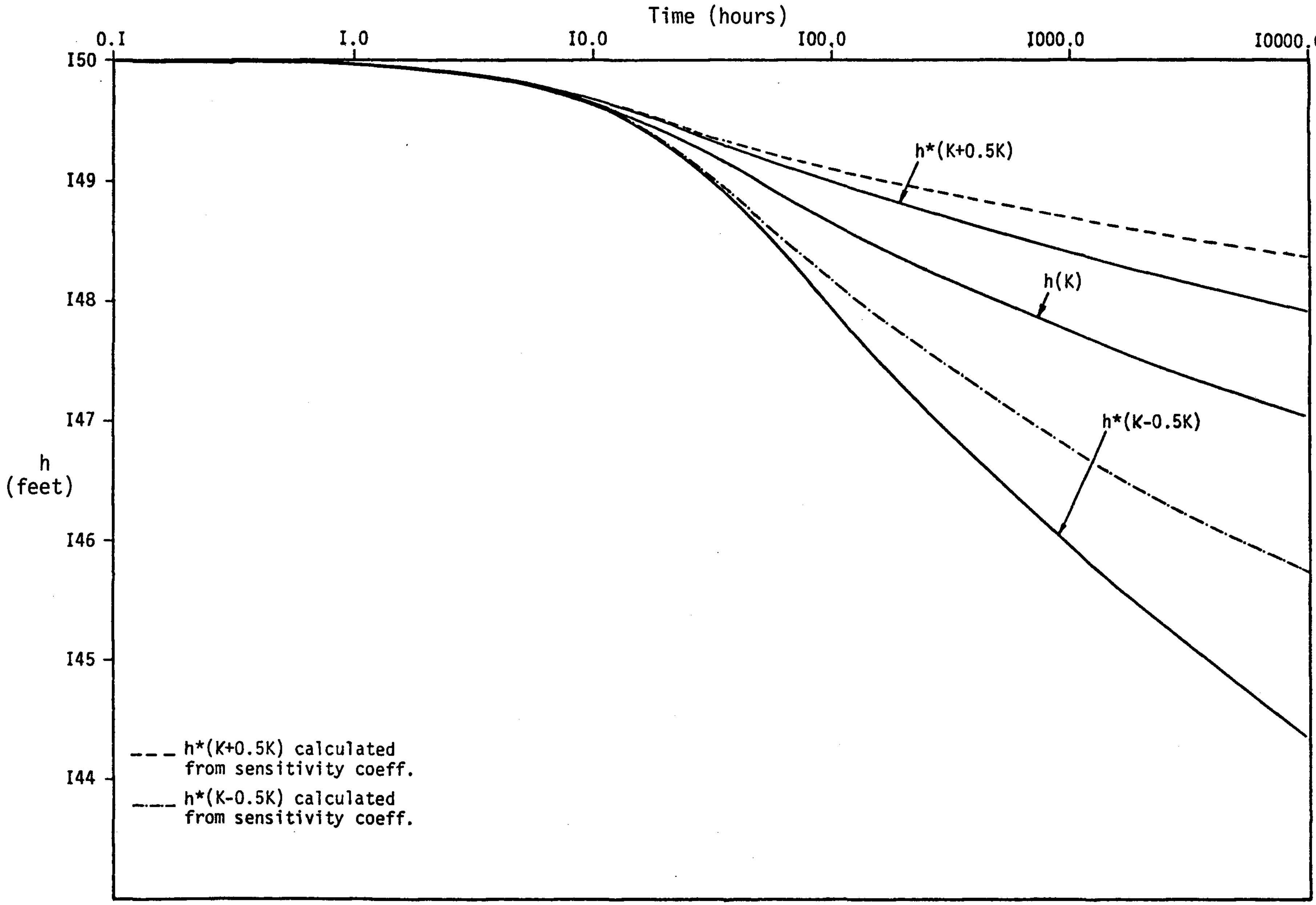


Figure 49. Comparison of the hydraulic head values determined at the well by direct solution and calculated from the sensitivity coefficients for ±50% disturbance in the hydraulic conductivity.

SOLUTION OF THE SENSITIVITY EQUATION

The sensitivity equation (32) is also solved using Crank-Nicolson scheme and time extrapolation routine (Appendix E).

CASE I. NO FLOW BOUNDARIES

The same unconfined aquifer with the aquifer parameters K_2, h_0, and Q is studied. The performance of the system is determined with the original and perturbed specific yield values (Figure 50). The change in the specific yield caused phase differences in the water level curves with time similar to what is observed in the confined flow case.

The water levels are determined from the flow equation (2) and calculated from equation (17) (Figure 50). The difference between these water levels would give the truncation errors, R. The truncation errors are large at the early times and gradually decrease with time as observed in the confined case. The percentage truncation errors, percentage of drawdown, are less than 14% for $\pm$ 50%, less than 6% for $\pm$ 20%, and less than 3% for $\pm$ 10% disturbances in specific yield.

The sensitivity coefficient values show a similar trend as observed in the confined case (Figure 51).

CASE II. CONSTANT HEAD BOUNDARIES

The initial water levels are constant, 150 feet, and the sensitivity coefficients are zero along the boundaries. The aquifer parameters are K_2, h_0, Sy_2, and Q. Since the calculations are stopped before the boundary effects became important in the system, the water levels and the sensitivity coefficient values are similar to what is observed in Case I.

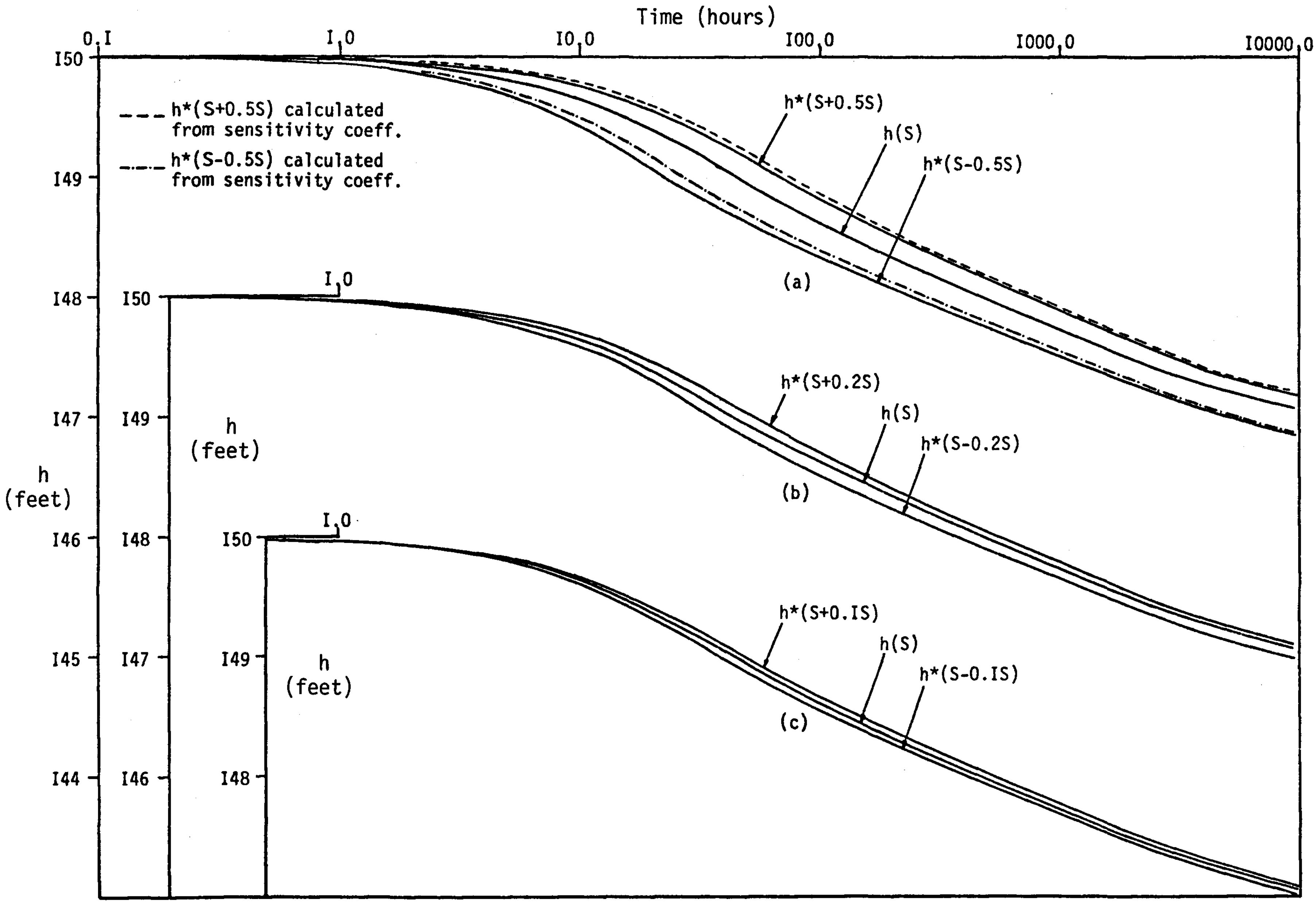


Figure 50. Comparison of the hydraulic head values determined at the well by direct solution and calculated from the sensitivity coefficients for (a) ±50%,(b) ±20%,(c) ±10% disturbances in the hydraulic conductivity,respectively.

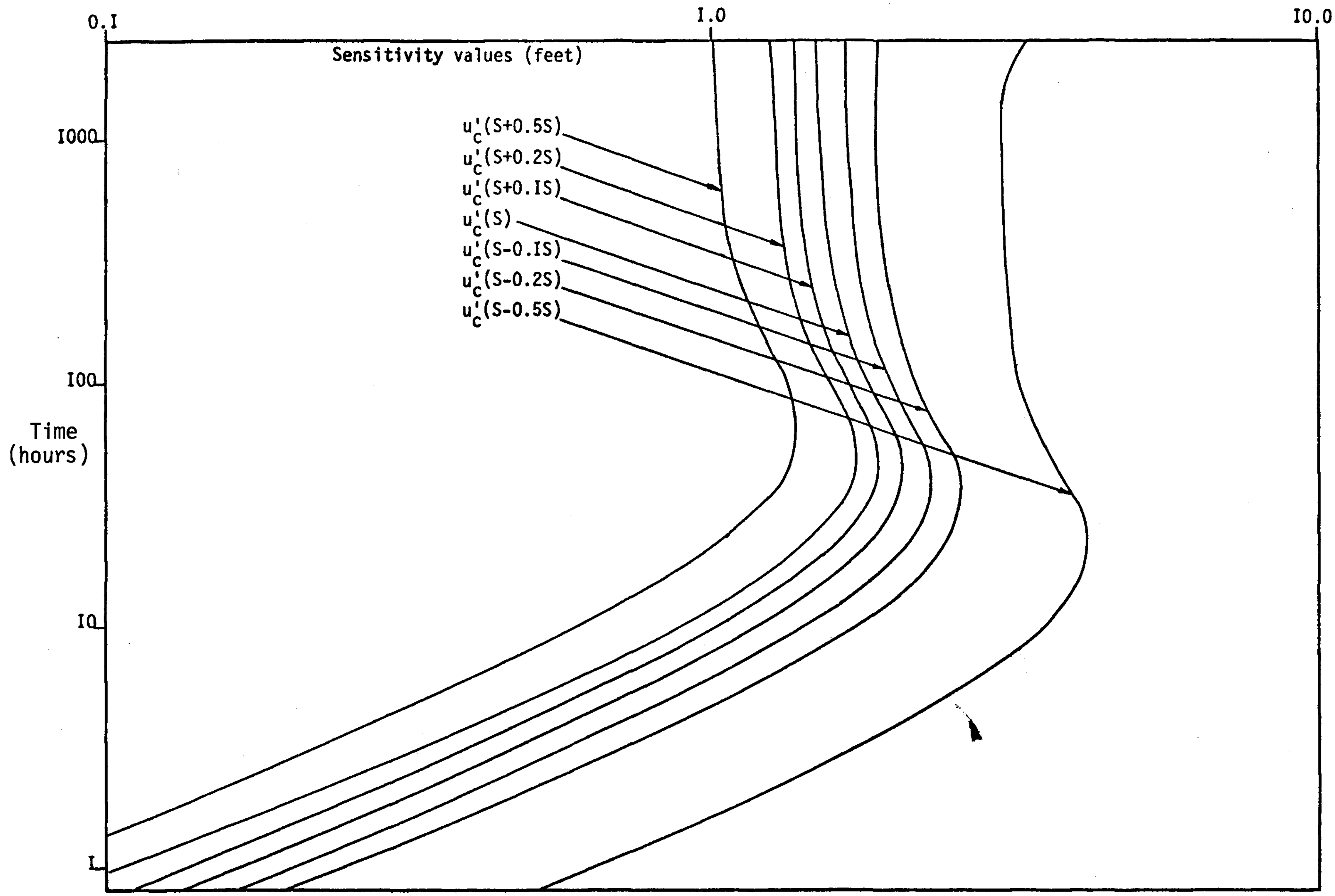


Figure 5I. Variations in the sensitivity coefficient values with changes in the storage coefficient.

APPLICATION OF COMPUTER MODELLING AND SENSITIVITY ANALYSIS TO WEST-CENTRAL KANSAS

STUDY AREA

The study area lies in southern Wallace County, central Greeley County, Wichita and Scott Counties, and central Lane County, all in Kansas. The total area is 3,960 square miles (Figure 52). Topography in the area is characterized by gently rolling upland plains that are dissected by the valleys of the Smoky Hill River and its tributaries, Ladder and Whitewoman Creeks and their tributaries. Common features of the upland plains in this area are shallow undrained depressions, which range from a few tens of feet to about half a mile in diameter.

The principal water-bearing unit is the Ogallala Formation. Its lithology varies sharply both vertically and laterally. The Ogallala is a heterogeneous complex of predominantly clastic deposits. Its texture ranges from very coarse gravel and pebbles to clay, and the sorting from good to poor. There is virtually no distinctive bed that can be traced an appreciable distance in the field.

This water-bearing unit is essentially an unconfined aquifer. Confining conditions occur locally in the area. The degree of confinement generally is governed by the areal extent of the impermeable units. The transmissivity of the unconsolidated aquifer ranges from 21,028 gal/day-ft (2,960 ft^2/day) to 70,330 gal/day-ft (9,900 ft^2/day). The storage coefficient ranges from 0.001 to 0.18. The specific yield is generally between 0.15 and 0.20. Subsurface movement of water is predominantly in an easterly direction under a hydraulic gradient of about 10 feet per mile.

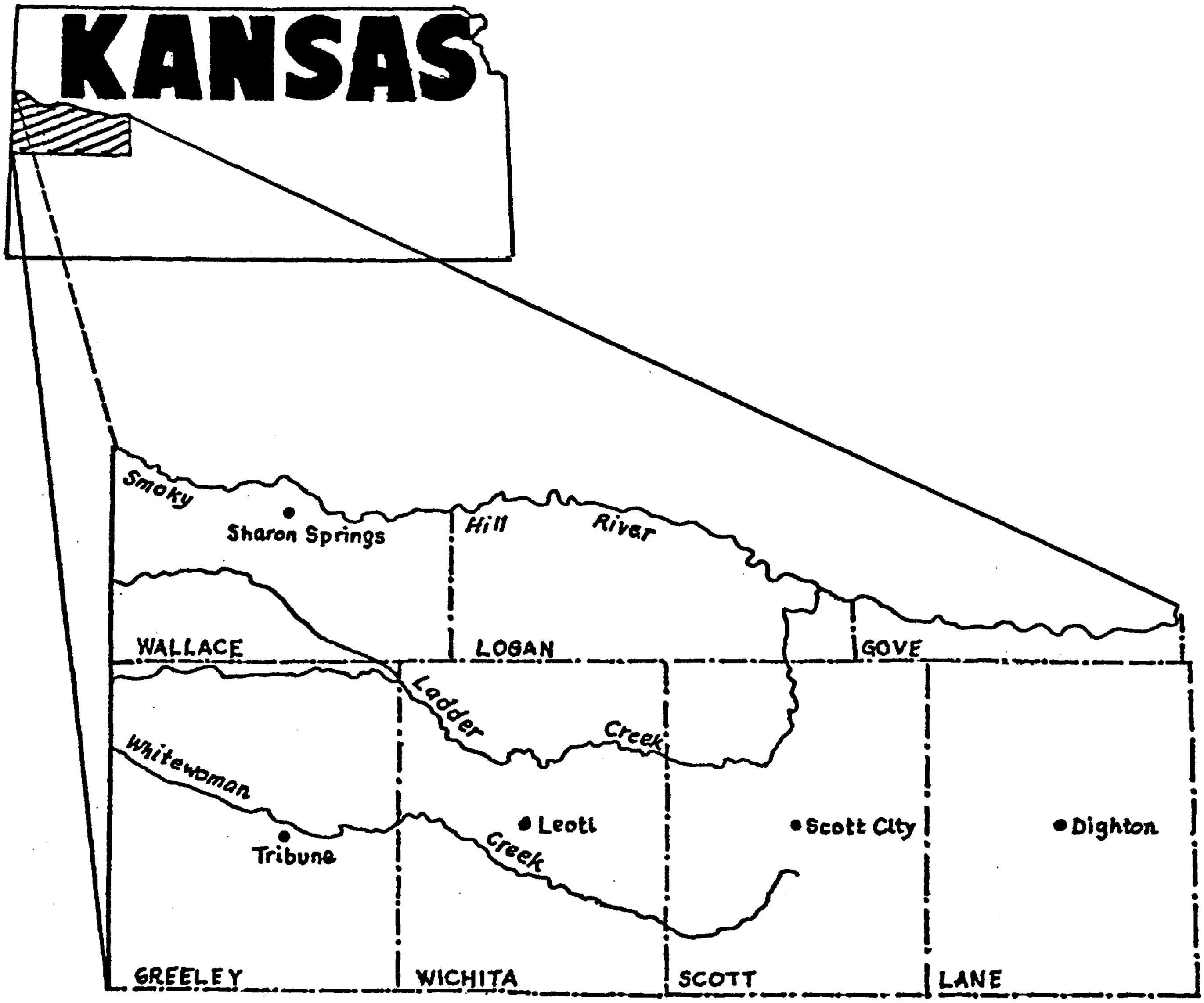


Figure 52. The study area.

Most of the water used in the area is withdrawn by wells tapping the Ogallala Formation. With increased wells, more water is removed annually from storage in the aquifer. As a result, the water level is declining in areas of heavy withdrawals for irrigation, industrial, and municipal uses (figure 53).

GEOLOGY OF THE STUDY AREA

Oldest rocks underlying the study area that are a potential source of groundwater to wells are the undifferentiated rocks of late Jurassic Age (Plate 1). Rocks underlying the Jurassic unit are of Permian Age, and consist primarily of thick shale and some sandstone that contain very highly mineralized water.

Lower Cretaceous rocks consist of shale interbedded with sandstone. Sandstones form one of the three aquifers underlying the study area. Consolidated rocks of Upper Cretaceous include the Graneros Shale, Greenhorn Limestone, Carlile Shale, Niobrara Chalk, and Pierre Shale. The Niobrara Chalk is a significant aquifer in Lane and Scott Counties due to fractures and solution openings in it (Waite, 1947; Prescott, 1951; Gutentag and Stullken, 1974).

The Ogallala Formation of Pliocene age unconformably overlies the Pierre Shale in Wallace County and the Niobrara throughout Greeley, Wichita, Scott, and Lane Counties. The Ogallala Formation, consisting chiefly of alluvial deposits, is the principal aquifer in the study area.

Thin, dissected and isolated deposits of sand and gravel of Pleistocene age occur along the larger streams of the study area. These deposits have been derived from the Ogallala Formation and lithologically are

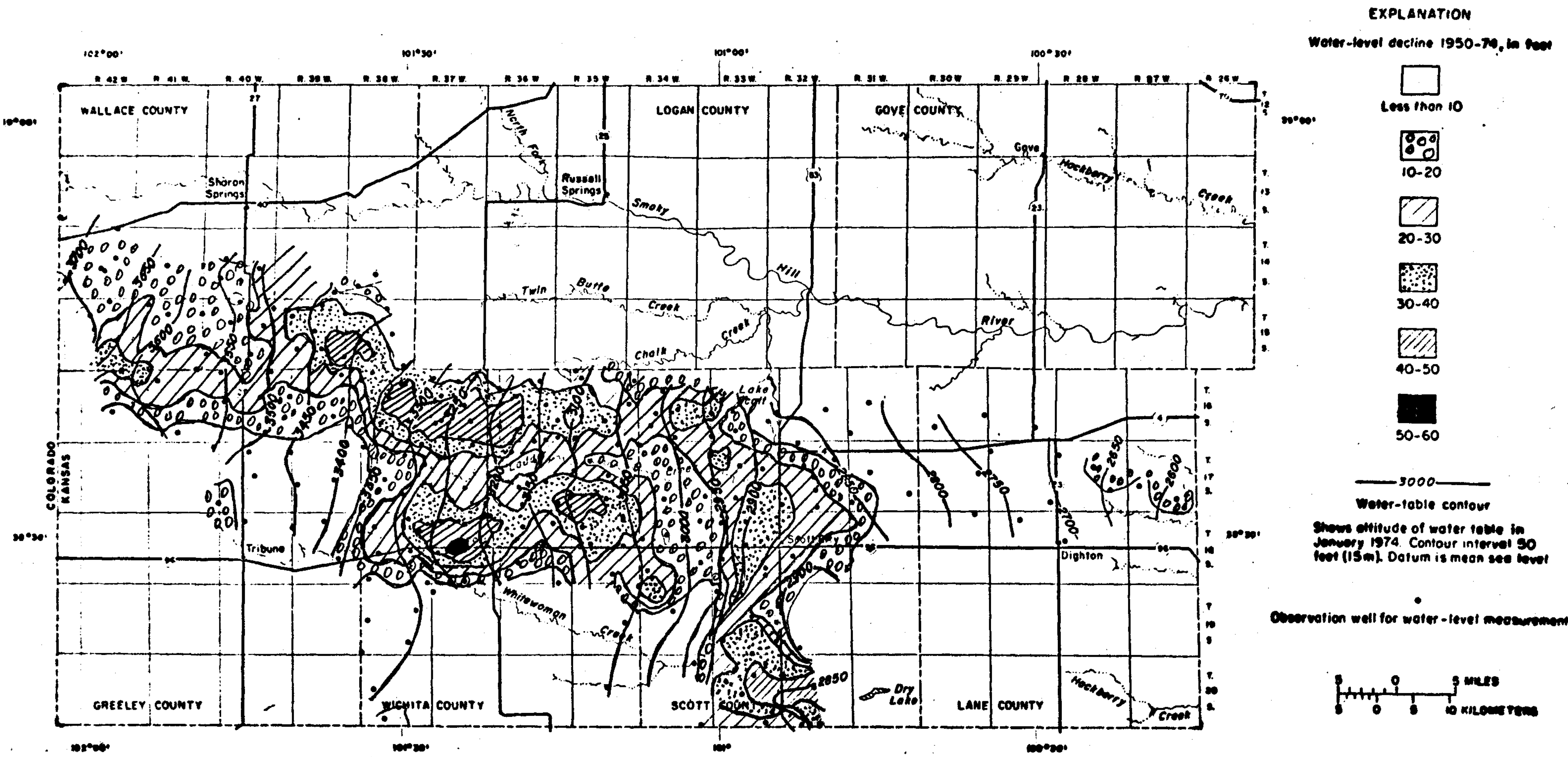


Figure 53. Water table configuration, January 1974, and water level decline, 1950-1974, west-central Kansas (Stullken et. al., 1974).

System	Series	Geologic Unit	Thickness, in feet	Physical Character	Water Supply
Quaternary	Pleistocene	Loess and dune sand	0-40	Silt and fine sand, mostly eolian. Mantles most of the upland and masks much of the valley walls.	Most of the deposits are above water table, but locally yield 5 to 10 gpm to wells. 2/
		Alluvium	0-50	Stream-laid deposits ranging from sand and gravel to silt and clay. Occurs along principal stream valleys.	Generally above the water table. Locally, yields about 250 gpm to irrigation wells.
		Undifferentiated deposits	0-200 Median 23	Medium to very coarse sand and gravel interbedded with clay, silt, fine sand, and caliche. These deposits are in contact with the Upper Cretaceous rocks where the Ogallala Formation is absent.	Principal aquifer in the Scott-Finney depression in central Scott County. Yields to irrigation wells range from 250 to 1,500 gpm.
Tertiary	Pliocene	Ogallala Formation	0-220 Median 110	Sand, gravel, silt, clay, and caliche, commonly unconsolidated. Locally cemented by calcium carbonate (lime) or silica (opal) into mortar beds. Also contains thin freshwater limestone beds.	Principal aquifer in the four-county area. Yields to irrigation wells range from 100 to 2,000 gpm.
Cretaceous	Upper Cretaceous	Pierre Shale	0-50	Dark-gray fissile shale. Locally, in the subsurface, the upper few feet is a yellow weathered zone.	Not known to yield significant amounts of water to wells.
		Niobrara Chalk	0-410 Median 100	Upper unit (Smoky Hill Chalk Member) consists of yellow to orange-yellow chalk and light- to dark-gray beds of chalky shale that locally weathers to ochre-yellow. Lower unit (Fort Hays Limestone Member) consists of a white to yellow massive chalky limestone; contains thin beds of dark-gray to brownish-gray chalky shale.	Yields as much as 1,000 gpm to wells in southeastern Scott County where the rocks have been fractured.

Plate I. Generalized section of geologic units (Gutentag and Stullken,1975).

Cretaceous	Upper Cretaceous	Carlile Shale	200-295 Median 245	Upper part consists of a dark-gray to blue-black noncalcareous to slightly calcareous shale that locally is interbedded with calcareous silty very fine sandstone. Lower part consists of very calcareous dark-gray shale and thin interbedded limestone.	Sandstone in upper part may yield 5 to 10 gpm to wells.
		Greenhorn Limestone	70-160 Median 105	Alternating light- to dark-gray thin-bedded chalky limestone and calcareous shale. Contains layers of bentonite.	Not known to yield significant amounts of water to wells.
		Graneros Shale	25-60 Median 40	Dark-gray calcareous shale interbedded with black noncalcareous shale. Contains thin beds of bentonite, gray limestone, and fine-grained silty sandstone.	Not known to yield significant amounts of water to wells.
	Lower Cretaceous	Undifferentiated rocks	300-680 Median 480	Upper unit (Dakota Formation)--brown to gray fine- to medium-grained sandstone interbedded with gray sandy shale and varicolored shale. Middle unit (Kiowa Formation)--dark-gray to black shale interbedded with tan and gray sandstone. Lower unit (Cheyenne Sandstone)--gray and brown fine- to medium-grained sandstone interbedded with dark-gray shale.	Yields of 30 to 300 gpm may be available to wells in sandstone beds. Yields of more than 1,000 gpm are reported from wells in counties farther south, but no irrigation wells tap these rocks within the report area.
Jurassic	Upper Jurassic	Undifferentiated rocks	0-200 Median 70	Gray, noncalcareous shale, interbedded with gray-green and blue-green calcareous shale. Contains fine-grained silty sandstone, and thin limestone beds.	The sandstone beds, although untested, may be a potential aquifer.

Plate I. Generalized section of geologic units (Gutentag and Stullken,1975).

very similar to the Ogallala. Because of this similarity it is difficult to distinguish the Pleistocene deposits from the Ogallala. The aquifer considered in the study is the Ogallala Formation. Since the Pleistocene deposits cannot be separated from the Ogallala Formation, they are treated as a part of the aquifer in order to construct a hydrologically connected system.

BOUNDARIES OF THE OGALLALA FORMATION

The Pierre Shale is the dominant bedrock in Wallace County and crops out in the central and northern parts of the County forming a barrier boundary (Hodson, 1963). The Niobrara Formation crops out in western Greeley County along Whitewoman Creek (Prescott et. al., 1954) and forms barrier boundaries in Wichita, Scott and Lane Counties (Gutentag and Stullken, 1974) (Figure 54).

There is a flow of groundwater into the Ogallala Formation from the Colorado border and some flow out the system through the Scott-Finney border and the Wichita-Kearny border.

BEDROCK PROPERTIES

The bedrock surface in the area shows a general slope from west to east. The major features on the bedrock surface are tributary valleys believed to have been formed in early Pleistocene and a north-south valley in central Scott County, believed to be a reflection of a northward plunging syncline formed as a result of deformation that occurred during post-Cretaceous and pre-Pleistocene time.

The bedrock surface slopes eastward at about 15 feet per mile in southern Wallace, Greeley, Wichita Counties and in western Scott

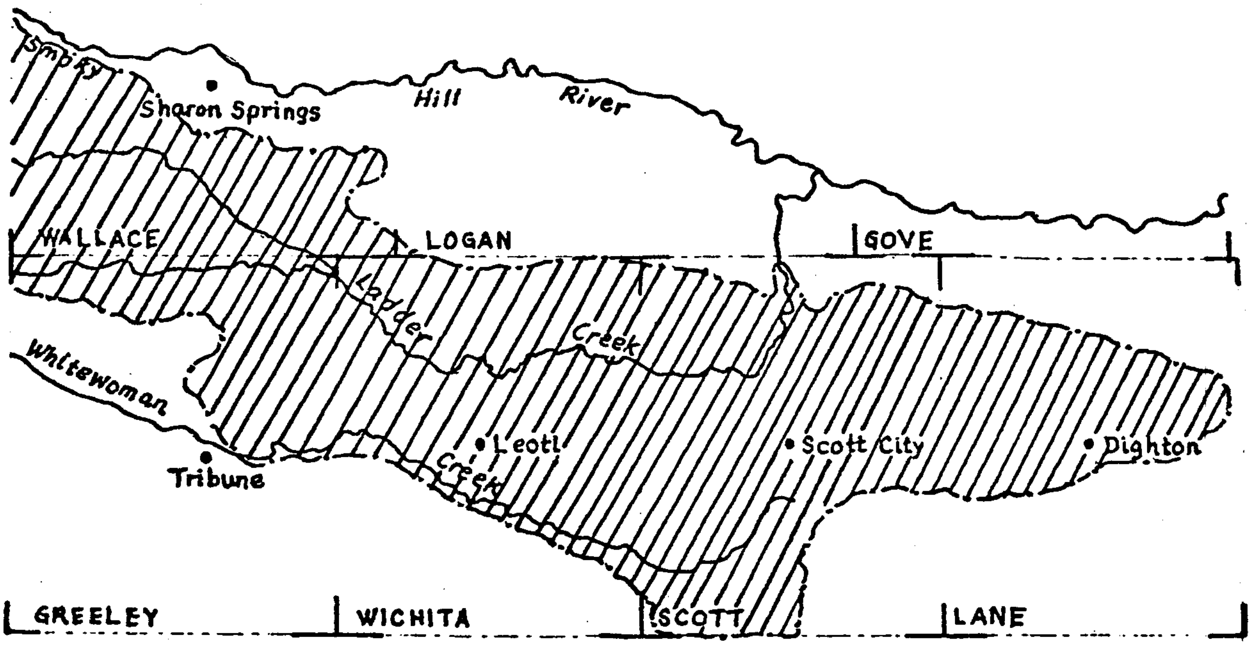


Figure 54. The boundaries of the unconfined aquifer (Ogallala Formation) in the study area. All the boundaries are no flow boundaries ~~except for the boundary on the Kansas and Colorado border~~. There is subsurface inflow from Colorado into the aquifer.

County and at an average rate of 7 feet per mile across eastern Scott County and Lane County. The land surface slopes eastward about 12 feet per mile in southern Wallace County, about 15 feet per mile in Greeley and Wichita Counties, and about 11.5 feet per mile in Scott and Lane Counties.

RECHARGE AND DISCHARGE TO AND FROM THE AQUIFER

The rate of recharge from precipitation in the study area varies with vertical hydraulic conductivity, topography, vegetation, agricultural practices, condition, type, and thickness of deposits overlying the aquifer, and intensity, duration, and seasonal distribution of precipitation.

Gutentag and Stullken (1974) estimated that 10 percent of the precipitation on irrigated land during the growing season and 1 percent on unirrigated land during the growing season becomes recharge to the aquifer. Also, 20 percent of the irrigation water is assumed to be recharged to the aquifer. There is also some recharge from the streams in the study area, but it is considered to be too small to be significant.

The discharge from the area is to Ladder Creek, Hackberry Creek, Walnut Creek, and to numerous tributaries of the Smoky Hill River. There is also subsurface outflow to bedrock channels and fractures. By far the largest discharge is the withdrawal of water by wells, especially for irrigation. A small amount is discharged by evapotranspiration. This becomes important when the water table is close to the surface.

COMPUTER MODELLING OF THE STUDY AREA AND THE APPLICATION OF SENSITIVITY ANALYSIS

DATA COMPILATION

The well location and water level data from 1948 to 1975 have been obtained from the files of the USGS set up by the cooperative program with Kansas Geological Survey. The water rights data from 1948 to 1975 have been furnished by Kansas Water Resources Board.

A grid system is set up for the study area with grid points two miles apart (Figure 55). The available data (bedrock elevation, transmissivity, specific yield, storage coefficient, water levels) are plotted and an average value is stored at each grid point.

The volume of water pumped by the discharging wells is not well known in the study area and must be estimated by various schemes. In this study the scheme adopted by Carl McElwee (Ground Water Seminar,

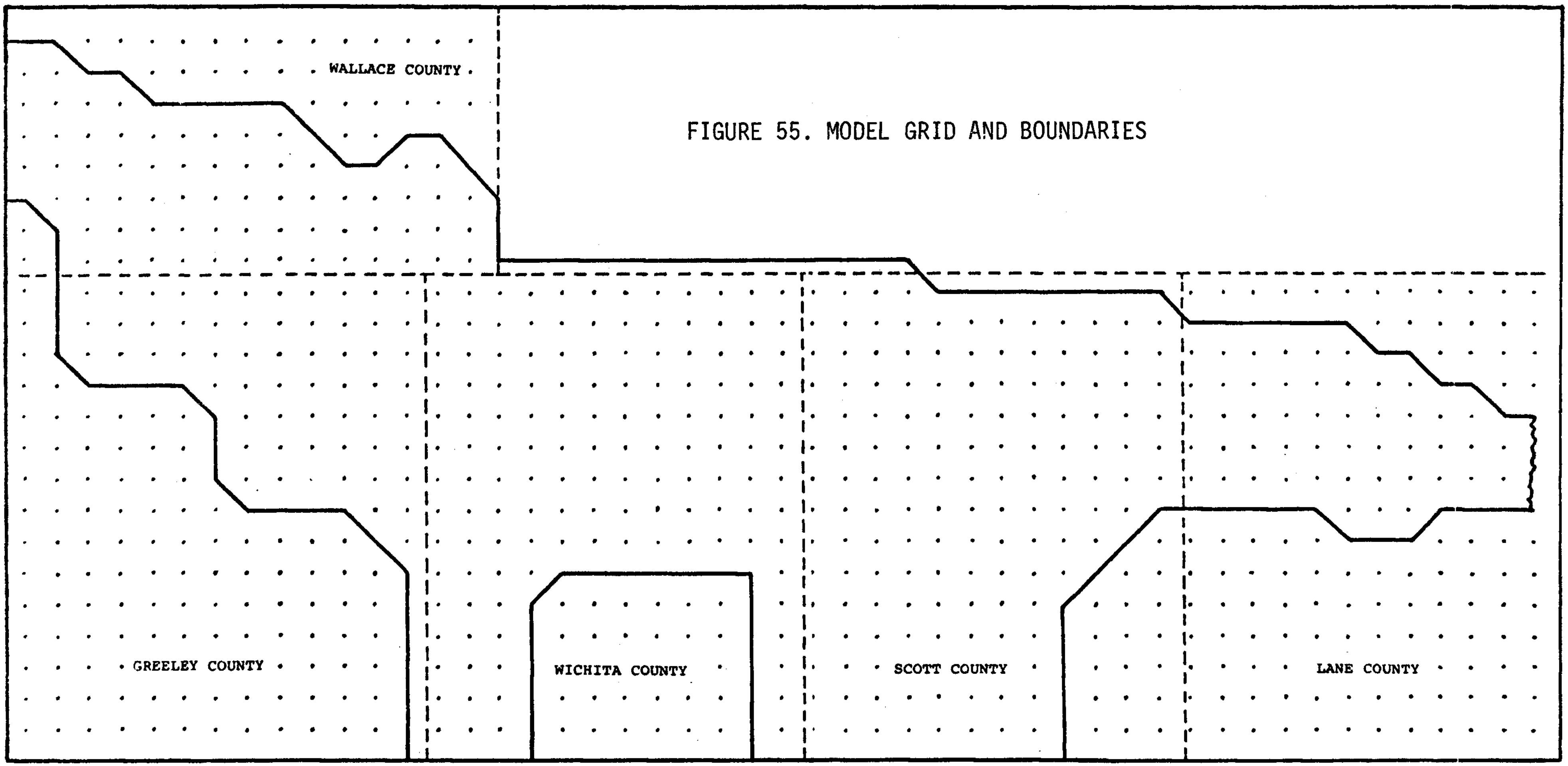


FIGURE 55. MODEL GRID AND BOUNDARIES

Scott City, 1975) is used (Figure 56). This is a statistical estimate from spotty reported pumpage.

COMPUTER MODEL

The Ogallala Formation is an unconfined aquifer and the unconfined flow equation (2) is used to determine the flow of groundwater in the system. The flow equation is solved by the finite-difference techniques described in Appendix B. There are 1127 grid points in the system. Therefore, there are 1127 equations and 1127 unknown water level values at each time step. These 1127 equations are set up in the computer and are solved simultaneously.

The sensitivity equations for changes in the hydraulic conductivity, equation (30), and changes in specific yield, equation (32), are written in algebraic form using the finite-difference technique described in Appendix D. First the sensitivity equation for the hydraulic conductivity is considered and again 1127 equations are set up in the computer. Then both the flow equation and the sensitivity equation are solved simultaneously. The water levels and the sensitivity coefficients at every grid point are determined for every year from 1948 to 1975, and projections are made into 1980. Then the sensitivity equation for specific yield is considered and the whole procedure is repeated.

ANALYSIS OF THE RESULTS

The most complete water level data are for the years 1948 and 1972, hence, the starting time for the computations is chosen as 1948. An average hydraulic conductivity of 355.2 gal/day-ft^2 (50 ft/day) and specific yield, 0.15, is considered as a first guess throughout

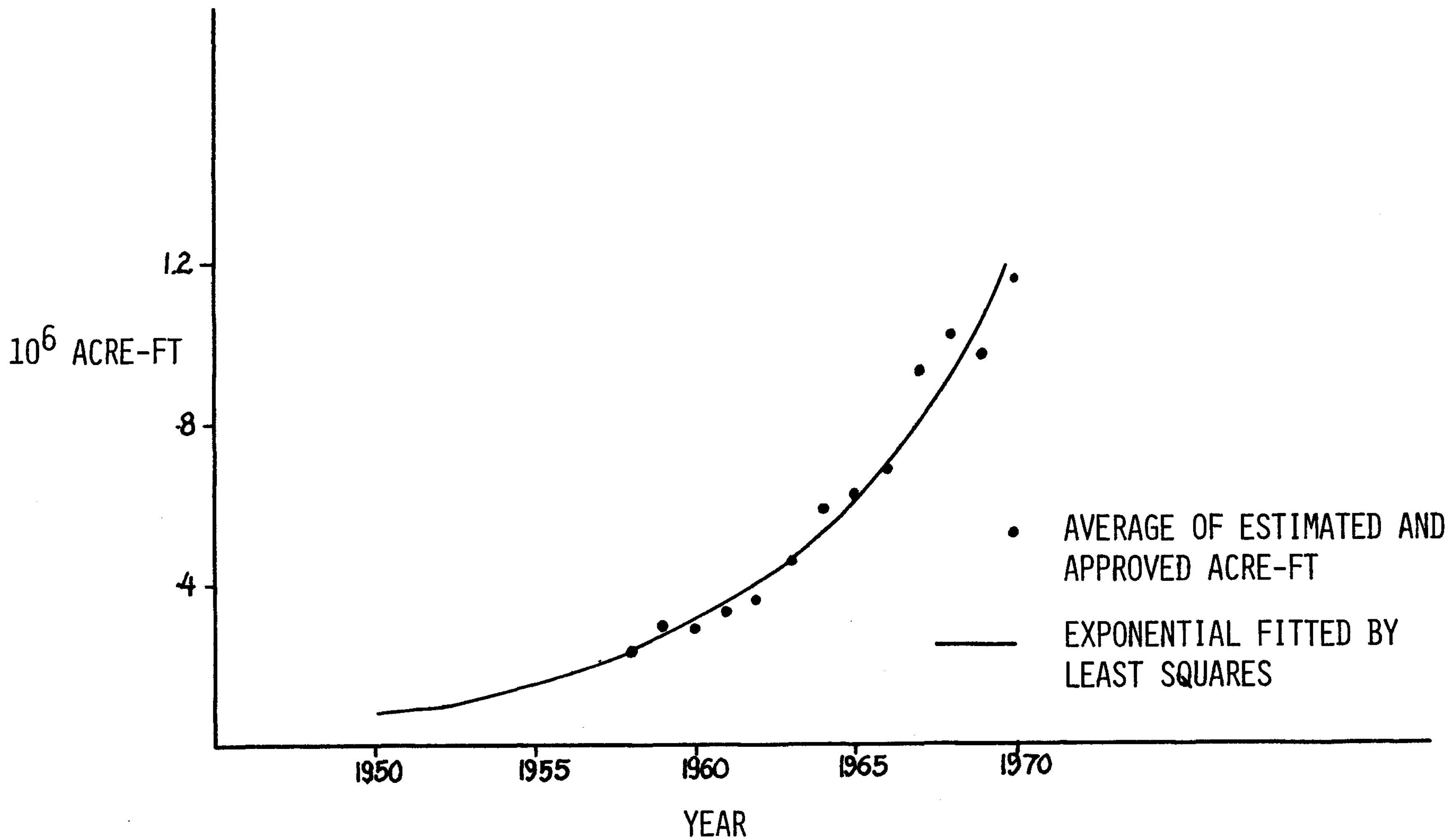


Figure 56. Estimated water pumpage from the study area (McElwee,1975).

the study area. The system is assumed steady state in 1948. This is justified by the very small pumpage in this year.

1972 water levels are computed from the unconfined flow equation (2) using the above aquifer parameters and the discharge obtained from figure 56 (Figure 57). The results are checked against the observed 1972 water levels (Figure 58). The areas which do not match well are the Colorado border, central Greeley County and central Scott County. Then 1972 water levels are computed from the confined flow equation (1) using the same aquifer parameters and discharge values (Figure 59). Figures 57 and 59 yield similar values and the same areas do not match with the observed 1972 water levels (Figure 58).

The sensitivity coefficients for the hydraulic conductivity variations have been computed and are shown in figure 60. The examination of figure 60 shows that the sensitivity coefficient values are quite high near the Colorado border and reasonably high in south-central Scott and Lane Counties. These are the areas that are very sensitive to variations in the hydraulic conductivity. The errors in the calculated 1972 water levels are found by subtracting the observed 1972 water levels from the computed water levels (Figure 61).

$$\Delta h = h^{1972}_{obs} - h^{1972}_{calc}$$

This map, figure 61, also shows that large errors are at and near the Colorado border and in central and south-central Scott County. Then the error in hydraulic conductivity, ΔK, is determined at each node point from

$$\Delta K = \frac{\Delta h}{u}$$

where u is the sensitivity coefficient

ΔK is plotted in figure 62. This map shows that the hydraulic conductivity

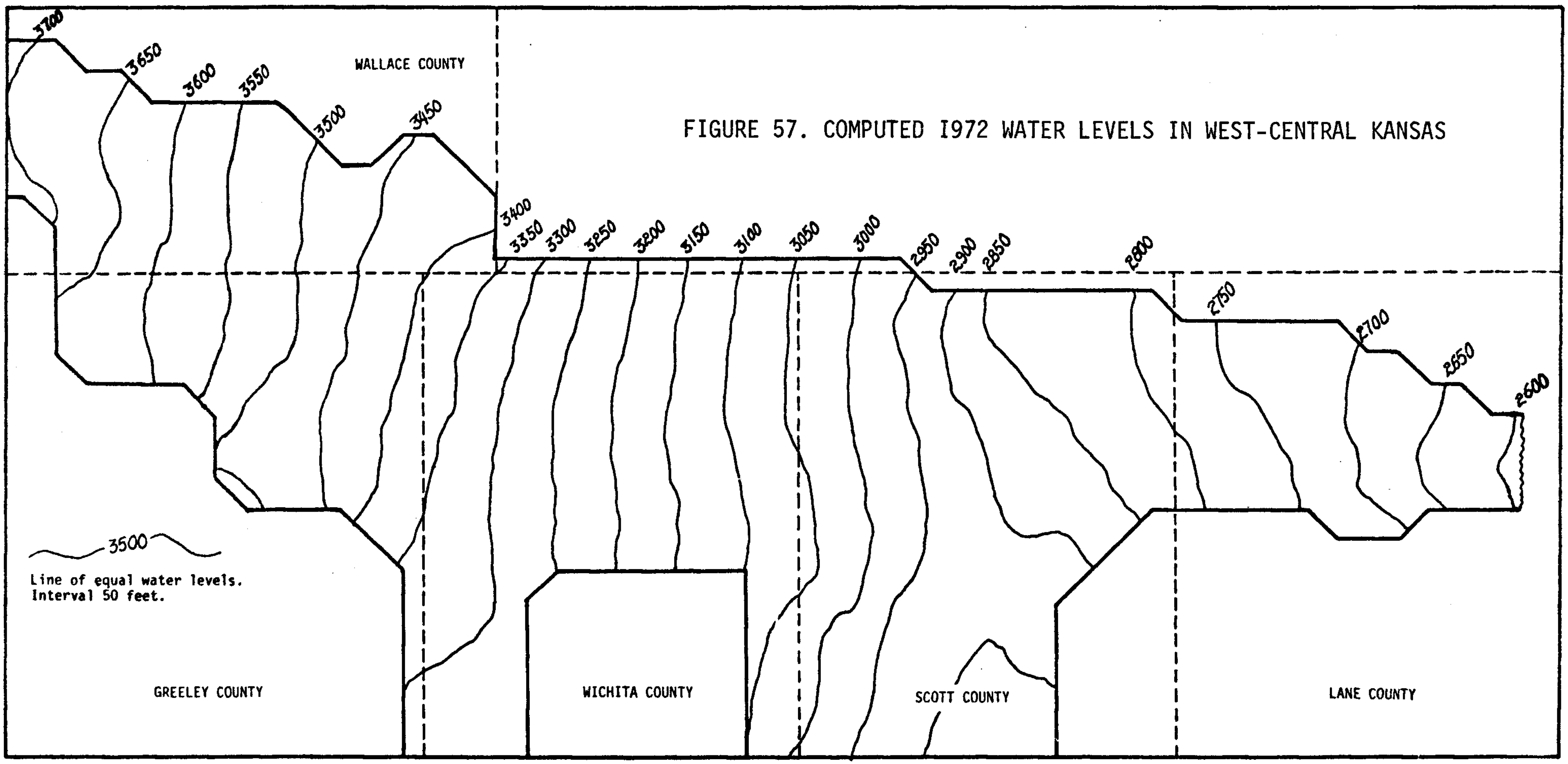


FIGURE 57. COMPUTED 1972 WATER LEVELS IN WEST-CENTRAL KANSAS

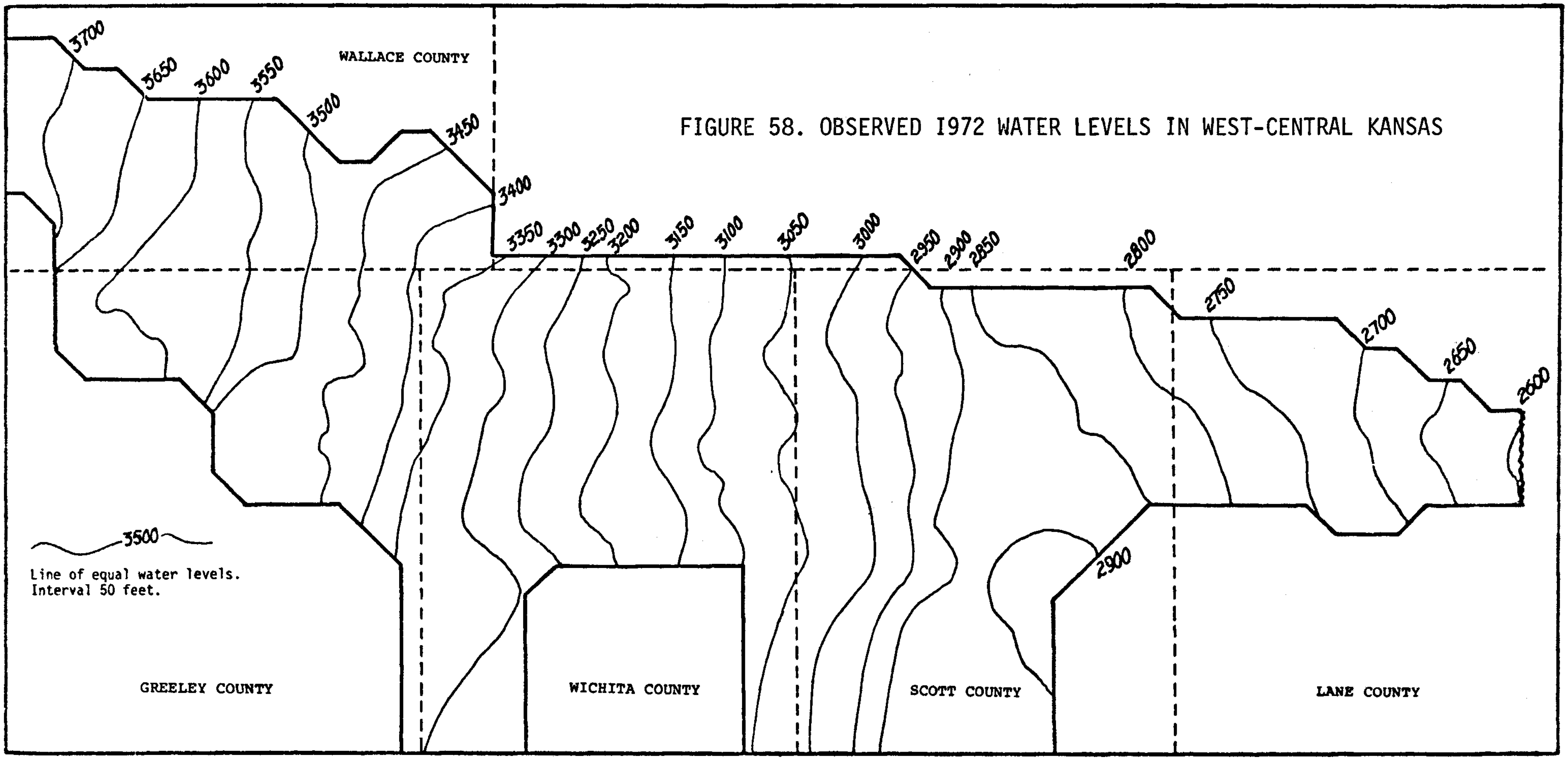


FIGURE 58. OBSERVED 1972 WATER LEVELS IN WEST-CENTRAL KANSAS

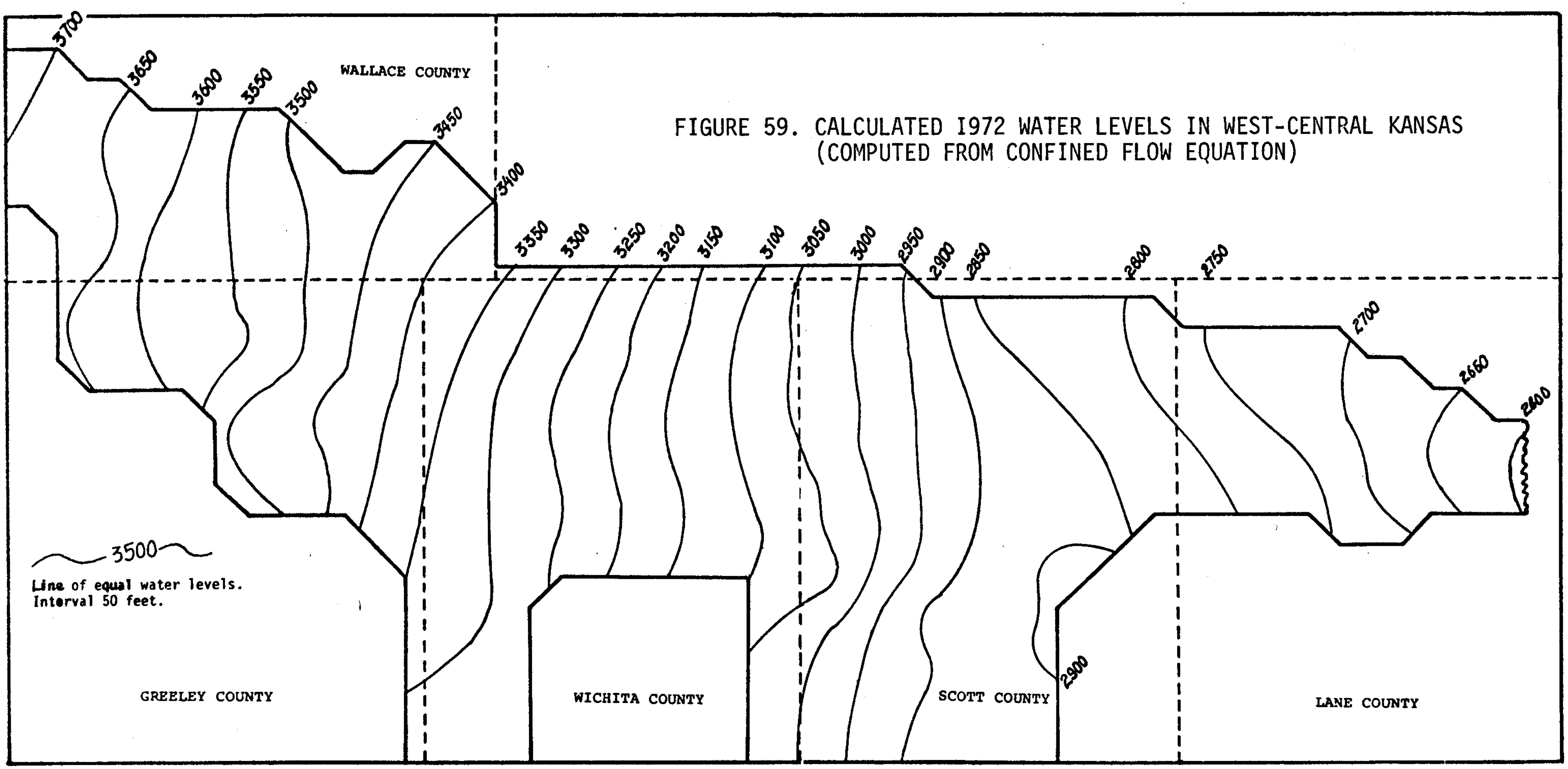
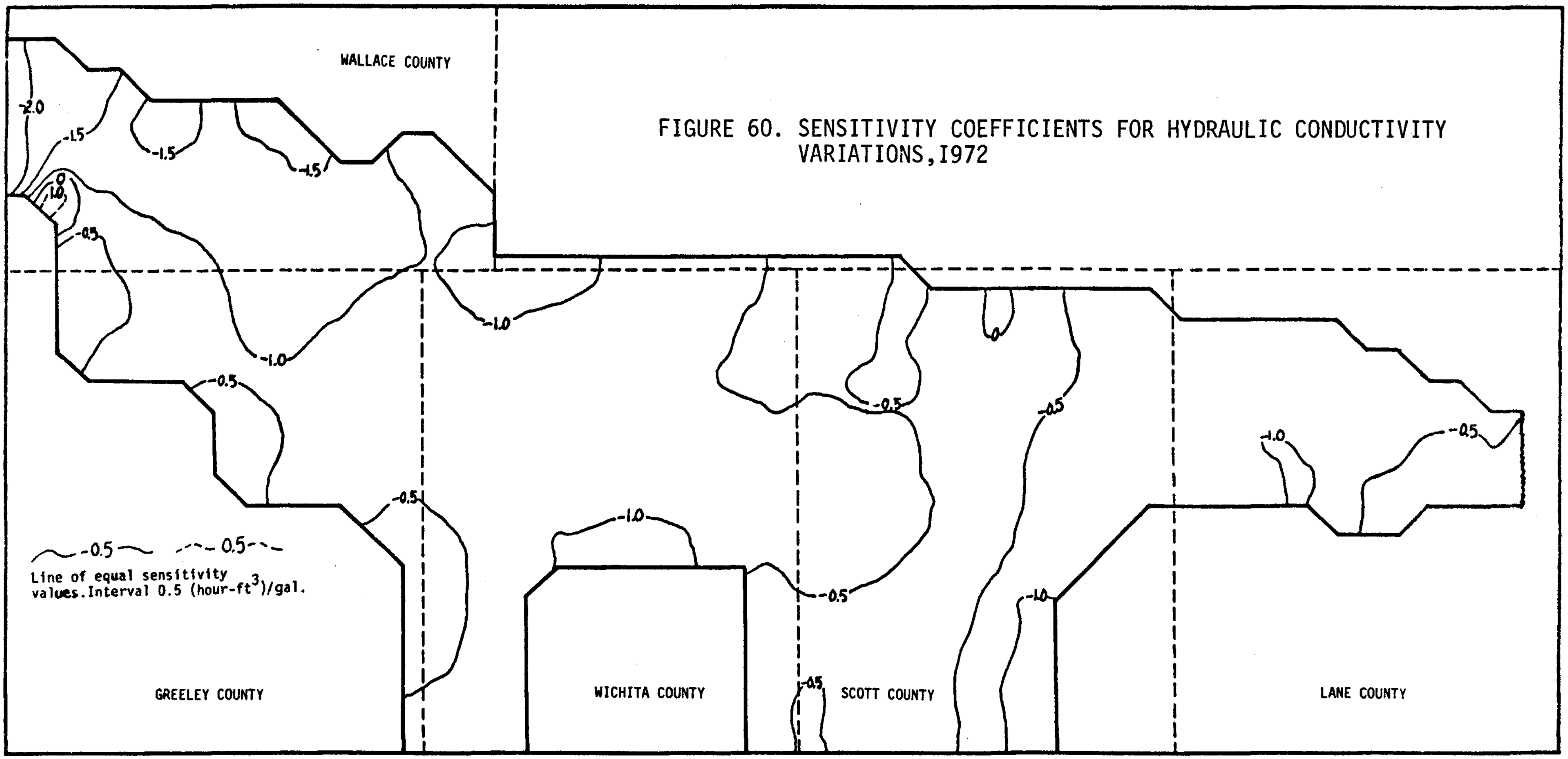


FIGURE 59. CALCULATED 1972 WATER LEVELS IN WEST-CENTRAL KANSAS (COMPUTED FROM CONFINED FLOW EQUATION)

FIGURE 60. SENSITIVITY COEFFICIENTS FOR HYDRAULIC CONDUCTIVITY VARIATIONS, 1972



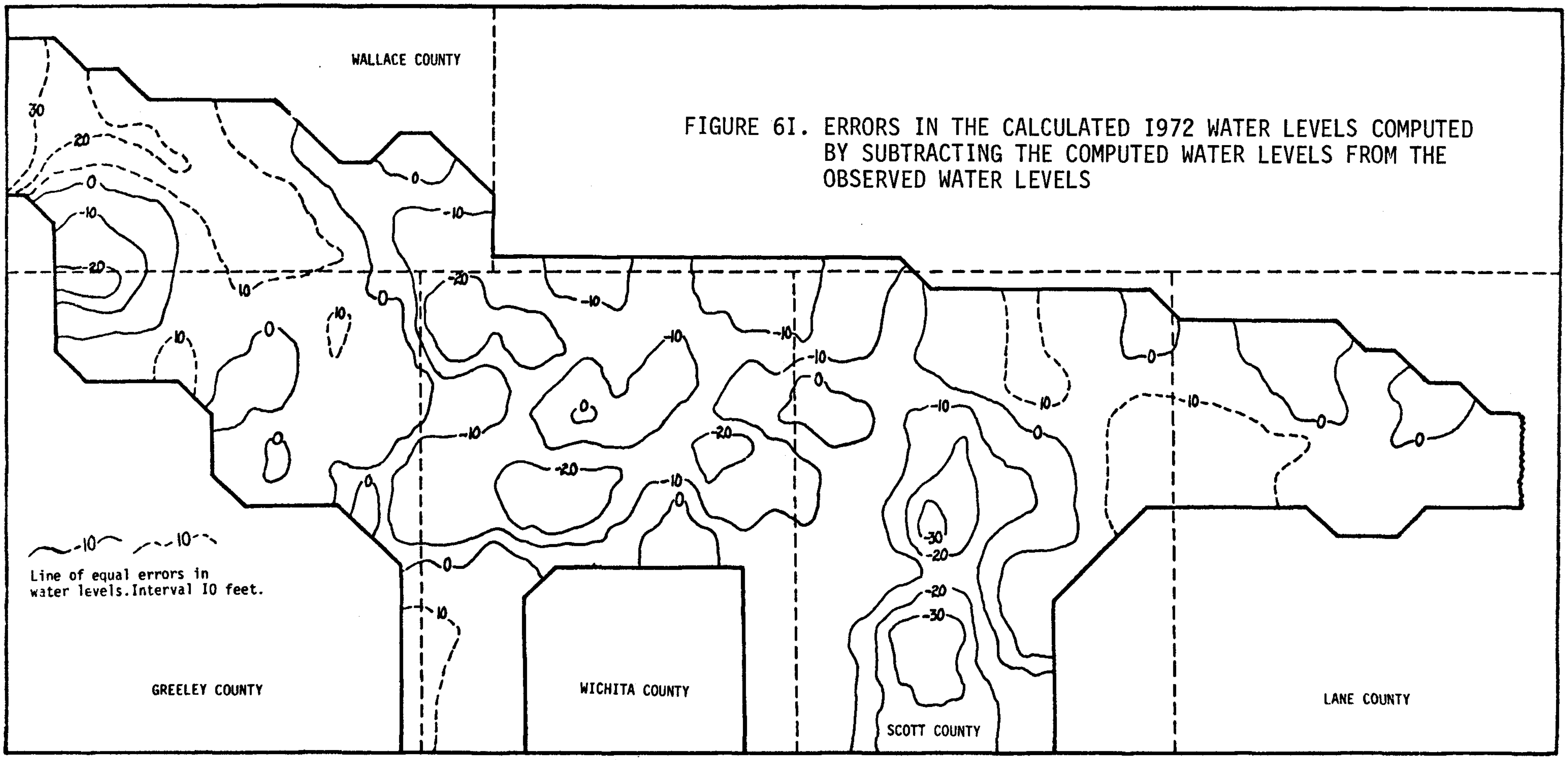
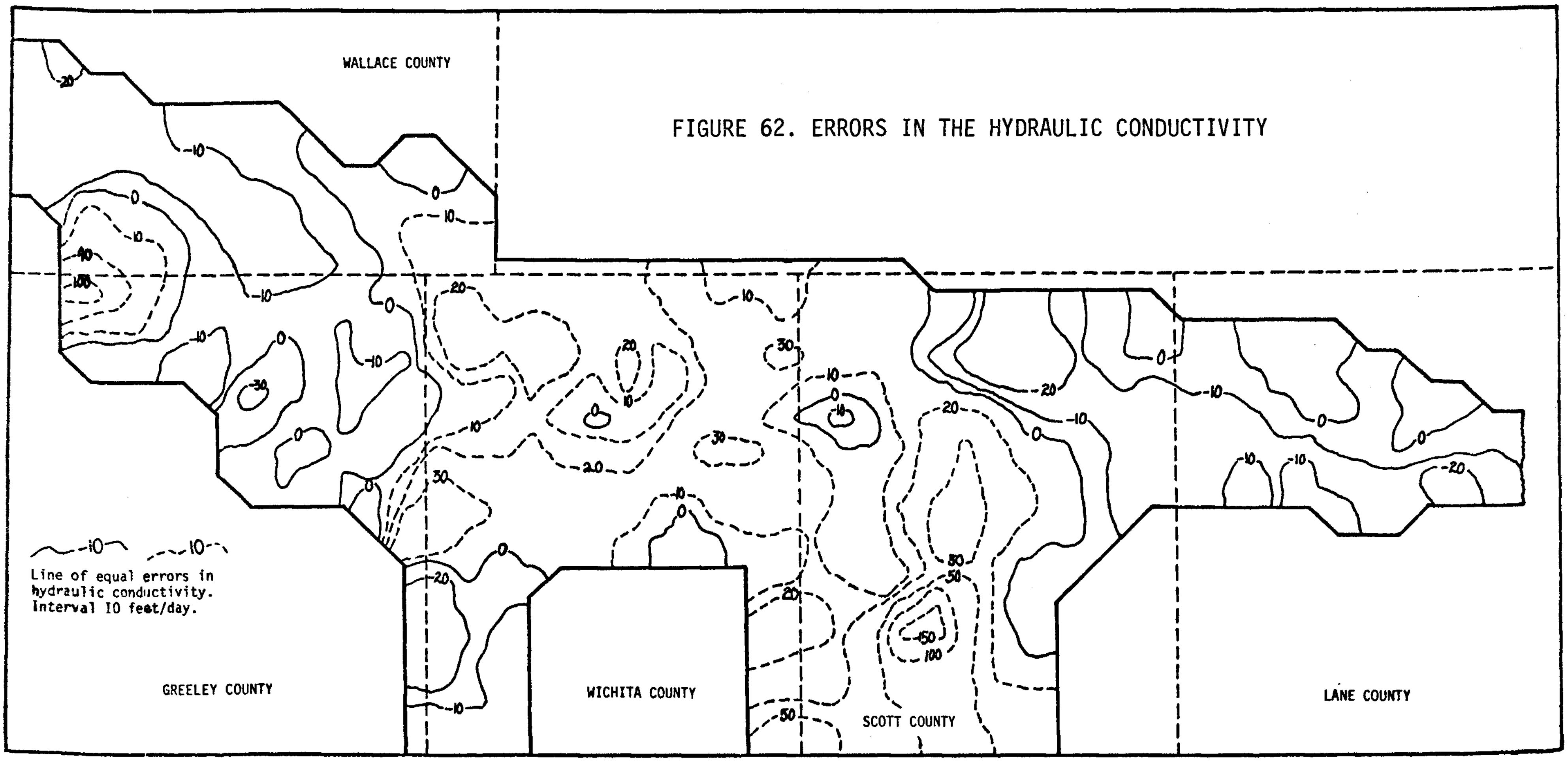


FIGURE 61. ERRORS IN THE CALCULATED 1972 WATER LEVELS COMPUTED BY SUBTRACTING THE COMPUTED WATER LEVELS FROM THE OBSERVED WATER LEVELS

FIGURE 62. ERRORS IN THE HYDRAULIC CONDUCTIVITY



values are overestimated in Greeley, Wichita, and Scott Counties, and overestimated in central Lane County. The hydraulic conductivity values in central and southern Scott County are underestimated by 100% to 300%. This area lies on the north-south bedrock channel in central Scott County. The deposits consist of medium to very coarse sand and gravel interbedded with layers of silt and clay (Gutentag and Stullken, 1974) and are thicker in the southern than in the northern part of the bedrock channel. Large ΔK values show that the hydraulic conductivity is quite high in this area. The other area where large ΔK values are observed is in western Greeley County. The deposits in this area consist of medium to coarse sand and gravels (Prescott et. al., 1954); thus, have high conductivity. Unreasonably low ΔK values (-500% to -1000% of the estimated hydraulic conductivity) occur near the Colorado border, the Wichita-Kearny border, and in north-central Scott County. It is believed that these physically wrong hydraulic conductivity values are due to erroneous discharge values and the wrong boundary configurations.

The sensitivity coefficient values for changes in specific yield are also found for the study area (Figure 63). The sensitivity coefficient values exhibit a symmetric configuration with respect to boundaries. The examination of Figure 52 shows that these values are roughly symmetric with respect to Ladder Creek. This is an interesting result showing that the system is very sensitive to variations in specific yield along Ladder Creek and the sensitivity decreases away from it towards the boundaries. Notice that contour lines are closed to the east, at central Scott County, where Ladder Creek turns north. The interaction of the aquifer with Ladder Creek was ignored in this study. The large

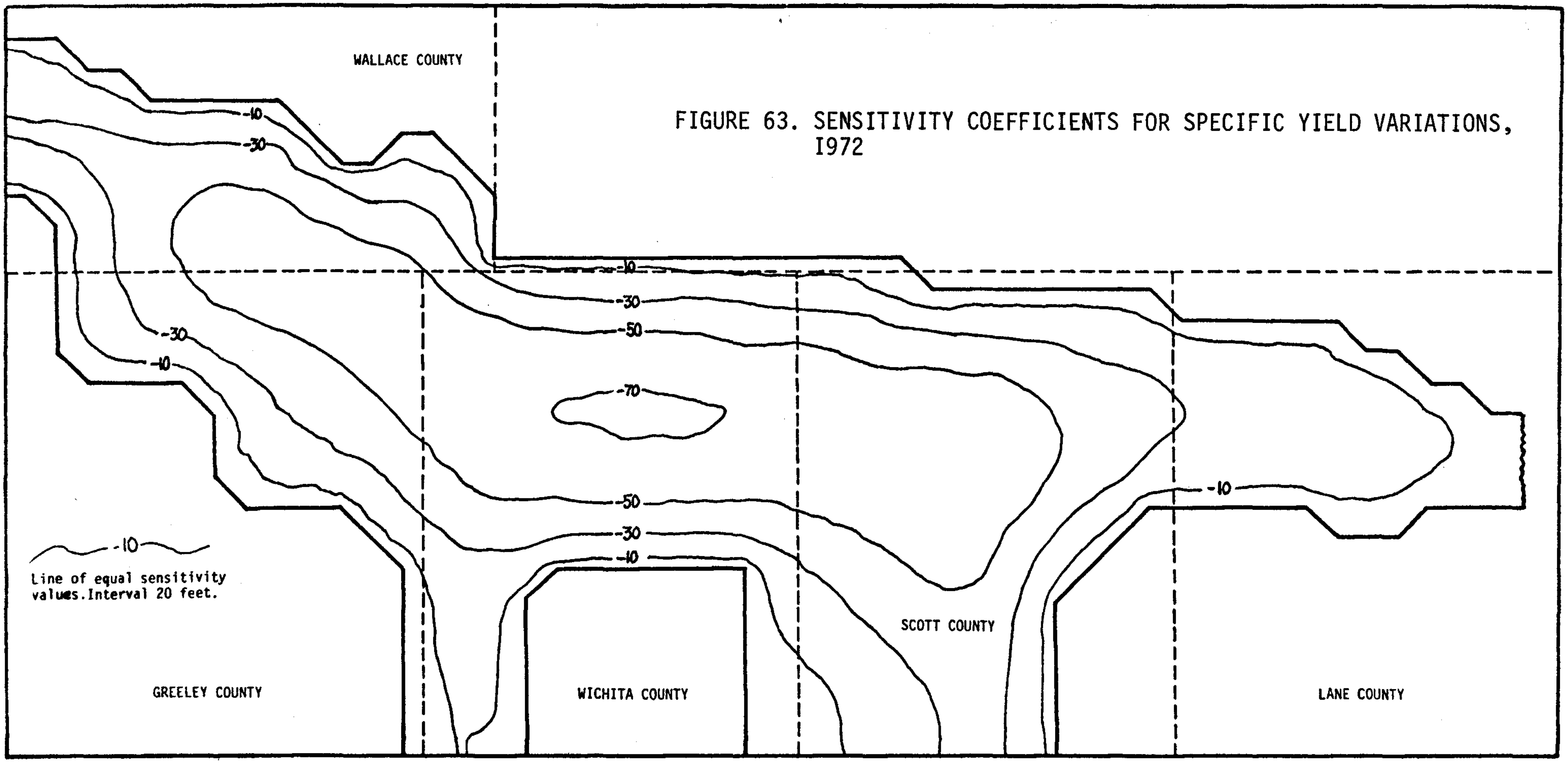


FIGURE 63. SENSITIVITY COEFFICIENTS FOR SPECIFIC YIELD VARIATIONS, 1972

sensitivity values along this river illustrate the error made due to this assumption. The error in water level is given by

$$\Delta h = \Delta Sy . u'_c$$

where ΔSy is the error in specific yield

u'_c is the sensitivity coefficient,

the error in water level, Δh, can be neglected because ΔSy values are quite small.

Water levels for 1975 are computed with the first estimate of the aquifer parameters (Figure 64). Then 1975 water levels are determined using the corrected hydraulic conductivity, $K + \Delta K$, and observed water level values at 1972 (Figure 65). Comparison of figure 64 and 65 with the observed 1975 water levels (Figure 66) illustrates that figure 65 matches fairly well with figure 66 except in areas where physically wrong hydraulic conductivity values are computed. Projections into 1980 are made with the uncorrected and corrected aquifer parameters, respectively (Figures 67 and 68). The difference in water levels is quite obvious.

The application of sensitivity analysis to a groundwater flow system is illustrated by this simplified model. A more complete study of this area is still under way and will be published by the Kansas Geological Survey with the title, "Ground-Water Model Studies, West-Central Kansas."

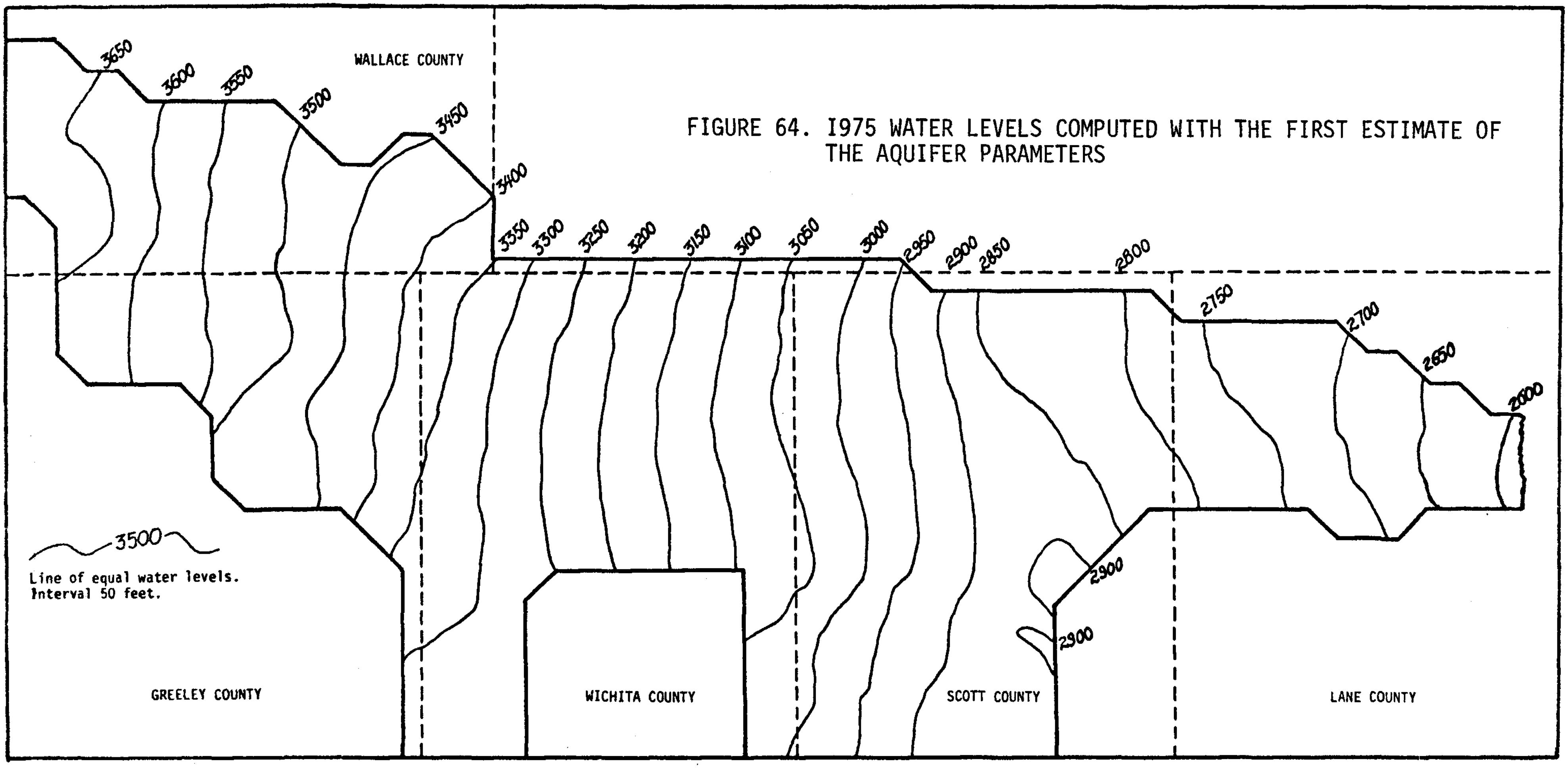
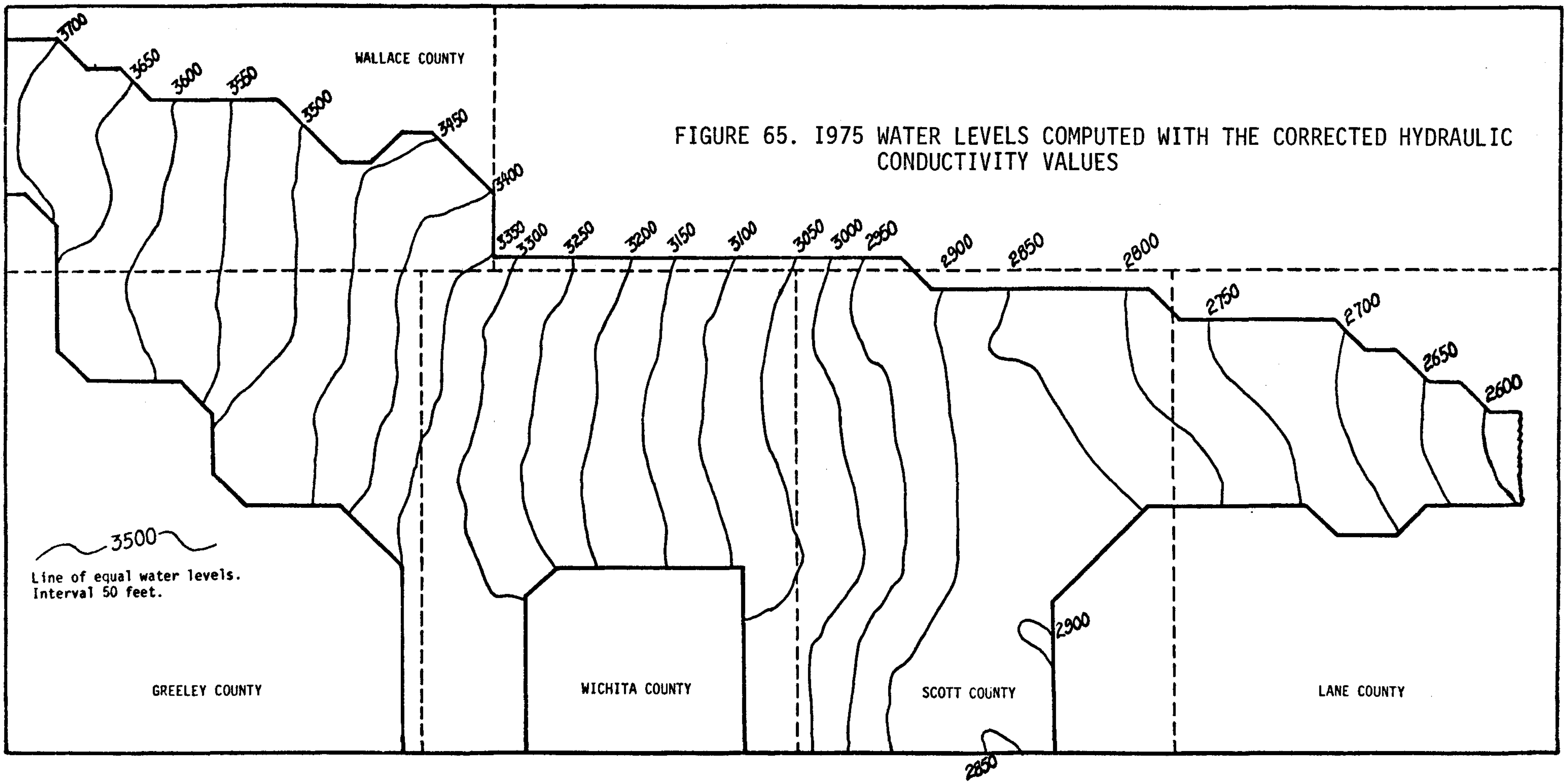


FIGURE 64. 1975 WATER LEVELS COMPUTED WITH THE FIRST ESTIMATE OF THE AQUIFER PARAMETERS

FIGURE 65. 1975 WATER LEVELS COMPUTED WITH THE CORRECTED HYDRAULIC CONDUCTIVITY VALUES



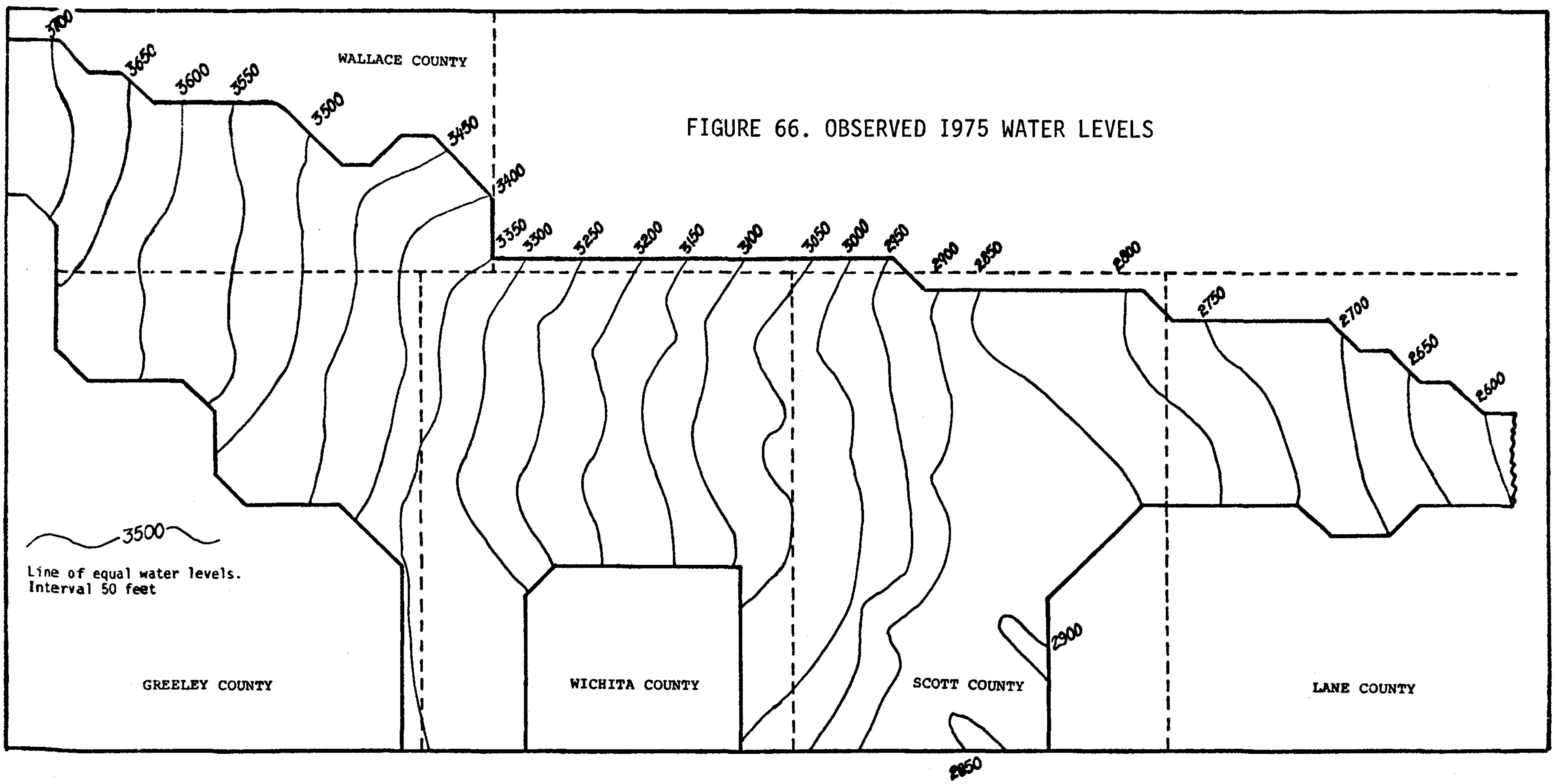


FIGURE 66. OBSERVED 1975 WATER LEVELS

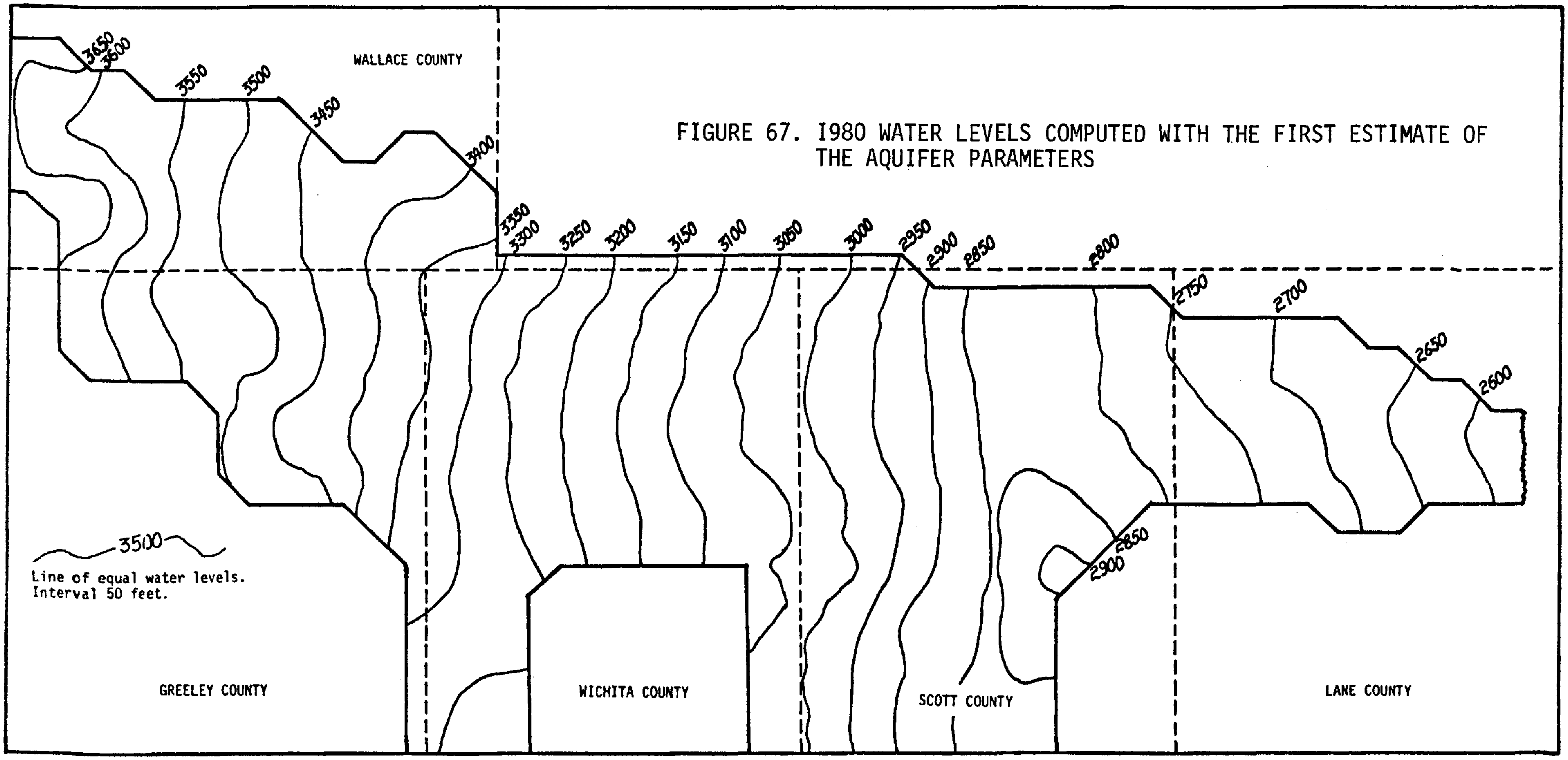
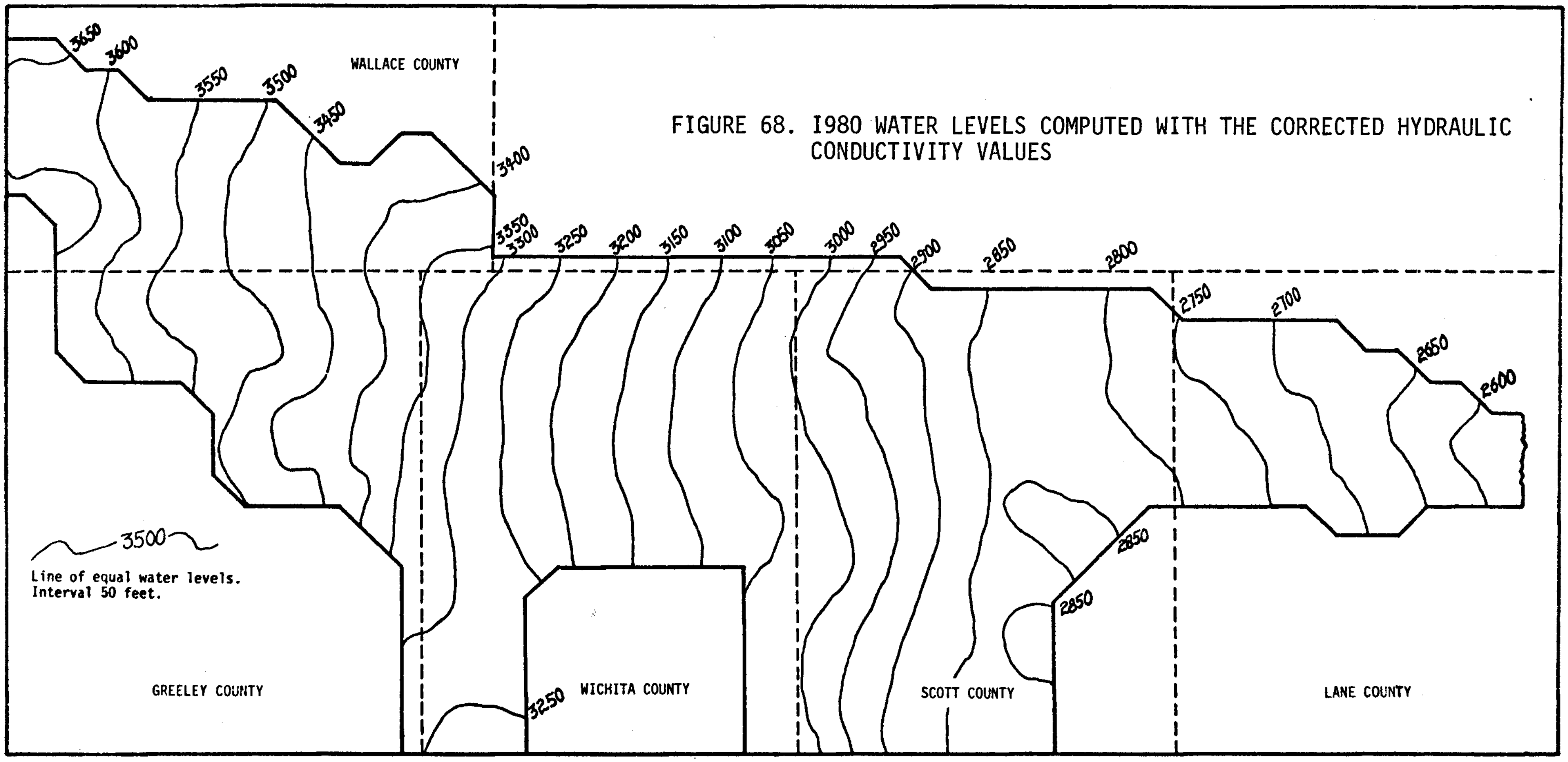


FIGURE 67. 1980 WATER LEVELS COMPUTED WITH THE FIRST ESTIMATE OF THE AQUIFER PARAMETERS

FIGURE 68. 1980 WATER LEVELS COMPUTED WITH THE CORRECTED HYDRAULIC CONDUCTIVITY VALUES



CONCLUSION AND RECOMMENDATIONS

This study was made to determine the errors in computer modelling of the groundwater flow systems. Two major sources of error considered in this work are; (a) validity of the theories and the assumptions made in the derivation of groundwater flow equations, (b) the errors in hydraulic head due to incorrect estimation of aquifer parameters.

Mathematical description of groundwater flow in a porous medium is obtained from the law of conservation of mass and Darcy's law. Darcy (1856) performed a simple experiment and stated that the velocity of flow is proportional to the first power of the hydraulic gradient. This statement has been accepted as a 'law' both in petroleum engineering and hydrogeology. Although it has been proven that this 'law' is not valid under every condition, a better mathematical expression of the flow has not been developed. I believe that Darcy's law is valid in modelling large groundwater systems. The velocity of flow is averaged over a large area and stored as an average velocity at a grid point. Since transitional flow is encountered only at the well where velocities are relatively high, flow can be accepted as laminar at average velocities over most of the area of a large system.

An equation was derived in this study for the unconfined flow case which more accurately describes the movement of the free surface in two-dimensions. This is an approximation to the three-dimensional flow system. The terms expressing the water-table movement are similar to the terms obtained by Jacob (1950) describing the compressibility of the fluid. Transmissivity is dependent upon time through the variations in the density of the fluid, water. The magnitude of these terms is determined. Results show that errors due to neglect of these terms are small. Thus, we can conclude that water-table movement and compressibility of water can be ignored in the computer modelling of large groundwater flow systems.

The most important source of error in model preparation is the lack of accurate measurements of storage coefficient and transmissivity. A common method to determine these parameters is the history match approach. The history match approach is the minimization of the error between the calculated and observed head values by the adjustment of the aquifer parameters. This is also called an inverse problem. Inverse problem is an improperly posed problem and has no unique solution. Nevertheless, if the physical structure of the desired solution is available a unique solution can be obtained (Emsellem and deMarsily, 1971). In most cases, however, the physical structure of the solution can not be determined due to spotty data in the system.

A new method is developed to analyze the errors in the head values due to errors in aquifer parameters, using the sensitivity coefficients. Although sensitivity analysis was known as early as the 1930's, its application was in controllable systems. The application and the potential of sensitivity analysis are discussed intensively in the text.

The sensitivity analysis of groundwater flow systems with the equations derived in the study proved to be a valuable tool. Water levels in an area in west-central Kansas are computed from 1948 to 1980 and calculated water levels are compared with observed water levels from 1948 to 1975. Then the spatial distribution of the sensitivity coefficients for both transmissivity and specific yield are determined.

Errors in calculated water levels are computed by subtracting calculated water levels from the observed water levels. The errors in aquifer parameters are calculated by dividing the errors in water levels by the sensitivity coefficients. Now we will give two examples to show how useful the sensitivity analysis is.

1. An average hydraulic conductivity of 355.2 gal/day-ft^2 (50 ft/day) is assumed throughout the aquifer. The errors in hydraulic conductivity showed that the area in central Scott County has much larger hydraulic conductivities than the first estimate (Gutentag and Stullken, 1974).

2. The interaction between the aquifer and Ladder Creek was assumed to be negligible in the design of the model. The sensitivity map for variations in specific yield showed symmetric contour lines with respect to Ladder Creek. The system appeared very sensitive along Ladder Creek. Thus, specific yield values have to be determined accurately along this creek.

The potential use and advantage of the sensitivity equations derived in this study are discussed below;

1. The method should be helpful in the management of groundwater resources. A criteria can be adopted for the accuracy of water levels to be determined from computer models. The sensitivity coefficients can be computed from the equations derived in this study. Then permissible error in the aquifer parameters will be determined from the division of the allowable error in water level by the sensitivity of the particular parameter. Management organization can decide if it is feasible to gather the necessary data and how much it will cost. If the budget is limited then the accuracy of the computer model results can be

estimated from the accuracy of the data that can be gathered.

2. If the study of the groundwater flow system depends on existing data sensitivity coefficients can be used to obtain a better estimate of the aquifer parameters. The errors in aquifer parameters could be found from the division of the errors in water levels (observed minus calculated water levels) by the sensitivity coefficients for the particular parameter. Then a better estimate of the parameters could be obtained.

3. The method is cheaper than the perturbation methods. The groundwater flow equation and the sensitivity equation will be solved only once. There is no trial and error process involved so it is fast and also cheap.

The following recommendations are made for future studies;

1. A better expression of flow equations has to be made in areas where Darcy's law is invalid.

2. The discharge from groundwater systems must be known with greater accuracy.

3. Sensitivity analysis should be applied to more complex problems.

REFERENCES

Biot, M. A., General Theory of Three-Dimensional Consolidation, Journal of Appl. Phys., Vol. 12(2), pp. 155-164, 1941.

_______, Theory of Elasticity and Consolidation for a Porous Anisotropic Solid, Journal of Appl. Phys., Vol. 26(2), pp. 182-185, 1955.

Boulton, N. S., The Drawdown of the Water-Table under Non-Steady Conditions near a Pumped Well in an Unconfined Formation, Proc., Instn. Civ. Engrs., Vol. 3, part. 3, pp. 564-578, 1955.

Carnahan, B., Luther, H. A., and Wilkes, J. O., Applied Numerical Methods, John Wiley and Sons, Inc., New York, 1969.

Coats, K. H., Dempsey, J. R., and Henderson, J. H., A New Technique for Determining Reservoir Description from Field Performance Data, Soc. Petr. Engr. Journal, Vol. 10, No. 2, pp. 66-74, 1970.

Collins, R. G., Flow of Fluids Through Porous Materials, Reinhold Publication Corporation, New York, 270 pp., 1961.

Cooper, H. H., Jr., The Equation of Groundwater Flow in Fixed and Deforming Coordinates, Journal Geophys. Res., Vol. 71, No. 20, pp. 4785-4790, 1966.

Darcy, H., Les Fontaines Publiques de la Ville de Dijon, 647 pp., Victor Dalmond, Paris, 1856.

Davis, S. N. and DeWiest, R. J. M., Hydrogeology, John Wiley and Sons, Inc., pp. 463, 1967.

Dupuit, J., Etudes Theoriques et Pratiques sur le Mouvement de Eaux, Dunod, Paris, 2nd edition, 1863.

Dupuy, M., C. Jain, and Ph. Besset, Possibilites d'Adjustement Automatique des Modeles de Reservoir dans la Periode Monophasique de Production, Rev. Inst. Fr. Petrol., Vol. 26, No. 4, 269 pp., 1971.

Elnaggar, H., Karadi, G., Krizek, R. J., Effect of Non-Darcian Behavior on the Characteristics of Transient Flow, Journal of Hydr., Vol. 13, pp. 127-138, 1971.

Emsellem, Y., and DeMarsily, G., An Automatic Solution for the Inverse Problem, Water Resour. Res., Vol. 7, No. 5, pp. 1264-1283, 1971.

Forchheimer, P., Heber die Ergiebigkeit von Brunnun-Anlagen und sickerschlitzen, Zeitschrift Archiv Ingenieur Verein, Hannover, 1886.

Freeze, R. A., Three-Dimensional, Transient, Saturated-Unsaturated Flow in a Groundwater Basin, Water Resour. Res., Vol. 7, No. 2, pp. 347-366, 1971.

Gambolati, G., Equation of One-Dimensional Vertical Flow of Groundwater 2. Validity Range of the Diffusion Equation, Water Resour. Res., Vol. 9, No. 5, pp. 1385-1395, 1973.

Gambolati, G., and Freeze, R. A., Mathematical Simulation of the Subsidence of Venice, Water Resour. Res., Vol. 9, No. 3, pp. 721-733, 1973.

Gutentag, E. D., and Stullken, L. E., Ground Water Resources of Lane and Scott Counties, Western Kansas, Kansas Geol. Survey Irrigation Series Number 1 (in print).

Halepaska, J. C., Comment on "Galerkin Solution of the Inverse Problem for Aquifer Transmissivity" by Emil O. Frind and George F. Pinder, Water Resour. Res., Vol. 10, No. 3, pp. 609, 1974.

Halepaska, J. C., and Hartman, F. W., Numerical Solution of the Three-Dimensional Heat Flow Equation, Short Papers on Research in 1971, Bull. 204, part 1, pp. 4.

Hanbush, M. S., Hydraulics of Wells, Advances in Hydroscience (Van Te Chow, ed.), pp. 281-96, Academic Press, 1964.

Hodson, W. G., Geology and Ground-Water Resources of Wallace County, Kansas, State Geol. Survey of Kansas, Bull. 161, pp. 108, 1963.

Hubbert, M. K., The Theory of Groundwater Motion, Journal of Geol., Vol. XLVIII, No. 8, 1940.

Jacob, C. E., The Flow of Water in an Elastic Artesian Aquifer, Transactions A.G.U., Vol. 21, pp. 583, 1940.

________, Flow of Ground Water, Engineering Hydraulic (H. Rouse, ed.), John Wiley and Sons, New York, pp. 321-328, 1950.

Jackquard, P. and Jain, C., Permeability Distribution from Field Pressure Data, Soc. of Petrol. Engr. Journal, Vol. 5, No. 4, pp. 281-289, 1965.

Jahns, H. O., A Rapid Method for Obtaining a Two-Dimensional Reservoir Description from Field Pressure Data, Soc. of Petrol. Engr. Journal, Vol. 6, No. 4, pp. 315-327, 1966.

Kleinecke, D., Use of Linear Programming for Estimating Geohydrologic Parameters of Groundwater Basins, Water Resour. Res., pp. 367-374, 1971.

Lovell, R. E., Duckstein, L., and Kisiel, C. C., Use of Subjective Information in Estimation of Aquifer Parameters, Water Resour. Res., Vol. 8, No. 3, pp. 680-693, 1972.

Margenan, H., The Mathematics of Physics and Chemistry, D. Van Nostrand Comp. Inc., Princeton, New Jersey, pp. 604, 1968.

Matthews, C. S., and Russell, D. G., Pressure Buildup and Flow Tests in Wells, Soc. of Petrol. Engr., Henry L. Doberty Series, Monograph Vol. 1, 1964.

McElwee, Carl, Groundwater Seminar Notes, Scott City Conference, Oct. 1975.

McCuen, R. H., Component Sensitivity: A Tool for the Analysis of Complex Water Resources Systems, Water Resour. Res., Vol. 9, No. 1, 1973.

Meinzer, D. E., Compressibility and Elasticity of Artesian Aquifers, Econ. Geol. Vol. 23, pp. 263-241, 1928.

Nelson, R. W., In Place Determination of Permeability Distribution for Heterogeneous Porous Media through Analysis of Energy Dissipation, Soc. of Petrol. Engr., Vol. 8, No. 1, pp. 33-47, 1968.

Nelson, R. W., and McCollum, W. L., Transient Energy Dissipation Methods of Measuring Permeability Distributions in Heterogeneous Porous Materials, Rep. CSC 691229, Water Resour. Div., U.S. Geol. Survey, Wash., D.C., 1969.

Neumann, S. P., and Witherspoon, P. A., Finite Element Method of Analyzing Steady Seepage with a Free Surface, Water Resour. Res., Vol. 6, No. 3, pp. 889-907, 1970.

Neumann, S. P., Theory of Flow in Unconfined Aquifers Considering Delayed Response of the Water Table, Water Resour. Res., Vol. 8, No. 4, pp. 1031-1045, 1972.

________, Effect of Partial Penetration on Flow in Unconfined Aquifers Considering Delayed Gravity Response, Water Resour. Res., Vol. 10, No. 2, pp. 303-312, 1974.

Paaswell, R., Thermal Influence on Flow from a Compressible Porous Medium, Water Resour. Res., Vol. 3, No. 1, pp. 271-278, 1967.

Prescott, G. C., Jr., Geology and Ground-Water Resources of Cheyenne County, Kansas, Kansas Geol. Survey, Bull. 100, pp. 1-106, 1951.

Prescott, G. C., Jr., Branch, J. R., and Wilson, W. W., Geology and Ground-Water Resources of Wichita and Greeley Counties, Kansas, Kansas Geol. Survey, Bull. 108, pp. 134, 1954.

Riley, J. P., Computer Simulation of Water Resource Systems, Proc. the 2nd Intr. Seminar for Hydrology Prof., Utah State Univ., Logan, August, 1970, pp. 249-274.

Slatter, G. E., and Durrar, E. J., Adjustment of Reservoir Simulation Models to Match Field Performance, Soc. of Petrol. Engr., Vol. 11, No. 3, pp. 295-305, 1971.

Smith, G. D., Numerical Solution of Partial Differential Equations, Oxford University Press, New York, pp. 201, 1965.

Smith, J. E., Shale Compaction, Pap. SPE 3633, AIME Soc. of Petrol. Engr., New Orleans, pp. 9, 1971.

Stallman, R. W., Numerical Analysis of Regional Water Levels to Define Aquifer Hydrology, Eos. Trans. AGU, Vol. 37, No. 4, pp. 451-462, 1956.

Streltsova, T. D., Flow Near a Pumped Well in an Unconfined Aquifer under Non-Steady Conditions, Water Resour. Res., Vol. 9, No. 1, pp. 227-235, 1972.

________, Unsteady Radial Flow in an Unconfined Aquifer, Water Resour. Res., Vol. 8, No. 4, pp. 1059-1066, 1973.

Stullken, L. E., Weakly, E. C., Gutentag, E. D., and Slagle, S. E., Hydrogeologic Data from Greeley, Wichita, Scott, and Lane Counties, Ground Water Release No. 4, Kansas Geol. Survey, pp. 58, 1974.

Theis, C. V., The Relation Between the Lowering of the Piezometric Surface and the Rate and Duration of Discharge of a Well Using Groundwater Storage, Eos. Trans. AGU, Vol. 2, pp. 519-524, 1935.

Todd, D. K., Ground-Water Hydrology, Wiley Corp., pp. 386, 1959.

Tomovic, R., Sensitivity Analysis of Dynamic Systems, McGraw-Hill Book Comp. Inc., pp. 142, 1963.

Vemuri, V., Dracup, A., Erdmann, R. C., and Vemuri, N., Sensitivity Analysis Method of System Identification and its Potential in Hydrologic Research, Water Resour. Res., Vol. 5, No. 2, pp. 341-348, 1969.

Vemuri, V., and Karplus, W. J., Identification of Non-Linear Parameters of Groundwater Basins by Hybrid Computation, Water Resour. Res., Vol. 5, No. 1, pp. 17-24, 1970.

Von Rosenberg, D. U., Methods for the Numerical Solution of Partial Differential Equations, American Elsevier Publishing Co., New York, pp. 118, 1969.

Waite, H. A., Geology and Ground-Water Resources of Scott County, Kansas, Kansas Geol. Survey, Bull. 66, pp. 1-216, 1947.

Wenzel, L. K., Methods for Determining Permeability of Water-Bearing Materials with Special Reference to Discharging-Well Methods, U.S. Geol. Survey, Water-Supply Paper 887, pp. 1-192, 1942.

Zienkiewicz, O. C., The Finite Element Method in Structural and Continuum Mechanics, McGraw-Hill Book Co., New York, 1967.

ADDITIONAL REFERENCES

Andreev, V. B., 1971, On stability with respect to the initial data of difference schemes for parabolic equations: USSR Math and Phys. Math., vol. 11-6, pp. 129-145.

Aris, Rutherford, 1962, Vectors, tensors, and the basic equations of fluid mechanics.

Babuska, Iro, March 1972, Numerical stability in problems of linear algebra: SIAM T. Num. Anal., pp. 55-77.

Chandhari, Narayan M., June 1971, An improved numerical technique for solving multidimensional discible displacement equations: Soc. of Petr. Eng. Jour., pp. 277-284.

Cheng, Sim-I, 1969, Accuracy of difference formulation of Navier-Stokes equations: The PHysics of Fluids Supplement II, pp. II-34 - II-41.

Clegg, M. W., and Mills, M., Dec. 1970, A study of the behavior or partially penetrating wells: Soc. of Petr. Eng., pp. 189-203.

DeVries, R. N., and Kent, D. C., Oct. 1973, Sensitivity of groundwater flow models to verical variability of aquifer constants: Water Res. Bull., pp. 998-1005.

Dipuma, R. C., Rogers, G. H., 1969, Computing problems in nonlinear hydrodynamic stability: The PHysics of Fluids Supp. II, pp. II-155 - II-165.

Fread, D. L., April 1973, Effects of tune step size in implicit dynamic routing, Water Res. Bull., pp. 338-351.

Fromm, T. E., 1969, Practical investigation of convective difference approximations of reduced dispersion: The Physics of Fluids Supplement II, pp. II-3 - II-12.

Hirasaki, G. J., and O'Dell, P. M., Dec. 1970, Representation of reservoir geometry for numerical simulation: Soc. of Petr. Eng., pp. 303-404.

Huang, Edward T. S., Feb. 1972, A sensitivity study of reservoir performance using a compositional reservoir simulator: Soc. of Petr. Eng., pp. 3-12.

Jackson, Donald R. and Aron, Gert, June 1971, Parameter estimation in hydrology: The state of the art Water Res. Rull., vol. 7, no. 3, pp. 457-472.

Karlini, H., Dec. 1969, Pressure transient analysis of naturally fractured reservoirs with uniform fracture distribution: Soc. of Petr. Eng., pp. 451-462.

Kazenir, H., Seth, M. S., and Thomas, G. W., Dec. 1969, The interpretation of interference tests in naturally fractured reservoirs with uniform fracture distribution: Soc. of Petr. Eng., pp. 463-472.

Lantz, R. B., Nov. 1971, Quantitative evaluation of numerical diffusion (truncation error): Soc. of Petr. Eng. Jour., pp. 315-320.

Loeh, Stefan, 1969, Examination of the stability of disturbed boundary - layer flow by a numerical method: The Physics of Fluids Supplement II, pp. II-139 - II-145.

Meyer, H. D., Dept. 1973, The numerical solution of nonlinear parabolic problems by variational methods: SIAM T. Num. Anal., pp. 700-722.

Nolen, James S. and Berry, D. W., June 1972, Tests of the stability and time-step sensitivity of semi-implicit reservoir stimulation techniques: Soc. of Petr. Eng. Jour., pp. 253-266.

Odeh, A. S., McMillen, T. M., Oct. 1972, Pulse testing: Mathematical analysis and experimental verification: Soc. of Petr. Eng., pp. 403-409.

Ortega, James M., 1972, Computer Science and Applied Mathematics.

Pinder, George F., Frind, Emil O., Feb. 1972, Application of Galerkins' Procedure to aquifer analysis: Water Resources Research, pp. 108-120.

Pinder, George F., Frind, Emil O., and Papadopulos, Stavros P., Feb. 1973, Functional coefficients in the analysis of groundwater flow: Water Resources Research, pp. 222-226.

Plair, P. M., and Weinaug, C. F., Dec. 1969, Solution of two-phase flow problems using implicit difference equations: Soc. of Petr. Eng., pp. 412-423.

Roebuck, I. F., Henderson, G. E., Douglas Ligor, Jr., and Ford, W. T., March 1969, The compositional reservoir simulator: Case I - The linear model: Soc. of Petr. Eng., pp. 114-130.

Settari, A., and Aziz, Khalid, April 1972, Use of irregular grid in reservoir simulation: Soc. of Petr. Eng. Jour., pp. 103-109.

Shtoian, G., 1971, Stability of additive difference schemes to the boundary value: USSR Math and Phys. Math., vol. 11-4, pp. 149-165.

Snyder, M. Willard, and Decoursey, Doug G., Computer-oriented of optimizing hydrologic model parameters: Jour. of Hydrology 9, pp. 34-56.

Spivak, A., Sept. 1970, Numerical simulation of county using implicit production terms: Soc. of Petr. Eng., pp. 257-267.

Thomas, Kent L., et. al., Dec. 1972, A nonlinear automatic history matching techniques for reservoir simulation models: Soc. of Petr. Eng., pp. 508-514.

Todd, M. R., et. al., Dec. 1972, Methods of increased accuracy in numerical reservoir simulators: Soc. of Petr. Eng. Jour., pp. 515-530.

Vela, Saul and McKineley, R. M., Feb. 1970, How aerial heterogeneities affect pulse-test results: Soc. of Petr. Jour., pp. 181-191.

Weinberger, H. F., March 1971, An optional numerical solution of partial differential equations: SIAM T. Num. Anal., pp. 182-197.

Wessel, W. Roy, 1969, Numerical stability of the collapse of a perturbation in an infinite density stratified fluid: The Physics of Fluids Supplement II, pp. II-171 - II-176.

Yen, William, W. G., and Touxe, George W., April 1971, Quasilinearization and the identification of aquifer parameters: Water Resources Research, pp. 375-381.

APPENDIX A

BASIC EQUATIONS OF GROUNDWATER MOVEMENT

i. CONFINED AQUIFER

Mathematical description of fluid flow in a porous media is obtained from the following physical principles: (1) the law of conservation of mass; (2) Darcy's law.

If we consider an elemental volume, ΔV, of the aquifer oriented in the field of flow, the net inward flux or the storage must be equal to the rate at which the water is accumulating within that volume due to the law of conservation of mass. Therefore, the net inward flux will be the sum of the difference of the incoming and outgoing mass of water from the pair of the faces of the elemental volume (Fig. 1):

on $\Delta x \Delta y$ face $\quad \rho u_z \Delta x \Delta y - (\rho u_z + \frac{\partial \rho u_z}{\partial z} \Delta z) \Delta x \Delta y$

on $\Delta x \Delta z$ face $\quad \rho u_y \Delta x \Delta z - (\rho u_y + \frac{\partial \rho u_y}{\partial y} \Delta y) \Delta x \Delta z$

on $\Delta y \Delta z$ face $\quad \rho u_x \Delta y \Delta z - (\rho u_x + \frac{\partial \rho u_x}{\partial x} \Delta x) \Delta y \Delta z$

Thus, the net inward flux is: $\quad - \frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} + \frac{\partial \rho u_z}{\partial z} \Delta x \Delta y \Delta z \qquad (1)$

The net inward flux has to be equal to the change of the mass of water present in the elemental volume with respect to time. It can be expressed as:

change in storage = $\frac{\partial(\Delta M)}{\partial t}$

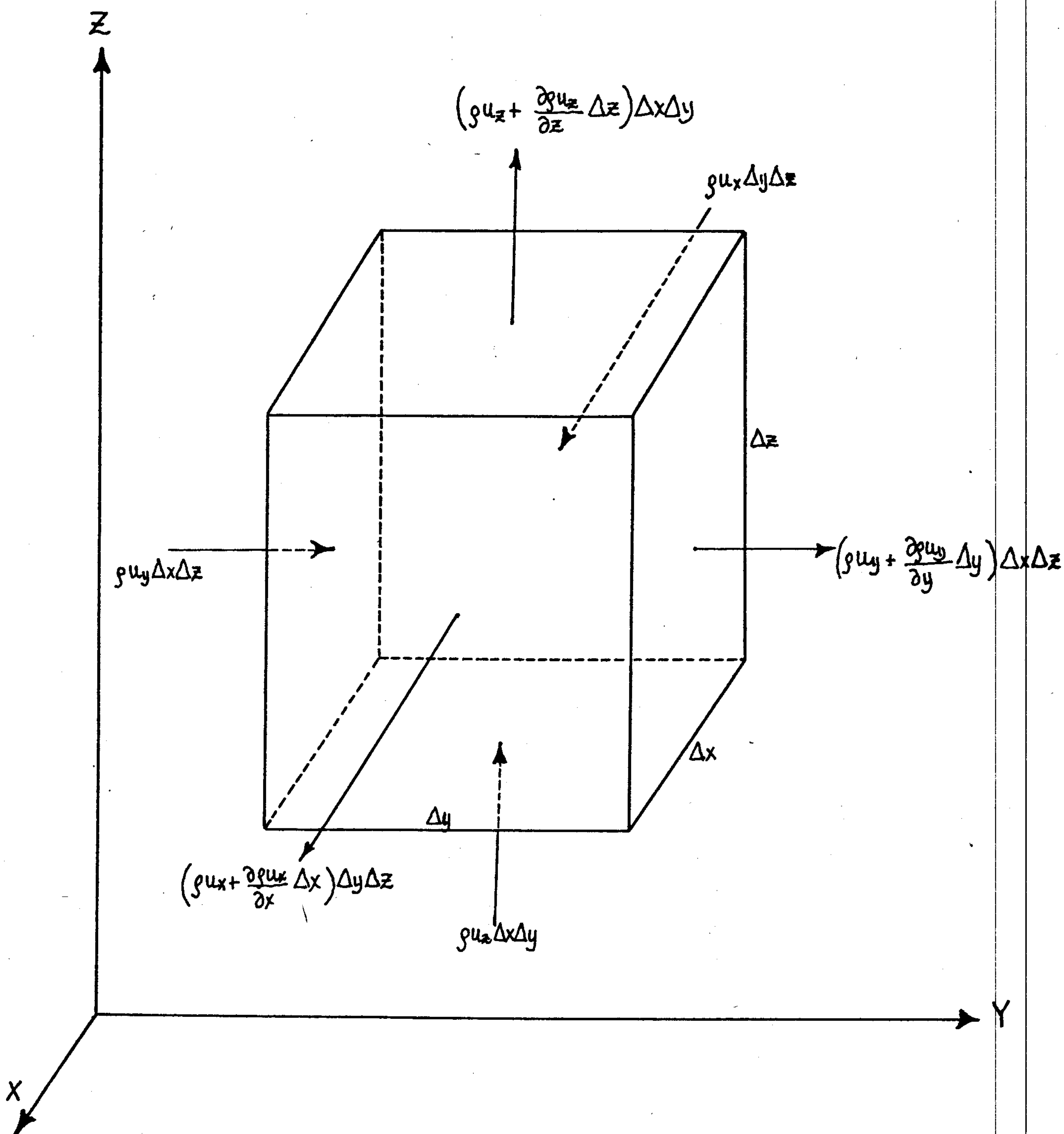


Fig. 1. Elemental volume in three dimensions with incoming and outgoing mass of water from its faces.

The mass of water present in the elemental volume can be expressed as the product of the porosity of the elemental volume, density of the water and the volume of the elemental volume:

$$\Delta M = \rho\theta\Delta x\Delta y\Delta z$$

Compressibility and elasticity of aquifers and compressibility of fluids have long been established by laboratory and field observations. The vertical compressibility of solid skeleton of the elemental volume can be expressed as the reciprocal of the Young's modulus ($\sigma_z = 1/E$) and $- d\sigma_z = d\Delta z/\Delta z$, where $d\sigma_z$ is the change in stress and $d\Delta z/\Delta z$ is the change in strain. Volume of the solid skeleton can be written as:

$$\Delta V_s = (1-\theta)\ \Delta x\Delta y\Delta z$$

If the compressibility of the grains is assumed to be negligible compared to change in porosity then we can write:

$$d\Delta V_s = [(1-\theta)\ \Delta x\Delta y\Delta z] = 0$$

$$\Delta z d\ (1-\theta) + (1-\theta)\ d\Delta z = 0 \qquad \text{or} \qquad d\theta = \frac{(1-\theta)}{\Delta z}\ d\Delta z$$

Differentiating with time

$$\frac{\partial\theta}{\partial t} = \frac{(1-\theta)}{\Delta z}\ \frac{\partial\Delta z}{\partial t} \qquad \text{or} \qquad \frac{\partial\theta}{\partial t} = \frac{(1-\theta)}{\Delta z}\ -\alpha\ \frac{\partial\sigma z}{\partial t}\cdot\Delta z = -\alpha\ (1-\theta)\ \frac{\partial\sigma z}{\partial t}$$

On the other hand, compressibility of fluid can be given with the formula:

$$\beta = \frac{1}{Ew} \qquad \text{or} \qquad -\beta dp = \frac{d\rho}{\rho o} \qquad\qquad d\rho = -\ \beta\rho_o dp$$

Differentiating with respect to time

$$\frac{\partial\rho}{\partial t} = -\ \beta\rho_o\ \frac{\partial p}{\partial t}$$

If we assume that the elemental volume is in static equilibrium with the overburden weight and atmospheric pressure and that these external forces are constant, that is;

$$p + \sigma_z = \text{const.} \qquad \frac{\partial p}{\partial t} = -\frac{\partial \sigma_z}{\partial t}$$

Now we can apply Darcy's law which states that the volumetric rate of flow per unit cross-sectional area at any point in a uniform porous medium is proportional to the gradient of hydraulic head in the direction of flow at that point. Hubbert showed the head as

$$h = \int_{p_o}^{p} \frac{dp}{\gamma} + gz$$

From Darcy's law we can express velocities as

$$V_x = -K_x \frac{\partial h}{\partial x} \qquad V_y = -K_y \frac{\partial h}{\partial y} \qquad V_z = -K_z \frac{\partial h}{\partial z}$$

Put all these terms in equation (I)

$$\left[\frac{\partial}{\partial x} \rho K_x \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} \rho K_y \frac{\partial h}{\partial y} + \frac{\partial}{\partial z} \rho K_z \frac{\partial h}{\partial z}\right] \Delta x \Delta y \Delta z$$

$$= \left[\rho \Delta z \left(-\alpha\ (1-\theta)\right)\frac{\partial \sigma z}{\partial t} + \rho\theta \left(-\alpha \Delta z\ \frac{\partial \sigma z}{\partial t}\right) + \theta \Delta z \left(-\beta \rho_o \frac{\partial p}{\partial t}\right)\right] \Delta x \Delta y$$

Since

$$-\frac{\partial p}{\partial t} = \frac{\partial \sigma z}{\partial t}$$

and

$$\frac{\partial p}{\partial t} = \gamma \frac{\partial h}{\partial t}$$

then from Hubberts expression of hydraulic head

$$\frac{\partial}{\partial x} \rho K_x \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} \rho K_y \frac{\partial h}{\partial y} + \frac{\partial}{\partial z} \rho K_z \frac{\partial h}{\partial z} = \gamma(\theta\beta\rho_o + \rho\alpha) \frac{\partial h}{\partial t}$$

If the change in density is assumed negligible, then the equation can be written

$$\frac{\partial}{\partial x} K_x \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} K_y \frac{\partial h}{\partial y} + \frac{\partial}{\partial z} K_z \frac{\partial h}{\partial z} = \gamma\theta\beta\left(1 + \frac{\alpha}{\beta\theta}\right)\frac{\partial h}{\partial t}$$

If the equation is expressed in terms of transmissivity and storativity, it can be written

$$\frac{\partial}{\partial x} T \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} T \frac{\partial h}{\partial y} + \frac{\partial}{\partial z} T \frac{\partial h}{\partial z} = S \frac{\partial h}{\partial t}$$

where

E =	Elasticity modulus (stress/strain)	$(ML^{-1}T^2)$
h =	$(\frac{P}{\gamma} - z)$, hydraulic head	(L)
K =	hydraulic conductivity	(LT^{-1})
p =	pressure	$(ML^{-1}T^{-2})$
S =	storativity	(L^{-1})
t =	time	(T)
V =	velocity	(LT^{-1})
α =	compressibility of the solid volume	(LT^2M^{-1})
β =	compressibility of water	(LT^2M^{-1})
γ =	specific weight	$(ML^{-2}T^{-2})$
θ =	porosity	(0)
σ_z =	vertical stress	$(ML^{-1}T^{-2})$

ii. UNCONFINED AQUIFER

Let's consider the elemental volume, dv, of the aquifer oriental in the flow field (Fig. 2). Since the upper surface of the aquifer is the water table, Hantush (1964) suggested that this case may be treated in terms of a head distribution that is averaged in the vertical directions. Therefore, the equation obtained for the confined case should be integrated between the bottom and the top of the aquifer.

The base of the aquifer is considered horizontal and taken as the reference plane. The thickness of the aquifer is given by the term b(x,y,t). Integration of the continuity equation between the limits o to b(x,y,t) with respect to z, without putting in Darcy's velocity expressions,

$$-\left[\int_0^{b(x,y,t)} \frac{\partial}{\partial x} u dz + \int_0^{b(x,y,t)} \frac{\partial v}{\partial y} dz + \int_0^{b(x,y,t)} \frac{\partial}{\partial z} w dz\right] = S_s \int_0^{b(x,y,t)} \frac{\partial h}{\partial t} dz \quad (1)$$

Differentiation under the integral sign

$$-\left[\frac{\partial}{\partial x} \int_0^{b(x,y,t)} udz - u(b) \frac{\partial b}{\partial x} + \frac{\partial}{\partial y} \int_0^{b(x,y,t)} vdz - v(b) \frac{\partial b}{\partial y} + w(b)\right]$$

$$= S_s \left[\frac{\partial}{\partial t} \int_0^{b(x,y,t)} hdz - h(b) \frac{\partial b}{\partial t}\right] \quad (2)$$

Where u(b), v(b), and w(b) are velocities in x, y, and z direction, respectively at the top of the aquifer.

Hydraulic head, h, is equal to the saturated thickness b(x,y,t) at the top of the aquifer.

$h(b) = b(x,y,t)$

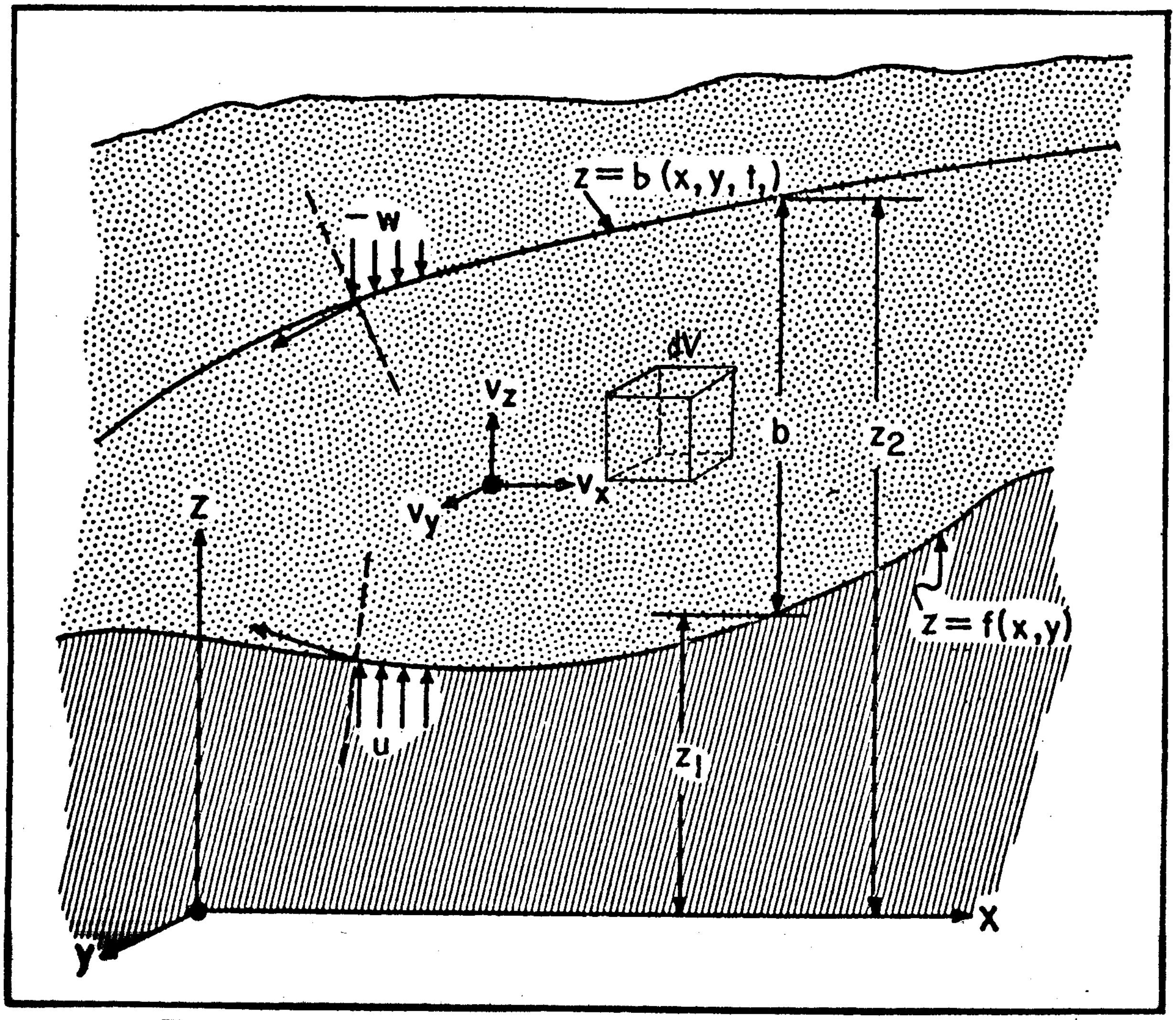


Figure 2. Illustration of an elemental volume in an unconfined flow field.

Since the upper surface of the aquifer, water table, is moving with change in the saturated thickness, b(x,y,t), differentiation with respect to time give velocity in the z direction. Now the velocity in the z direction can be expressed as:

$$\frac{db}{dt} = \frac{\partial b}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial b}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial b}{\partial t} \tag{3}$$

where $\frac{db}{dt}, \frac{dx}{dt}, \frac{dy}{dt}$ are defined as effective velocity, $V_{eff} = V_{bulk} / \varepsilon$

ε is the specific yield, v_{bulk} is the bulk velocity, or Darcy velocity. Replacing eq. 3 in the eq. 2,

$$-\left[\frac{\partial}{\partial x} \int_0^{b(x,y,t)} udz - u(b)\frac{\partial b}{\partial x} + \frac{\partial}{\partial y} \int_0^{b(x,y,t)} vdz - v(b)\frac{\partial b}{\partial y} + \varepsilon \frac{\partial b}{\partial t} + u(b)\frac{\partial b}{\partial x} + v(b)\frac{\partial b}{\partial y}\right] = S_s\left[\frac{\partial}{\partial t} \int_0^{b(x,y,t)} hdz - h(b)\frac{\partial b}{\partial t}\right] \tag{4}$$

$$-\left[\frac{\partial}{\partial x} \int_0^{b(x,y,t)} udz + \frac{\partial}{\partial y} \int_0^{b(x,y,t)} vdz + \varepsilon \frac{\partial b}{\partial t}\right] = S_s\left[\frac{\partial}{\partial t} \int_0^{b(x,y,t)} hdz - h(b)\frac{\partial b}{\partial t}\right] \tag{5}$$

Since

$$u = -K_x \frac{\partial h}{\partial x} \qquad v = -K_y \frac{\partial h}{\partial y}$$

$$\frac{\partial}{\partial x} \int_0^{b(x,y,t)} K_x \frac{\partial h}{\partial x} dz + \frac{\partial}{\partial y} \int_0^{b(x,y,t)} K_y \frac{\partial h}{\partial y} dz - \varepsilon \frac{\partial b}{\partial t} = S_s\left[\frac{\partial}{\partial t} \int_0^{b(x,y,t)} hdz - h(b)\frac{\partial b}{\partial t}\right] \tag{6}$$

Differentiation under the sign

$$\frac{\partial}{\partial x}\left[K_x\left(\frac{\partial}{\partial x}\int_0^{b(x,y,t)} hdz - h(b)\frac{\partial b}{\partial x}\right)\right] +$$

$$\frac{\partial}{\partial y}\left[K_y\left(\frac{\partial}{\partial y}\int_0^{b(x,y,t)} hdz - h(b)\frac{\partial b}{\partial y}\right)\right] - \varepsilon\frac{\partial b}{\partial t}$$

$$= S_s\, h(b)\frac{\partial b}{\partial t} + S_s\frac{\partial}{\partial t}\int_0^{b(x,y,t)} hdz \qquad (7)$$

or

$$\frac{\partial}{\partial x} K_x \frac{\partial}{\partial x}\int_0^{b(x,y,t)} hdz - \frac{\partial}{\partial x} K_x\, h(b)\frac{\partial b}{\partial x} + \frac{\partial}{\partial y} K_y \frac{\partial}{\partial y}\int_0^{b(x,y,t)} hdz -$$

$$\frac{\partial}{\partial y} K_y \frac{\partial}{\partial y}\int_0^{b(x,y,t)} hdz + \varepsilon\frac{\partial b}{\partial t} = S_s\, h(b)\frac{\partial b}{\partial t} - S_s\frac{\partial}{\partial t}\int_0^{b(x,y,t)} hdz \qquad (8)$$

Multiplying both sides by -1.

$$\frac{\partial}{\partial x} K_x\, h(b)\frac{\partial b}{\partial x} + \frac{\partial}{\partial y} K_y\, h(b)\frac{\partial b}{\partial y} - \frac{\partial}{\partial x} K_x \frac{\partial}{\partial x}\int_0^{b(x,y,t)} hdz -$$

$$\frac{\partial}{\partial y} K_y \frac{\partial}{\partial y}\int_0^{b(x,y,t)} hdz + \varepsilon\frac{\partial b}{\partial t} = S_s\, h(b)\frac{\partial b}{\partial t} - S_s\frac{\partial}{\partial t}\int_0^{b(x,y,t)} hdz \qquad (9)$$

If the average head is defined as

$$\bar{h} = \frac{1}{b}\int_0^{b(x,y,t)} hdz$$

Then,

$$\frac{\partial}{\partial x} K_x\, h(b)\frac{\partial b}{\partial x} + \frac{\partial}{\partial y} K_y\, h(b)\frac{\partial b}{\partial y} - \frac{\partial}{\partial x} K_x \frac{\partial b\bar{h}}{\partial x} - \frac{\partial}{\partial y} K_y \frac{\partial b\bar{h}}{\partial y}$$

$$= S_s\, h(b)\frac{\partial b}{\partial t} - S_s\frac{\partial}{\partial t} b\bar{h} - \varepsilon\frac{\partial b}{\partial t}$$

or

$$\frac{\partial}{\partial x} K_x \frac{\partial}{\partial x} b\bar{h} + \frac{\partial}{\partial y} K_y \frac{\partial}{\partial y} b\bar{h} - \frac{\partial}{\partial x} K_x b \frac{\partial b}{\partial x} - \frac{\partial}{\partial y} K_y b \frac{\partial b}{\partial y}$$

$$= S_s \frac{\partial}{\partial t} b\bar{h} + \varepsilon \frac{\partial b}{\partial t} - S_s b \frac{\partial b}{\partial t} \tag{10}$$

This is Hantush's derivation of equation of groundwater flow in unconfined case.

The other equation can be derived using Dupuit-Forchheimer assumptions, and by using it over a small volume of aquifer with base of unit area and height being the saturated thickness. Saturated thickness is set equal to h.

The equation that is derived under these assumptions for two dimensional case is given by (Jacob, 1950)

$$\frac{K}{2} \left[\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} \right] = S \frac{\partial h}{\partial t}$$

Expanding the equation and assuming k is not constant,

$$\frac{\partial}{\partial x} Kh \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Kh \frac{\partial h}{\partial y} = S \frac{\partial h}{\partial t} \tag{11}$$

Hantush's equation can be stated similarly if average head is assumed to be equal to the saturated thickness. Expansion of eq. 10 and cancelling yields.

$$\frac{\partial}{\partial x} Kh \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Kh \frac{\partial h}{\partial y} = S \frac{\partial h}{\partial t} + \varepsilon \frac{\partial h}{\partial t} \tag{12}$$

iii. OTHER APPROACH FOR UNCONFINED CASE

A small volume of the aquifer is considered with base being a unit area and the thickness is the saturated thickness b(x,y,t) which is equal to h(x,y,z,t) on the free surface (Fig. 3).

$$Q_x = -K_x \frac{\partial h}{\partial x} b\Delta y \quad , \quad Q_{x+\Delta x} = -K_x \frac{\partial h}{\partial x} b\Delta y + \frac{\partial}{\partial x}\left[-K_x \frac{\partial h}{\partial x} \Delta x b\right] \Delta y$$

$$Q_y = -K_y \frac{\partial h}{\partial y} b\Delta x \quad , \quad Q_{y+\Delta y} = -K_y \frac{\partial h}{\partial y} b\Delta x + \frac{\partial}{\partial y}\left[-K_y \frac{\partial h}{\partial y} \Delta y b\right] \Delta x$$

$$Q_z = -\left[\varepsilon \frac{\partial b}{t} + u(b) \frac{\partial b}{x} + v(b) \frac{\partial b}{y}\right] \Delta x \Delta y \quad , \quad Q_{z+\Delta z} = 0$$

Where velocity in the direction is assumed to be the total derivative of saturated thickness, which can be taken as a displacement term.

$$\Delta Q_x + \Delta Q_y + \Delta Q_z = \frac{\partial \Delta M}{\partial t}$$

$$\frac{\partial}{\partial x} K_x b \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} K_y b \frac{\partial h}{\partial y} + K_x \frac{\partial b}{\partial x} \cdot \frac{\partial b}{\partial x} + K_y \frac{\partial b}{\partial y} \cdot \frac{\partial b}{\partial y} = S \frac{\partial b}{\partial t} + \varepsilon \frac{\partial b}{\partial t}$$

If b is set equal to h,

$$\frac{\partial}{\partial x} K_x h \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} K_y h \frac{\partial h}{\partial y} + K_x \left(\frac{\partial h}{\partial x}\right)^2 + K_y \left(\frac{\partial h}{\partial y}\right)^2 = S\frac{\partial h}{\partial t} + \varepsilon \frac{\partial h}{\partial t} \qquad (13)$$

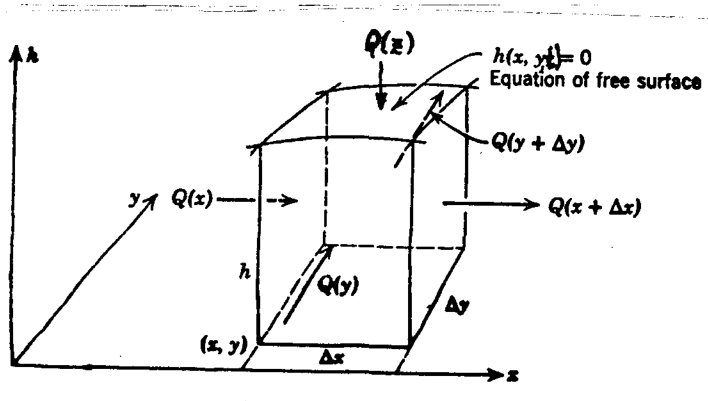


Figure 3. Elemental volume in the flow field.

There are two extra terms in this equation due to movement of the free surface. Both of these terms go to zero faster than the other two terms on the left hand side.

APPENDIX B

SOLUTION OF THE EQUATION OF FLOW OF GROUNDWATER

The partial differential equation derived will be solved by the Crank-Nicolson scheme. This scheme is chosen because it is proved to be unconditionally stable and convergent. Therefore, there is no restriction on the time step and on the space interval from the point of stability.

The basic idea of the finite-difference methods is to replace derivatives at a point by ratios of the changes in appropriate variables over a small but finite interval.

Thus

$$\frac{dh}{dx} = \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} \cong \frac{\Delta h}{\Delta x} \tag{1}$$

This type of approximation is made at finite number of points and reduces a continuous boundary-value problem to a set of algebraic equations which are easy to solve.

When a smooth function h(x) and its derivatives are single-valued, finite and continuous functions of x, then this function can be expanded into Taylor's series at a point which is in the neighborhood of the point considered.

$$h(x+\Delta x) = h(x) + \frac{\Delta x}{1!}\frac{\partial h}{\partial x} + \frac{(\Delta x)}{2!}\frac{\partial^2 h}{\partial x^2} + \cdots\cdots \tag{2}$$

For the function h(x,y,t),

$$h(x\pm\Delta x,\ y\pm\Delta y,\ z\pm\Delta z) = h(x,y,z) + \sum_{i=1}^{\infty} (\pm 1)^i \ \frac{\Delta x}{i!}\frac{\partial h}{\partial x}(x,y,z) +$$

$$\frac{\Delta y}{i!}\frac{\partial h}{\partial y}(x,y,z) + \frac{\Delta z}{i!}\frac{\partial h}{\partial z}(x,y,z) \tag{3}$$

If the terms higher than first order are neglected in equation (3).

$$h(x+\Delta x) = h(x) + \frac{\Delta x}{1!} \frac{\partial h}{\partial x}$$

Therefore

$$\frac{\partial h}{\partial x} \cong \frac{h(x+\Delta x) - h(x)}{\Delta x} + O(\Delta x) \qquad (4)$$

This is the approximation to the slope $\frac{\partial h}{\partial x}$ with an error of order x. This is called forward-difference formula.

If $h(x-\Delta x)$ is expanded into Taylor series at point x.

$$\frac{\partial h}{\partial x} \cong \frac{h(x) - h(x-\Delta x)}{\Delta x} + O(\Delta x) \qquad \text{(backward-difference formula)} \qquad (5)$$

Subtraction of the terms obtained by expanding $h(x-\Delta x)$ to Taylor series at x yields,

$$\frac{\partial h}{\partial x} \cong \frac{h(x+\Delta x) - h(x-\Delta x)}{2\Delta x} + O(\Delta x)^2 \qquad (6)$$

This is the central difference formula and the error is order of x^2.

All these relations are shown in figure 1.

Addition of the terms obtained by expanding $h(x+\Delta x)$ and $h(x-\Delta x)$ to Taylor series at point x yields.

$$h(x+\Delta x) = h(x) + \Delta x \frac{\partial h}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 h}{\partial x^2} + \frac{(\Delta x)^3}{3!} \frac{\partial^3 h}{\partial x^3} + \frac{(\Delta x)^4}{4!} \frac{\partial^4 h}{\partial x^4}$$

$$+ \quad h(x- x) = h(x) = \Delta x \frac{\partial h}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 h}{\partial x^2} - \frac{(\Delta x)^3}{3!} \frac{\partial^3 h}{\partial x^3} + \frac{(\Delta x)^4}{4!} \frac{\partial^4 h}{\partial x^4}$$

$$h(x+\Delta x) + h(x-\Delta x) = 2h(x) + (\Delta x)^2 \frac{\partial^2 h}{\partial x^2} + 2 \left[\frac{(\Delta x)^4}{4!} \frac{\partial^4 h}{\partial x^4}\right]$$

Neglecting terms higher than second order and arranging the equation

$$h(x+\Delta x) - 2h(x) + h(x-\Delta x) = (\Delta x)^2 \frac{\partial^2 h}{\partial x^2} + O(\Delta x^4)$$

$$\frac{\partial^2 h}{\partial x^2} \simeq \frac{h(x+\Delta x) - 2h(x) + h(x-\Delta x)}{(\Delta x)^2} + O(\Delta x^2) \qquad (7)$$

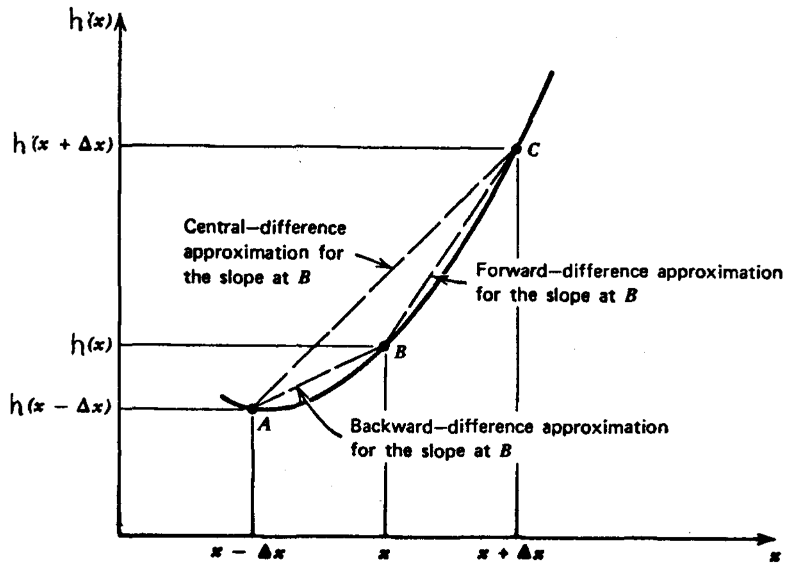


Figure 1. Illustration of different approximations to the slope term.

Similar terms can be obtained for partial derivatives with respect to time.

Two types of errors are encountered when partial differential equations are solved using finite difference techniques by the computers.

1. Discretization error, resulting from neglecting higher order terms.
2. Round-off error, due to computer capability of handling only a certain amount of digits.

Round-off errors are assumed negligible. The major error is the discretization error. This is minimized by choosing very short intervals.

Now, the finite-difference solution technique can be applied to the partial differential equation. Crank Nicolson scheme is a semi-implicit scheme. The partial differential equation is

$$T\frac{\partial^2 h}{\partial x^2} + T\frac{\partial^2 h}{\partial y^2} = S\frac{\partial h}{\partial t}$$

For simplicity T is assumed constant.

$$\frac{\partial^2 h}{\partial x^2} = \frac{h_{i-1,j}^{n+\frac{1}{2}} - 2h_{i,j}^{n+\frac{1}{2}} + h_{i+1,j}^{n+\frac{1}{2}}}{(\Delta x)^2}$$

$$= \frac{1}{2}\left[\frac{h_{i-1,j}^{n+1} - 2h_{i,j}^{n+1} + h_{i+1,j}^{n+1}}{(\Delta x)^2} + \frac{h_{i-1,j}^{n} - 2h_{i,j}^{n} + h_{i+1,j}^{n}}{(\Delta x)^2}\right]$$

This relationship can be shown in figure 2.

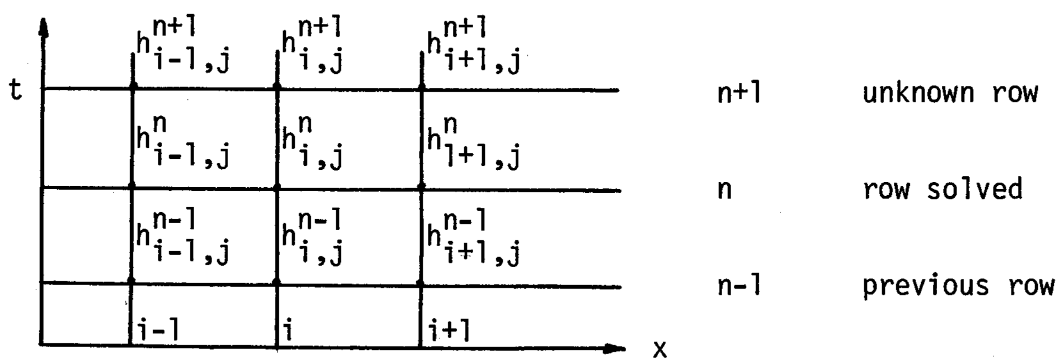


Figure 2. Grid points in the solution domain.

y axis is taken perpendicular to x-t plane. The values of h(x,y,t) will be calculated at points a, b, and c. Since the values of *h* at time n are known values (from initial conditions), the unknown values of h(x,y,t) are at the time level n+1.

Similarly,

$$\frac{\partial^2 h}{\partial y^2} = \frac{1}{2}\left[\frac{h^{n+1}_{i,j-1} - 2h^{n+1}_{i,j} + h^{n+1}_{i,j+1}}{(\Delta y)^2} + \frac{h^{n}_{i,j-1} - 2h^{n}_{i,j} + h^{n}_{i,j+1}}{(\Delta y)^2}\right]$$

Also

$$\frac{\partial h}{\partial t} = \frac{h^{n+1}_{i,j} - h^{n}_{i,j}}{\Delta t}$$

Putting all these terms into the partial differential equation and keeping unknowns at the left hand side.

$$T\left[\frac{h^{n+1}_{i-1,j} - 2h^{n+1}_{i,j} + h^{n+1}_{i+1,j}}{(\Delta x)^2}\right] + T\left[\frac{h^{n+1}_{i,j-1} - 2h^{n+1}_{h,j} + h^{n+1}_{i,j+1}}{(\Delta y)^2}\right] - \frac{2S}{\Delta t}h^{n+1}_{i,j}$$

$$= -T\left[\frac{h^{n}_{i-1,j} - 2h^{n}_{i,j} + h^{n}_{i-1,j}}{(\Delta x)^2}\right] - T\left[\frac{h^{n}_{i,j-1} - 2h^{n}_{i,j} + h^{n}_{i,j+1}}{(\Delta y)^2}\right] -$$

$$\frac{2S}{\Delta t}\ h^{n}_{i,j}$$

Let $\Delta x = \Delta y$. The right hand side of equation 8 is known. Let's call it D(x,y). Arranging the left hand side,

$$Th^{n+1}_{i-1,j} - \left[4T + \frac{2S(\Delta x)^2}{\Delta t}\right]h^{n+1}_{i,j} + T^{n+1}_{i+1,j} + T^{n+1}_{i,j-1} + T^{n+1}_{i,j+1} = D(x,y)$$

This leaves us with five unknowns on the left hand side. There is no simple algorithm to solve these five unknowns. Iterative methods have been introduced to solve this kind of problems. In this study extrapolated time routine introduced by Halepaska et. al. (1971) will be used. This routine will be applied to $h^{n+1}_{i,j-1}$ and $h^{n+1}_{i,j+1}$. $h^{n+1}_{i,j-1}$ can be known from the boundary condition so only $h^{n+1}_{i,j+1}$ should be calculated. $h^{n+1}_{i,j+1}$ is redefined as $hK^{n+1}_{i,j+1}$.

The extrapolated time routine is

$$hK^{n+1}_{i,j+1} = hK^{n}_{i,j+1} + \frac{t^{n+\frac{1}{2}}}{t^{n-\frac{1}{2}}}\left[hK^{n}_{i,j+1} - hK^{n-1}_{i,j+1}\right]$$

Since the value of $h^{n}_{i,j+1}$ is known at the time n and also the value of $h^{n-1}_{i,j+1}$ is known at the previous time n-1, the value of $h^{n+1}_{i,j+1}$ can be found easily.

Convergence criteria is given as

$$\left[\frac{h^{m+1}_{i,j+1} - h^{m}_{i,j+1}}{h^{m}_{i,j+1}}\right] \leqslant \varepsilon \qquad \text{m, iteration parameter.}$$

Where ε is a small number.

Then the algebraic equation reduces to algebraic equation with three unknowns

$$Th^{n+1}_{i-1,j} - \left[4T + \frac{2S(\Delta x)^2}{\Delta t}\right] h^{n+1}_{i,j} + Th^{n+1}_{i+1,j} = D(x,y) + h^{n+1}_{i,j-1} + h^{n+1}_{i,j+1}$$

The right hand side is known and there are only three unknowns on the left hand side. This is a tri-diagonal matrix and can be solved using Thomas algorithm.

However, in the study T is assumed as a function of x and y. The finite difference formulation is obtained for the center node instread of the corner nodes as shown in the above derivation (Halepaska et. al., 1971). This formulation is second order correct in time and space.

The difference equation of the differential equation of groundwater flow is:

$$\frac{1}{\Delta X_i}\left[T_{i+\frac{1}{2},j}\frac{h^{n+\frac{1}{2}}_{i+1,j}-h^{n+\frac{1}{2}}_{i,j}}{\Delta X_{i+\frac{1}{2}}}-T_{i-\frac{1}{2},j}\frac{h^{n+\frac{1}{2}}_{i,j}-h^{n+\frac{1}{2}}_{i-1,j}}{\Delta X_{i-\frac{1}{2}}}\right]+$$

$$\frac{1}{\Delta y_j}\left[T_{i,j+\frac{1}{2}}\frac{h^{n+\frac{1}{2}}_{i,j+1}-h^{n+\frac{1}{2}}_{i,j}}{\Delta y_{j+\frac{1}{2}}}-T_{i,j-\frac{1}{2}}\frac{h^{n+\frac{1}{2}}_{i,j}-h^{n+\frac{1}{2}}_{i,j-1}}{\Delta y_{j-\frac{1}{2}}}\right]+$$

$$\frac{Q}{\Delta x\Delta y} = S_{i,j}\left[\frac{h^{n+1}_{i,j}-h^{n}_{i,j}}{\Delta t}\right]$$

APPENDIX C

DERIVATION OF THE SENSITIVITY EQUATION FOR THE CONFINED CASE

The equation that describes the confined groundwater flow system is (Appendix A),

$$\frac{\partial}{\partial x} \; T \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} \; T \frac{\partial h}{\partial y} + \frac{Q}{\Delta x \Delta y} = S \frac{\partial h}{\partial t} \tag{1}$$

where T = T(x,y)

The sensitivity equations are derived for the aquifer parameters, transmissivity (T) and storage coefficient(s).

i. THE SENSITIVITY EQUATION FOR TRANSMISSIVITY VARIATIONS

The sensitivity equation can be found on the basis of definition

$$u(x,y,t;\; T,S,Q) = \frac{\partial h}{\partial T} \tag{2}$$

analogous to what is introduced by Tomovic', 1963. The assumptions made in the derivation of the sensitivity equations are;

a. Solution of the flow equation depends analytically on the parameters T and S.

b. The parameters T, S, and Q (also k, Sy, and Q) are independent of each other.

c. The function h and its derivatives, $\frac{\partial h}{\partial x}$, $\frac{\partial h}{\partial y}$, $\frac{\partial h}{\partial t}$, $\frac{\partial h}{\partial T}$, and $\frac{\partial h}{\partial S}$ are continuous so that the order of differentiation is interchangeable (Margenau, 1968), i.e.,

$$\frac{\partial}{\partial T} \left[\frac{\partial h}{\partial x}\right] = \frac{\partial}{\partial x} \left[\frac{\partial h}{\partial T}\right] \tag{3(a)}$$

$$\frac{\partial}{\partial T}\left[\frac{\partial h}{\partial y}\right] = \frac{\partial}{\partial y}\left[\frac{\partial h}{\partial T}\right] \qquad 3(b)$$

$$\frac{\partial}{\partial T}\left[\frac{\partial h}{\partial t}\right] = \frac{\partial}{\partial t}\left[\frac{\partial h}{\partial T}\right] \qquad 3(c)$$

$$\frac{\partial}{\partial S}\left[\frac{\partial h}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial h}{\partial S}\right] \qquad 3(d)$$

$$\frac{\partial}{\partial S}\left[\frac{\partial h}{\partial y}\right] = \frac{\partial}{\partial y}\left[\frac{\partial h}{\partial S}\right] \qquad 3(e)$$

$$\frac{\partial}{\partial S}\left[\frac{\partial h}{\partial t}\right] = \frac{\partial}{\partial t}\left[\frac{\partial h}{\partial S}\right] \qquad 3(f)$$

The sensitivity equation is obtained from the partial derivative of the equation (1) with respect to T.

$$\frac{\partial}{\partial T}\left[\frac{\partial}{\partial x} T \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} T \frac{\partial h}{\partial y} + \frac{Q}{\Delta x \Delta y}\right] = \frac{\partial}{\partial T}\left[S \frac{\partial h}{\partial t}\right]$$

Expand the terms in the brackets

$$\frac{\partial}{\partial T}\left[\frac{\partial T}{\partial x} \cdot \frac{\partial h}{\partial x} + T\frac{\partial^2 h}{\partial x^2} + \frac{\partial T}{\partial y} \cdot \frac{\partial h}{\partial y} + T \frac{\partial^2 h}{\partial y^2} + \frac{Q}{\Delta x \Delta y}\right] = \frac{\partial}{\partial T}\left[S \frac{\partial h}{\partial t}\right]$$

and differentiate with respect to T

$$\frac{\partial^2 T}{\partial T \partial x} \cdot \frac{\partial h}{\partial x} + \frac{\partial T}{\partial x} \cdot \frac{\partial^2 h}{\partial T \partial x} + \frac{\partial^2 h}{\partial x^2} + T \frac{\partial^3 h}{\partial T \partial x^2} + \frac{\partial^2 T}{\partial T \partial y} \cdot \frac{\partial h}{\partial y} + \frac{\partial T}{\partial y} \cdot \frac{\partial^2 h}{\partial T \partial y} +$$

$$\frac{\partial^2 h}{\partial y^2} + T \frac{\partial^2 h}{\partial T \partial y^2} + \frac{\partial Q}{\partial T} \cdot \frac{1}{\Delta x \Delta y} = \frac{\partial S}{\partial T} \cdot \frac{\partial h}{\partial t} + S \frac{\partial^2 h}{\partial T \partial t}$$

From assumption (b)

$$\frac{\partial Q}{\partial T} = 0 \qquad \text{and} \qquad \frac{\partial S}{\partial T} = 0$$

Interchange the order of differentiation, T and x, T and y, and T and t

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial T}\right) \cdot \frac{\partial h}{\partial x} + \frac{\partial T}{\partial x} \cdot \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial T}\right) + \frac{\partial^2 h}{\partial x^2} + T \frac{\partial^2}{\partial x^2}\left(\frac{\partial h}{\partial T}\right) + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial T}\right) \cdot \frac{\partial h}{\partial y} +$$

$$\frac{\partial T}{\partial y} \cdot \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial T}\right) + \frac{\partial^2 h}{\partial y^2} + T \frac{\partial^2}{\partial y^2}\left(\frac{\partial h}{\partial T}\right) = S \frac{\partial}{\partial t}\left(\frac{\partial h}{\partial T}\right)$$

remembering that

$$\frac{\partial h}{\partial T} = u$$

the equation can be written,

$$\frac{\partial T}{\partial x} \cdot \frac{\partial}{\partial x} u + \frac{\partial^2 h}{\partial x} + T \frac{\partial^2}{\partial x^2} u + \frac{\partial T}{\partial y_2} \cdot \frac{\partial}{\partial y} u + \frac{\partial^2 h}{\partial y^2} + T \frac{\partial^2}{\partial y^2} u = S \frac{\partial}{\partial t} u \qquad (4)$$

The terms

$$\frac{\partial T}{\partial x} \cdot \frac{\partial u}{\partial x} + T \frac{\partial^2}{\partial x^2} u = \frac{\partial}{\partial x} T \frac{\partial u}{\partial x} \qquad 5(a)$$

$$\frac{\partial T}{\partial y} \cdot \frac{\partial u}{\partial y} + T \frac{\partial^2}{\partial y^2} u = \frac{\partial}{\partial y} T \frac{\partial u}{\partial y} \qquad 5(b)$$

Put the terms in (5a) and (5b) in the equation (4) and arrange the equation

$$\frac{\partial}{\partial x} T \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} T \frac{\partial u}{\partial y} + \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = S \frac{\partial u}{\partial t} \qquad (6)$$

This is the sensitivity equation for transmissivity.

ii. THE SENSITIVITY EQUATION FOR STORAGE COEFFICIENT VARIATIONS

The sensitivity coefficient is defined by

$$u' \; (x,y,t;\; T,S,Q) = \frac{\partial h}{\partial S} \tag{7}$$

and the assumptions a, b, and c are used.

The sensitivity equation is obtained from the partial derivative of the equation (1) with respect to S.

$$\frac{\partial}{\partial S}\left[\frac{\partial}{\partial x} T \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} T \frac{\partial h}{\partial y} + \frac{Q}{\Delta x \Delta y}\right] = \frac{\partial}{\partial S}\left[S \frac{\partial h}{\partial t}\right]$$

Expand the terms in the brackets

$$\frac{\partial}{\partial S}\left[\frac{\partial T}{\partial x} \cdot \frac{\partial h}{\partial x} + T \frac{\partial^2 h}{\partial x^2} + \frac{\partial T}{\partial y} \cdot \frac{\partial h}{\partial y} + T \frac{\partial^2 h}{\partial y^2} + \frac{Q}{\Delta x \Delta y}\right] = \frac{\partial}{\partial S}\left[S \frac{\partial h}{\partial t}\right]$$

and differentiate with respect to S

$$\frac{\partial^2 T}{\partial S \partial x} \cdot \frac{\partial h}{\partial x} + \frac{\partial T}{\partial x} \cdot \frac{\partial^2 h}{\partial S \partial x} + \frac{\partial T}{\partial S} \cdot \frac{\partial^2 h}{\partial x^2} + T \frac{\partial^3 h}{\partial S \partial x^2} + \frac{\partial^2 T}{\partial S \partial y} \cdot \frac{\partial h}{\partial y} + \frac{\partial T}{\partial y} \cdot \frac{\partial^2 h}{\partial S \partial y} +$$

$$\frac{\partial T}{\partial S} \cdot \frac{\partial^2 h}{\partial y^2} + T \frac{\partial^3 h}{\partial S \partial y^2} + \frac{\partial Q}{\partial S} \cdot \frac{1}{\Delta x \Delta y} = \frac{\partial h}{\partial t} + S \frac{\partial^2 h}{\partial S \partial t}$$

From assumption (b)

$$\frac{\partial Q}{\partial S} = 0 \quad \text{and} \quad \frac{\partial T}{\partial S} = 0$$

Interchange the order of differentiation, S and x, S and y, and S and t

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial S}\right) \cdot \frac{\partial h}{\partial x} + \frac{\partial T}{\partial x} \cdot \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial S}\right) + T \frac{\partial^2}{\partial x^2}\left(\frac{\partial h}{\partial S}\right) + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial S}\right) \cdot \frac{\partial h}{\partial y} +$$

$$\frac{\partial T}{\partial y} \cdot \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial S}\right) + T \frac{\partial^2}{\partial y^2}\left(\frac{\partial h}{\partial S}\right) = \frac{\partial h}{\partial t} + S \frac{\partial}{\partial t}\left(\frac{\partial h}{\partial S}\right)$$

From expression (7)

$$\frac{\partial T}{\partial x} \cdot \frac{\partial u'}{\partial x} + T \frac{\partial^2}{\partial x^2} u' + \frac{\partial T}{\partial y} \cdot \frac{\partial}{\partial y} u' + T \frac{\partial^2}{\partial y^2} u' = \frac{\partial h}{\partial t} + S \frac{\partial}{\partial t} u'$$

Remembering expression (5)

$$\frac{\partial}{\partial x} T \frac{\partial}{\partial x} u' + \frac{\partial}{\partial y} T \frac{\partial}{\partial y} u' = \frac{\partial h}{\partial t} + S \frac{\partial u'}{\partial t}$$

This is the sensitivity equation for storage coefficient.

APPENDIX D

DERIVATION OF THE SENSITIVITY EQUATION FOR THE UNCONFINED CASE

The equation that describes the unconfined groundwater flow system is (Appendix A)

$$\frac{\partial}{\partial x} Kh \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Kh \frac{\partial h}{\partial y} + \frac{Q}{\Delta x \Delta y} = Sy \frac{\partial h}{\partial t} \qquad (1)$$

where $K = K(x,y)$

The sensitivity equations are derived for the aquifer parameters, hydraulic conductivity (K) and specific yield (Sy).

i. THE SENSITIVITY EQUATION FOR HYDRAULIC CONDUCTIVITY

The sensitivity coefficient is defined by

$$u_c \; (x,y,t;\, K,Sy,Q) = \frac{\partial h}{\partial K}$$

and the same assumptions a, b, and c in Appendix C are used.

The sensitivity equation is obtained from the partial derivative of the equation (1) with respect to K.

$$\frac{\partial}{\partial K} \left\{ \frac{\partial}{\partial x} Kh \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Kh \frac{\partial h}{\partial y} + \frac{Q}{\Delta x \Delta y} \right\} = \frac{\partial}{\partial K} \left\{ Sy \frac{\partial h}{\partial t} \right\}$$

Expand the terms in the brackets

$$\frac{\partial}{\partial K} \left\{ h \frac{\partial K}{\partial x} \cdot \frac{\partial h}{\partial x} + K \frac{\partial h}{\partial x} \cdot \frac{\partial h}{\partial x} + Kh \frac{\partial^2 h}{\partial x^2} + h \frac{\partial K}{\partial y} \cdot \frac{\partial h}{\partial y} + K \frac{\partial h}{\partial y} \cdot \frac{\partial h}{\partial y} + \right.$$

$$\left. Kh \frac{\partial^2 h}{\partial y^2} + \frac{Q}{\Delta x \Delta y} \right\} = \frac{\partial}{\partial K} \left\{ Sy \frac{\partial h}{\partial t} \right\}$$

and differentiate with respect to K

$$\frac{\partial h}{\partial K} \cdot \frac{\partial K}{\partial x} \cdot \frac{\partial h}{\partial x} + h \frac{\partial^2 K}{\partial K \partial x} \cdot \frac{\partial h}{\partial x} + h \frac{\partial K}{\partial x} \cdot \frac{\partial^2 h}{\partial K \partial x} + \frac{\partial h}{\partial x} \cdot \frac{\partial h}{\partial x} + K \frac{\partial^2 h}{\partial K \partial x} \cdot \frac{\partial h}{\partial x} +$$

$$K \frac{\partial h}{\partial x} \cdot \frac{\partial^2 h}{\partial K \partial x} + h \frac{\partial^2 h}{\partial x^2} + K \frac{\partial h}{\partial K} \cdot \frac{\partial^2 h}{\partial x^2} + Kh \frac{\partial^3 h}{\partial K \partial x^2} + \frac{\partial h}{\partial K} \cdot \frac{\partial K}{\partial y} \cdot \frac{\partial h}{\partial y} +$$

$$h \frac{\partial^2 K}{\partial K \partial y} \cdot \frac{\partial h}{\partial y} + h \frac{\partial K}{\partial y} \cdot \frac{\partial^2 h}{\partial K \partial y} + \frac{\partial h}{\partial y} \cdot \frac{\partial h}{\partial y} + K \frac{\partial^2 h}{\partial K \partial y} \cdot \frac{\partial h}{\partial y} + h \frac{\partial^2 h}{\partial y^2} +$$

$$K \frac{\partial h}{\partial K} \cdot \frac{\partial^2 h}{\partial y^2} + Kh \frac{\partial^3 h}{\partial K \partial y^2} + \frac{\partial Q}{\partial K} \cdot \frac{1}{\Delta x \Delta y} = \frac{\partial Sy}{\partial K} \cdot \frac{\partial h}{\partial t} + Sy \frac{\partial^2 h}{\partial K \partial t}$$

From assumption (b)

$$\frac{\partial Q}{\partial K} = 0 \quad \text{and} \quad \frac{\partial Sy}{\partial K} = 0$$

Interchange the order of differentiation, K and s, K and y, and K and t.

Replace $\frac{\partial h}{\partial K}$ by u_c .

$$u_c \frac{\partial K}{\partial x} \cdot \frac{\partial h}{\partial x} + h \frac{\partial}{\partial x} \left(\frac{\partial K}{\partial K} \right) \cdot \frac{\partial h}{\partial x} + h \frac{\partial K}{\partial x} \cdot \frac{\partial}{\partial x} u_c + \frac{\partial h}{\partial x} \cdot \frac{\partial h}{\partial x} + K \frac{\partial}{\partial x} u_c \frac{\partial h}{\partial x} +$$

$$K \frac{\partial h}{\partial x} \cdot \frac{\partial}{\partial x} u_c + h \frac{\partial^2 h}{\partial x^2} + Ku_c \frac{\partial^2 h}{\partial x^2} + Kh \frac{\partial^2}{\partial x^2} u_c + u_c \frac{\partial K}{\partial y} \cdot \frac{\partial h}{\partial y} +$$

$$h \frac{\partial}{\partial y} \left(\frac{\partial K}{\partial K} \right) \cdot \frac{\partial h}{\partial y} + h \frac{\partial K}{\partial y} \cdot \frac{\partial}{\partial y} u_c + \frac{\partial h}{\partial y} \cdot \frac{\partial h}{\partial y} + K \frac{\partial}{\partial y} u_c \cdot \frac{\partial h}{\partial y} +$$

$$K \frac{\partial h}{\partial y} \cdot \frac{\partial}{\partial y} u_c + h \frac{\partial^2 h}{\partial y^2} + Ku_c \frac{\partial^2 h}{\partial y^2} + Kh \frac{\partial^2}{\partial y^2} u_c = Sy \frac{\partial}{\partial t} u_c \qquad (2)$$

The terms

$$u_c \frac{\partial K}{\partial x} \cdot \frac{\partial h}{\partial x} + K \frac{\partial}{\partial x} u_c \cdot \frac{\partial h}{\partial x} + Ku_c \frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x} Ku_c \frac{\partial h}{\partial x} \qquad (3a)$$

$$h \frac{\partial K}{\partial x} \cdot \frac{\partial}{\partial x} u_c + K \frac{\partial h}{\partial x} \cdot \frac{\partial}{\partial x} u_c + Kh \frac{\partial^2}{\partial x^2} u_c = \frac{\partial}{\partial x} Kh \frac{\partial}{\partial x} u_c \qquad (3b)$$

$$\frac{\partial h}{\partial x} \cdot \frac{\partial h}{\partial x} + h \frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x} h \frac{\partial h}{\partial x} \qquad (3c)$$

$$u_c \frac{\partial K}{\partial y} \cdot \frac{\partial h}{\partial y} + K \frac{\partial u_c}{\partial y} \cdot \frac{\partial h}{\partial y} + Ku_c \frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y} Ku_c \frac{\partial h}{\partial y} \tag{3d}$$

$$h \frac{\partial K}{\partial y} \cdot \frac{\partial}{\partial y} u_c + K \frac{\partial h}{\partial y} \cdot \frac{\partial}{\partial y} u_c + Kh \frac{\partial^2}{\partial y^2} u_c = \frac{\partial}{\partial y} Kh \frac{\partial}{\partial y} u_c \tag{3e}$$

$$\frac{\partial h}{\partial y} \cdot \frac{\partial h}{\partial y} + h \frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y} h \frac{\partial h}{\partial y} \tag{3f}$$

Put the terms in 3a, 3b, 3c, 3d, 3e, and 3f into the equation (2) and rearrange the equation

$$\frac{\partial}{\partial x} Ku_c \frac{\partial h}{\partial x} + \frac{\partial}{\partial x} Kh \frac{\partial}{\partial x} u_c + \frac{\partial}{\partial x} h \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Ku_c \frac{\partial h}{\partial y} + \frac{\partial}{\partial y} Kh \frac{\partial u_c}{\partial y} + \frac{\partial}{\partial y} h \frac{\partial h}{\partial y} = Sy \frac{\partial}{\partial t} u_c \tag{4}$$

This is the sensitivity equation for hydraulic conductivity.

ii. THE SENSITIVITY EQUATION FOR SPECIFIC YIELD

The sensitivity coefficient is defined by

$$u_c' \, (x,y,t;\ T,S,Q) = \frac{\partial h}{\partial Sy}$$

and the same assumptions a, b, and c in Appendix C are used.

The sensitivity equation is obtained from the partial derivative of the equation (1) with respect to Sy.

$$\frac{\partial}{\partial Sy} \left\{ \frac{\partial}{\partial x} Kh \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Kh \frac{\partial h}{\partial y} + \frac{Q}{\Delta x \Delta y} \right\} = \frac{\partial}{\partial Sy} \left\{ Sy \frac{\partial h}{\partial t} \right\}$$

Expand the terms in the brackets

$$\frac{\partial}{\partial Sy} \left\{ K \frac{\partial h}{\partial x} \cdot \frac{\partial h}{\partial x} + h \frac{\partial K}{\partial x} \cdot \frac{\partial h}{\partial x} + Kh \frac{\partial^2 h}{\partial x^2} + K \frac{\partial h}{\partial y} \cdot \frac{\partial h}{\partial y} + h \frac{\partial K}{\partial y} \cdot \frac{\partial h}{\partial y} + Kh \frac{\partial^2 h}{\partial y^2} + \frac{Q}{\Delta x \Delta y} \right\} = \frac{\partial}{\partial Sy} \left\{ Sy \frac{\partial h}{\partial t} \right\}$$

and differentiate with respect to Sy

$$\frac{\partial K}{\partial Sy} \cdot \frac{\partial h}{\partial x} \cdot \frac{\partial h}{\partial x} + K \frac{\partial^2 h}{\partial Sy \partial x} \cdot \frac{\partial h}{\partial x} + K \frac{\partial h}{\partial x} \cdot \frac{\partial^2 h}{\partial Sy \partial x} + \frac{\partial h}{\partial Sy} \cdot \frac{\partial K}{\partial x} \cdot \frac{\partial h}{\partial x} +$$

$$h \frac{\partial^2 K}{\partial Sy \partial x} \cdot \frac{\partial h}{\partial x} + h \frac{\partial K}{\partial x} \cdot \frac{\partial^2 h}{\partial Sy \partial x} + \frac{\partial K}{\partial Sy} h \frac{\partial^2 h}{\partial x^2} + K \frac{\partial h}{\partial Sy} \cdot \frac{\partial^2 h}{\partial x^2} +$$

$$Kh \frac{\partial^3 h}{\partial Sy \partial x^2} + \frac{\partial K}{\partial Sy} \cdot \frac{\partial h}{\partial y} \cdot \frac{\partial h}{\partial y} + K \frac{\partial^2 h}{\partial Sy \partial y} \cdot \frac{\partial h}{\partial y} + K \frac{\partial h}{\partial y} \cdot \frac{\partial^2 h}{\partial Sy \partial y} +$$

$$\frac{\partial h}{\partial Sy} \cdot \frac{\partial K}{\partial y} \cdot \frac{\partial h}{\partial y} + h \frac{\partial^2 K}{\partial Sy \partial y} \cdot \frac{\partial h}{\partial y} + h \frac{\partial K}{\partial y} \cdot \frac{\partial^2 h}{\partial Sy \partial y} + \frac{\partial K}{\partial Sy} h \frac{\partial^2 h}{\partial y^2} +$$

$$K \frac{\partial h}{\partial Sy} \cdot \frac{\partial^2 h}{\partial y^2} + Kh \frac{\partial^3 h}{\partial Sy \partial y^2} + \frac{\partial Q}{\partial Sy} \cdot \frac{1}{\Delta x \Delta y} = \frac{\partial h}{\partial t} + Sy \frac{\partial^2 h}{\partial Sy \partial t}$$

From assumption (b)

$$\frac{\partial Q}{\partial Sy} = 0 \quad \text{and} \quad \frac{\partial K}{\partial Sy} = 0$$

Interchange the order of differentiation, Sy and x, Sy and y, and Sy and t. Replace $\frac{\partial h}{\partial Sy}$ by u_c'

$$K \frac{\partial}{\partial x} u_c' \frac{\partial h}{\partial x} + K \frac{\partial h}{\partial x} \cdot \frac{\partial}{\partial x} u_c' + u_c' \frac{\partial K}{\partial x} \cdot \frac{\partial h}{\partial x} + h \frac{\partial}{\partial x} u_c' \frac{\partial h}{\partial x} +$$

$$h \frac{\partial K}{\partial x} \cdot \frac{\partial}{\partial x} u_c' + K u_c' \frac{\partial^2 h}{\partial x^2} + Kh \frac{\partial^2}{\partial x^2} u_c' + K \frac{\partial}{\partial y} u_c' \cdot \frac{\partial h}{\partial y} +$$

$$K \frac{\partial h}{\partial y} \cdot \frac{\partial}{\partial y} u_c' + u_c' \frac{\partial K}{\partial y} \cdot \frac{\partial h}{\partial y} + h \frac{\partial}{\partial y} u_c' \frac{\partial h}{\partial y} + h \frac{\partial K}{\partial y} \cdot \frac{\partial}{\partial y} u_c' +$$

$$K u_c' \frac{\partial^2 h}{\partial y^2} + Kh \frac{\partial^2}{\partial y^2} u_c' = \frac{\partial h}{\partial t} + Sy \frac{\partial}{\partial t} u_c' \tag{5}$$

The terms

$$K \frac{\partial}{\partial x} u_c' \frac{\partial h}{\partial x} + u_c' \frac{\partial K}{\partial x} \cdot \frac{\partial h}{\partial x} + K u_c' \frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x} K u_c' \frac{\partial h}{\partial x} \tag{6a}$$

$$K \frac{\partial h}{\partial x} \cdot \frac{\partial}{\partial x} u_c' + h \frac{\partial}{\partial x} u_c' \frac{\partial h}{\partial x} + Kh \frac{\partial^2}{\partial x^2} u_c' = \frac{\partial}{\partial x} Kh \frac{\partial}{\partial x} u_c' \tag{3b}$$

$$K \frac{\partial}{\partial y} u_c' \frac{\partial h}{\partial y} + u_c' \frac{\partial K}{\partial y} \cdot \frac{\partial h}{\partial y} + K u_c' \frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y} K u_c' \frac{\partial h}{\partial y} \qquad (6c)$$

$$K \frac{\partial h}{\partial y} \cdot \frac{\partial}{\partial y} u_c' + h \frac{\partial}{\partial y} u_c' \frac{\partial h}{\partial y} + Kh \frac{\partial^2}{\partial y^2} u_c' = \frac{\partial}{\partial y} Kh \frac{\partial}{\partial y} u_c' \qquad (6d)$$

Put the terms in 6a, 6b, 6c, and 6d into the equation (5) and rearrange the equation;

$$\frac{\partial}{\partial x} K u_c' \frac{\partial h}{\partial x} + \frac{\partial}{\partial x} Kh \frac{\partial}{\partial x} u_c' + \frac{\partial}{\partial y} K u_c' \frac{\partial h}{\partial y} + \frac{\partial}{\partial y} Kh \frac{\partial}{\partial y} u_c'$$

$$= Sy \frac{\partial}{\partial t} u_c' + \frac{\partial h}{\partial t} \qquad (7)$$

This is the sensitivity equation for specific yield.

APPENDIX E

SOLUTION OF THE SENSITIVITY EQUATIONS

The partial differential equations derived are solved by the Crank-Nicolson scheme. This scheme is chosen because it is unconditionally stable and convergent.

CONFINED FLOW SYSTEMS

The sensitivity equations are derived with respect to variations in transmissivity and storage coefficient.

The sensitivity equation with respect to transmissivity is:

$$\frac{\partial}{\partial x} T \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} T \frac{\partial u}{\partial y} + \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = S \frac{\partial}{\partial t} u$$

The difference equation is:

$$\frac{1}{\Delta x_i}\left[T_{i+\frac{1}{2},j} \frac{u^{n+\frac{1}{2}}_{i+1,j} - u^{n+\frac{1}{2}}_{i,j}}{\Delta x_{i+\frac{1}{2}}} - T_{i-\frac{1}{2},j} \frac{u^{n+\frac{1}{2}}_{i,j} - u^{n+\frac{1}{2}}_{i-1,j}}{\Delta x_{i-\frac{1}{2}}}\right] +$$

$$\frac{1}{\Delta y_j}\left[T_{i,j+\frac{1}{2}} \frac{u^{n+\frac{1}{2}}_{i,j+1} - u^{n+\frac{1}{2}}_{i,j}}{\Delta y_{j+\frac{1}{2}}} - T_{i,j-\frac{1}{2}} \frac{u^{n+\frac{1}{2}}_{i,j} - u^{n+\frac{1}{2}}_{i,j-1}}{\Delta y_{j-\frac{1}{2}}}\right]$$

$$\frac{1}{\Delta x_i}\left[\frac{h^{n+\frac{1}{2}}_{i+1,j} - h^{n+\frac{1}{2}}_{i,j}}{\Delta x_{i+\frac{1}{2}}} - \frac{h^{n+\frac{1}{2}}_{i,j} - h^{n+\frac{1}{2}}_{i-1,j}}{\Delta x_{i-\frac{1}{2}}}\right] +$$

$$\frac{1}{\Delta y_j}\left[\frac{h^{n+\frac{1}{2}}_{i,j+1} - h^{n+\frac{1}{2}}_{i,j}}{\Delta y_{j+\frac{1}{2}}} - \frac{h^{n+\frac{1}{2}}_{i,j} - h^{n+\frac{1}{2}}_{i,j-1}}{\Delta y_{j-\frac{1}{2}}}\right] = S_{i,j} \frac{u^{n+1}_{i,j} - u^{n}_{i,j}}{\Delta t^{n+\frac{1}{2}}} \qquad (1)$$

where

$$T_{i+\frac{1}{2},j} = 2\frac{T_{i+1,j} * T_{i,j}}{T_{i+1,j} + T_{i,j}} = Ax_{i,j} \quad (2a)$$

$$T_{i-\frac{1}{2},j} = 2\frac{T_{i-1,j} * T_{i,j}}{T_{i-1,j} + T_{i,j}} = Bx_{i,j} \quad (2b)$$

$$T_{i,j+\frac{1}{2}} = 2\frac{T_{i,j+1} * T_{i,j}}{T_{i,j+1} + T_{i,j}} = Cx_{i,j} \quad (2c)$$

$$T_{i,j-\frac{1}{2}} = 2\frac{T_{i,j-1} * T_{i,j}}{T_{i,j-1} + T_{i,j}} = Dx_{i,j} \quad (2d)$$

We assumed equal spacing in each layer ($\Delta x = \Delta y$) and constant spacing in all directions ($\Delta x_i = \Delta x_{i+\frac{1}{2}}$, $\Delta y_j = \Delta y_{j+\frac{1}{2}}$).

The sensitivity and the head values are defined as:

$$u_{i,j}^{n+\frac{1}{2}} = \frac{1}{2} \; u_{i,j}^{n+1} + u_{i,j}^{n}$$

$$h_{i,j}^{n+\frac{1}{2}} = \frac{1}{2} \; h_{i,j}^{n+1} + h_{i,j}^{n}$$

where $u_{i,j}^{n+1}$ and $h_{i,j}^{n+1}$ are the unknown values, and $u_{i,j}^{n}$ and $h_{i,j}^{n}$ are the known values.

The value of the terms

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

are known from the solution of the flow equation. Let these terms be equal to $HEADFC_{i,j}$. Now we can rewrite equation (1) with unknowns on the left side.

$$AX_{i,j}\left\{u^{n+1}_{i+1,j} - u^{n+1}_{i,j}\right\} - BX_{i,j}\left\{u^{n+1}_{i,j} - u^{n+1}_{i-1,j}\right\} +$$

$$CX_{i,j}\left\{u^{n+1}_{i,j+1} - u^{n+1}_{i,j}\right\} - DX_{i,j}\left\{u^{n+1}_{i,j} - u^{n+1}_{i,j-1}\right\} -$$

$$\frac{2(\Delta x)\, S_{i,j}}{\Delta t^{n+\frac{1}{2}}}\; u^{n+1}_{i,j} = -AX_{i,j}\left\{u^{n}_{i+1,j} - u^{n}_{i,j}\right\} +$$

$$BX_{i,j}\left\{u^{n}_{i,j} - u^{n}_{i-1,j}\right\} - CX_{i,j}\left\{u^{n}_{i+1,j} - u^{n}_{i,j}\right\} +$$

$$DX_{i,j}\left\{u^{n}_{i,j} - u^{n}_{i,j-1}\right\} - \frac{2(\Delta x)\, S_{i,j}}{\Delta t^{n+\frac{1}{2}}}\; u^{n}_{i,j} - HEADFC_{i,j} \qquad (3)$$

Let the **right-hand side be equal to** $D_{i,j}$ **and rearrange equation (3),**

$$AX_{i,j}\, u^{n+1}_{i+1,j} - \left\{AX_{i,j} + BX_{i,j} + CX_{i,j} + DX_{i,j} + \frac{2(\Delta x)^2 S_{i,j}}{\Delta t^{n+\frac{1}{2}}}\right\} u^{n+1}_{i,j} +$$

$$BX_{i,j}\, u^{n+1}_{i-1,j} + CX_{i,j}\, u^{n+1}_{i,j+1} + DX_{i,j}\, u^{n+1}_{i,j-1} = D_{i,j}$$

We can now apply the time extrapolation routine stated in Appendix B to estimate $u^{n+1}_{i,j+1}$

$$u^{n+1}_{i,j+1} = u^{n}_{i,j+1} + \frac{\Delta t^{n+\frac{1}{2}}}{\Delta t^{n-\frac{1}{2}}}\left(u^{n}_{i,j+1} - u^{n}_{i,j-1}\right)$$

and $u^{n+1}_{i,j-1}$ is known from the boundary condition. Thus, we can rewrite equation (3) as:

$$AX_{i,j}\, u^{n+1}_{i+1,j} - \left\{ AX_{i,j} + BX_{i,j} + CX_{i,j} + DX_{i,j} + \frac{2\,(\Delta x)^2 S_{i,j}}{\Delta t^{n+\frac{1}{2}}} \right\} u^{n+1}_{i,j} +$$

$$BX_{i,j}\, u^{n+1}_{i-1,j} = D_{i,j} - CX_{i,j}\, u^{n+1}_{i,j+1} - DX_{i,j}\, u^{n+1}_{i,j-1} \qquad (4)$$

Now the right-hand side of the equation is known and we have three unknowns on the left-hand side. The system of equations as presented in the form of equation (4) is then solved using the Thomas algorithm for tridiagonal matrices (Von Rosenberg, 1969).

The sensitivity equation with respect to storage coefficient is:

$$\frac{\partial}{\partial x} T \frac{\partial}{\partial x} u' + \frac{\partial}{\partial y} T \frac{\partial}{\partial y} u' = \frac{\partial h}{\partial t} + S \frac{\partial}{\partial t} u' \qquad (5)$$

This is solved similar to equation (1) except that HEADFC = 0 and

$$\frac{\partial h}{\partial t} = \frac{h^{n+1}_{i,j} - h^{n}_{i,j}}{\Delta t^{n+\frac{1}{2}}} \ .$$

UNCONFINED FLOW SYSTEMS

The sensitivity equations are derived with respect to variations in hydraulic conductivity and specific yield.

The sensitivity equation with respect to hydraulic conductivity is:

$$\frac{\partial}{\partial x} Ku_c \frac{\partial h}{\partial x} + \frac{\partial}{\partial x} Kh \frac{\partial}{\partial x} u_c + \frac{\partial}{\partial x} h \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} Ku_c \frac{\partial h}{\partial y} + \frac{\partial}{\partial y} Kh \frac{\partial}{\partial y} u_c +$$

$$\frac{\partial}{\partial y} h \frac{\partial h}{\partial y} = Sy \frac{\partial}{\partial t} u_c \qquad (6)$$

The difference equation is:

$$\frac{1}{\Delta x_i}\left[K_{i+\frac{1}{2},j} \cdot u^{n+1}_{i+\frac{1}{2},j}\ \frac{h^{n+\frac{1}{2}}_{i+1,j} - h^{n+\frac{1}{2}}_{i,j}}{\Delta x_{i+\frac{1}{2}}} - K_{i-\frac{1}{2},j} \cdot u^{n+1}_{i-\frac{1}{2},j}\ \frac{h^{n+\frac{1}{2}}_{i,j} - h^{n+\frac{1}{2}}_{i-1,j}}{\Delta x_{i-\frac{1}{2}}}\right] +$$

$$\frac{1}{\Delta x_i}\left[K_{i+\frac{1}{2},j} \cdot h^{n+1}_{i+\frac{1}{2},j}\ \frac{u^{n+\frac{1}{2}}_{c_{i+1,j}} - u^{n+\frac{1}{2}}_{c_{i,j}}}{\Delta x_{i+\frac{1}{2}}} - \right.$$

$$\left. K_{i-\frac{1}{2},j} \cdot h^{n+1}_{i-\frac{1}{2},j}\ \frac{u^{n+\frac{1}{2}}_{c_{i,j}} - u^{n+\frac{1}{2}}_{c_{i-1,j}}}{\Delta x_{i-\frac{1}{2}}}\right] + \frac{1}{\Delta x_i}\left[h^{n+1}_{i+\frac{1}{2},j}\ \frac{h^{n+\frac{1}{2}}_{i+1,j} - h^{n+\frac{1}{2}}_{i,j}}{\Delta x_{i+\frac{1}{2}}} - \right.$$

$$\left. h^{n+1}_{i-\frac{1}{2},j}\ \frac{h^{n+\frac{1}{2}}_{i,j} - h^{n+\frac{1}{2}}_{i-1,j}}{\Delta x_{i-\frac{1}{2}}}\right] + \frac{1}{\Delta y_i}\left[K_{i,j+\frac{1}{2}} \cdot u^{n+1}_{c_{i,j+\frac{1}{2}}}\ \frac{h^{n+\frac{1}{2}}_{i,j+1}\ \ h^{n+\frac{1}{2}}_{i,j}}{\Delta y_{j+\frac{1}{2}}} - \right.$$

$$\left. K_{i,j-\frac{1}{2}} \cdot u^{n+1}_{c_{i,j-\frac{1}{2}}}\ \frac{h^{n+\frac{1}{2}}_{i,j} - h^{n+\frac{1}{2}}_{i,j-1}}{\Delta y_{j-\frac{1}{2}}}\right] +$$

$$\frac{1}{\Delta y_i}\left[K_{i,j+\frac{1}{2}} \cdot h^{n+1}_{i,j+\frac{1}{2}}\ \frac{u^{n+\frac{1}{2}}_{c_{i,j+1}} - u^{n+\frac{1}{2}}_{c_{i,j}}}{\Delta y_{j+\frac{1}{2}}} - \right.$$

$$\left. K_{i,j-\frac{1}{2}} \cdot h^{n+1}_{i,j-\frac{1}{2}}\ \frac{u^{n+\frac{1}{2}}_{c_{i,j}} - u^{n+\frac{1}{2}}_{c_{i,j-1}}}{\Delta y_{j-\frac{1}{2}}}\right] + \frac{1}{\Delta y_i}\left[h^{n+1}_{i,j+\frac{1}{2}}\ \frac{h^{n+\frac{1}{2}}_{i,j+1} - h^{n+\frac{1}{2}}_{i,j}}{\Delta y_{j+\frac{1}{2}}} - \right.$$

$$\left. h^{n+1}_{i,j-\frac{1}{2}}\ \frac{h^{n+\frac{1}{2}}_{i,j} - h^{n+\frac{1}{2}}_{i,j-1}}{\Delta y_{j-\frac{1}{2}}}\right] = Sy_{i,j}\ \frac{u^{n+1}_{c_{i,j}} - u^{n}_{c_{i,j}}}{\Delta t^{n+\frac{1}{2}}} \qquad (7)$$

This equation is reduced to a tridiagonal matrix as described above and solved using the Thomas algorithm.

The sensitivity equation with respect to specific yield is:

$$\frac{\partial}{\partial x} Ku_c' \frac{\partial h}{\partial x} + \frac{\partial}{\partial x} Kh \frac{\partial}{\partial x} u_c' + \frac{\partial h}{\partial y} + \frac{\partial}{\partial y} Kh \frac{\partial}{\partial y} u_c' = \frac{\partial h}{\partial t} + Sy \frac{\partial}{\partial t} u_c' \qquad (8)$$

This equation is similar to equation (6) except

$$\frac{\partial}{\partial x} h \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} h \frac{\partial h}{\partial y} = 0$$

and there is $\frac{\partial h}{\partial t}$ on the right-hand side of the equation. The difference equation for $\frac{\partial h}{\partial t}$ is:

$$\frac{\partial h}{\partial t} = \frac{h_{i,j}^{n+1} - h_{i,j}^{n}}{\Delta t^{n+\frac{1}{2}}}$$