Unconfined Radial Flow to a Well
Without Dupuit-Forchheimer Assumptions*

by
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Traditionally, unsteady groundwater flow has been treated using the equation shown at the top of the first slide. This equation is a result primarily of the work of Dupuit, Forchheimer, and Boussinesq. It is commonly called the Boussinesq equation. K is the hydraulic conductivity, h is the height of the water table, Sy is the specific yield, ∇ is the usual gradient operator, and t is the time. This equation is nonlinear and is difficult to solve analytically. However, it can be solved numerically without extreme difficulty. In addition to the flow equation it is necessary to specify the initial water table configuration and the boundary conditions. If no flow is allowed across the boundary, then the derivative of the hydraulic head in a direction normal to the boundary is zero. This is commonly called a barrier boundary condition. The other common boundary condition is where the hydraulic head is specified on the boundary.

This formalism for describing unconfined groundwater flow is only an approximation. The formalism is valid only to the extent the usual Dupuit assumptions are valid. First, the slope of the water table is assumed to be small. Second, the velocity is assumed to be horizontal; or, equivalently the vertical velocity is assumed negligible. Third, the velocity is assumed to be uniform with depth. Lastly, the velocity in any direction is assumed to be determined by the slope of the water table in that direction. These assumptions are violated for a large class of interesting problems. Most noteably, they are violated near a discharging well. Nevertheless, the formalism has traditionally been applied to most unconfined flow problems. The results have been quite adequate for many situations. We wish to address those situations where the Dupuit assumptions give inadequate results.

Other investigators have presented more rigorous discussions of unconfined flow. Szabo and McCaig have numerically studied unconfined flows in the X-Z plane using finite difference techniques. Taylor and Luthin have numerically attacked the unconfined flow problem by considering both the saturated and unsaturated flow regions. Neuman and Witherspoon have applied finite element techniques to a more rigorous description of unconfined flow.

Today the discussion of unconfined flow will develop along these lines. First, we shall define the model along with its inherent assumptions. Second, we will derive some of the basic equations describing the flow and show how they reduce to the Boussinesq equation under certain assumptions. Third, numerical procedures for solving the basic flow equations will be discussed. Lastly, the numerical results for this technique and the Boussinesq equation will be compared for a specific example.

The second slide shows the flow model we will deal with.
The top of the slide shows a vertical slice through a system
with radial symmetry. H_{rw} is the water level in a well of
radius r_w. At radius R we have a constant head boundary
condition, H_e. ϕ is the hydraulic head. h is a two dim-
ensional surface describing the free surface elevation. We
shall neglect non-Darcian flow and assume the velocity of
flow is given by the product of the hydraulic conductivity
and the negative gradient of ϕ. Also production of water
from elastic storage is assumed negligible in comparison to
that produced by dewatering. This means that ϕ obeys the
Laplace equation in the interior of the model. Laplace's
equation is the appropriate continuity equation. We shall
neglect the unsaturated zone and any delayed gravity drain-
age mechanism.

The boundary conditions for our model are shown on the
bottom of this slide. On the free surface from A to B it is
required that the hydraulic head simply equal the elevation
of the water table. On the seepage face EA the hydraulic
head must equal the vertical height Z. On the well face,
DE, and at the outer boundary, BC, we have the hydraulic
head specified as H_{rw} and H_e, respectively. The bottom of
the model CD is assumed a barrier boundary.

Even with ϕ satisfying Laplace's equation in the interior of
the model and boundary conditions on ϕ we still do not have
a complete description of the model because the position of
the free surface, h, is unknown. We need an additional
equation relating h and ϕ. On the next slide we show an
elemental volume at the surface of the water table. We have
a change in storage due to the movement of the free surface
and a change in storage due to fluid flow through the ele-
mental volume. Of course these two changes must balance
each other if water is not created in the elemental volume.
In the calculus limit as the elemental volume shrinks to
zero the equation at the bottom of the slide results. This
equation relates the movement of the free surface to grad-
ients of ϕ at the free surface.

This same equation for the free surface movement can be
derived easily from another point of view as shown on the
next slide. The free surface is a material surface so the
material derivative of the free surface must vanish. The
function F describes the free surface. Setting the material
derivative of F equal to zero results in the same free
surface equation.

An additional equation describing unconfined flow can be
derived by integrating Laplace's equation from the base of
the aquifer to the free surface. An alternate derivation
shown in the next slide considers flow through an elemental
annular volume of thickness dr. Considering the flow into

the volume, the flow out of the volume, the change in height
of the free surface, and the limit as dr → o gives the
equation shown at the bottom of this slide.

The next slide summarizes the equations we have to describe
the unconfined flow. They are not independent equations.
Any two can be used to obtain the third. Together with the
boundary conditions two of these equations provide a com-
plete description of this unconfined flow model. The first
two equations shown in this slide will be used in the numerical
procedure to be discussed later. The main reason the third
equation is discussed at all is to show how the usual theories
of unconfined flow result from additional assumptions.

On the next slide we have transferred the third equation of
the previous slide. We would like to get the partial de-
rivatives with respect to r outside the integral. To do
this it is necessary to apply Leibnitz's rule twice. The
resulting equation still involves both ϕ and h. If one
assumes that ϕ is invariant with depth and equal to h then
the integral of ϕ is just h^2. This is in the spirit of the
Dupuit assumptions. With this approximation the Boussinesq
equation results. It involves only h and is the usual
choice to model unconfined flow.

We wish to apply the more rigorous formalism discussed
previously to a specific example. The next slide shows the
example choosen. For simplicity we consider only flow in
the X-Z plane instead of radial symmetry. The model con-
sists of two water bodies initially with water heights of
1000 ft. separated by 1000 ft. of porous material. The
specific yield of the porous material is taken to be .15.
The hydraulic conductivity is assumed to be 15 ft/day. At
time equal to zero the water body on the left experiences a
sudden water level drop to 500 ft. after which it remains
constant. The problem is to find the position of the free
surface at any time. We shall numerically model this sit-
uation by introducing a grid spacing of 50 ft. in both the X
and Z directions.

Since the water table is moving we must solve Laplace's
equation in an irregular region that changes with time. The
next slide shows what is involved in solving Laplace's
equation in an irregular region. As the water table declines

some grid points will no longer be included in the model. Examples of this are shown at points A and B of this slide. For points of the model near the free surface boundary a special numerical form of Laplace's equation must be used that takes into account the distances $a\Delta Z$ and $b\Delta X$. For a grid point not adjacent to the free surface, a and b are equal to one and the usual numerical approximation to Laplace's equation results.

The next slide summarizes the numerical procedure used to solve the example. First, Laplace's equation is solved for ϕ for some initial configuration of the free surface. In the example the initial water table is flat with an elevation of 1000 feet. The boundary conditions involve an initial seepage face of 500 ft., a barrier lower boundary, and constant head boundaries of 500 ft. on the left and 1000 ft. on the right. After obtaining a solution of Laplace's equation for this initial configuration, the gradients of ϕ with respect to Z and r can be evaluated at the free surface. These gradients are then used in the free surface equation (step 2) to calculate a new position of the free surface after a time Δt.

At this new position of the free surface Laplace's equation is solved again and the gradients of ϕ are recalculated (step 3). These new values of the gradients are used in the free surface equation to calculate an improved position of the free surface (step 4). Steps 3 and 4 are repeated until no further significant improvement in the free surface, h, results.

The time is then incremented by Δt and the procedure is repeated starting at step 2 until the desired final time is reached. The increased difficulty of solution is seen to arise from the fact that two partial differential equations must be solved simultaneously.

The numerical results are shown in the last slide. The solid lines show the water table at two different times calculated with the numerical procedure just discussed. The dashed curve shows the water table for our example calculated from the Boussinesq equation at a time of 1.06 days. One notices immediately a great discrepancy between the two theories. This should not be too surprising when one realizes that all the Dupuit assumptions are rather badly violated in this example. In particular, when rapid drawdowns are experienced which are sizeable percentages of the saturated thickness, the Dupuit assumptions are invalid. The distance between the water bodies was only 1000 ft. in the example, the same as the saturated thickness. If the distance between water bodies had been many times the saturated thickness the Dupuit assumptions would have given much better results.

From these results it can be seen that the Dupuit assumptions are not very good for a large class of unconfined flow problems. The formalism discussed here gives much better results but is somewhat more difficult to program. The method seems to be stable and have good convergence properties. At the Kansas University computation center a run on the example problem for 35 time steps with an acceleration factor of 1.2 cost about $8.30. The application of this formalism to more complicated flow systems without radial or planar symmetry is somewhat more difficult and is probably not feasible at this time.

Dupuit-Forchheimer Approximation To Unconfined Flow

$$\tilde{\nabla} \cdot (kh\tilde{\nabla}h) = S_\gamma \frac{\partial h}{\partial t} \qquad \text{Boussinesq Equation 1904}$$
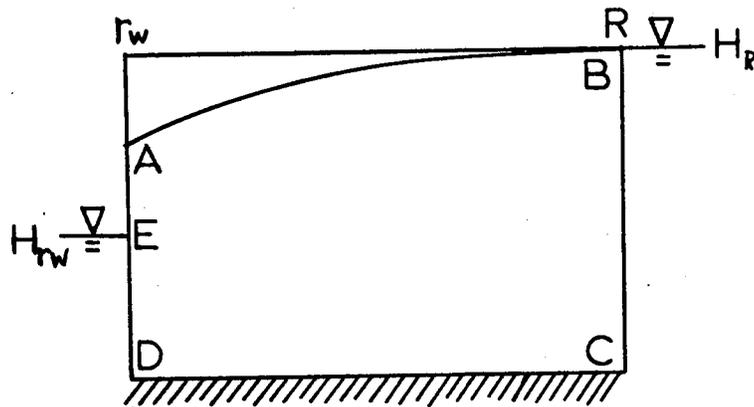
$h_o = $ Initial Hydraulic Head

Boundary Conditions:

(1). $\frac{\partial h}{\partial n} = 0$ Barrier Boundary

(2). h Prescribed On Boundary
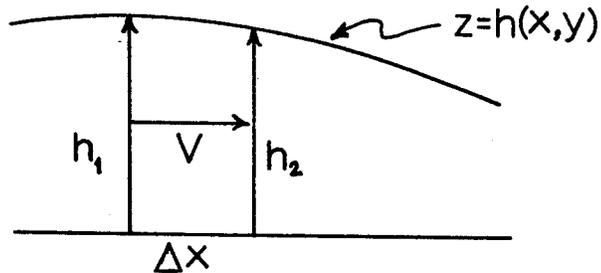
Assumptions:

(1). Water Table Slope Is Small

(2). Velocity Is Horizontal

(3). Velocity Is Invariant With Depth

(4). $V_x = -k\frac{\partial h}{\partial x}$

Unconfined Radial Flow

$\phi(r,z,t) = $ Hydraulic Head

$z = h(r,t) = $ Equation Of Free Surface

$\tilde{V} = -k\tilde{\nabla}\phi(r,z,t)$ Darcy's Law

Assume No Elastic Effects

$\nabla^2 \phi(r,z,t) = 0$ Continuity Equation

Boundary Conditions:

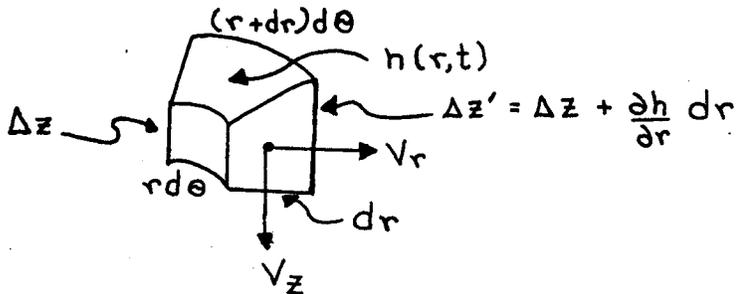$\phi(r, h(r,t),t) = h(r,t)$ On AB

$\phi(r_w, z,t) = z$ On EA

$\phi(r_w, z,t) = H_{r_w}$ On DE

$\phi(R,z,t) = H_R$ On BC

$\left. \dfrac{\partial \phi(r,z,t)}{\partial z} \right|_{z=0} = 0$ On CD

Free Surface Equation

Physical Derivation

S_y = Specific Yield

$S_y \dfrac{\partial h(r,t)}{\partial t} rd\theta dr \equiv$ change in storage due to surface movement

$-V_{r+dr}(r+dr)d\theta \Delta z' + V_r rd\theta \Delta z + V_z rd\theta dr \equiv$ change in storage due to fluid velocity

$$S_y \frac{\partial h}{\partial t} = -\left[\frac{V_{r+dr}(r+dr)\Delta z' - V_r r \Delta z}{rdr}\right] + V_z$$

$$S_y \frac{\partial h}{\partial t} = -\left[\frac{V_{r+dr}(r+dr)\left(\Delta z + \frac{\partial h}{\partial r}dr\right) - V_r r \Delta z}{rdr}\right] + V_z$$

In the limit as $d\theta, dr, \ \& \ \Delta z \longrightarrow 0$

$$S_y \frac{\partial h}{\partial t} = V_z - V_r \frac{\partial h}{\partial r} \qquad \text{on free surface}$$

$$S_y \frac{\partial h(r,t)}{\partial t} = -K\left[\frac{\partial \phi(r,z,t)}{\partial z}\right]_{z=h(r,t)} + K \frac{\partial h(r,t)}{\partial r}\left[\frac{\partial \phi(r,z,t)}{\partial r}\right]_{z=h(r,t)}$$

Free Surface Equation

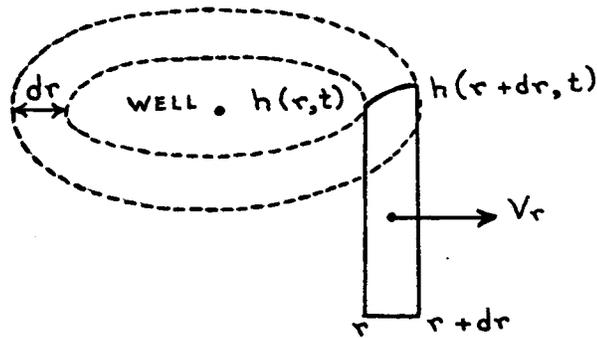The Material Derivative of the Free Surface
Must Vanish

$$F(r,z,t) = z - h(r,t) = 0 \qquad \text{on free surface}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla} \qquad \begin{array}{l} \text{material derivative} \\ V \text{ is fluid velocity} \end{array}$$

$$\frac{d}{dt} F(r,z,t) = -\frac{\partial h(r,t)}{\partial t} + \vec{V} \cdot \vec{\nabla}(z - h(r,t)) = 0$$

$$= -\frac{\partial h(r,t)}{\partial t} - \frac{V_r}{S_y} \frac{\partial h(r,t)}{\partial r} + \frac{V_z}{S_y} = 0$$

$$S_y \frac{\partial h(r,t)}{\partial t} = -K \left[\frac{\partial \phi(r,z,t)}{\partial z} \right]_{z=h(r,t)}$$

$$+ K \frac{\partial h(r,t)}{\partial r} \left[\frac{\partial \phi(r,z,t)}{\partial r} \right]_{z=h(r,t)}$$

Continuity Equation
(Incompressible Fluid)

$$-\int_{0}^{h(r+dr,t)} V_{r+dr}\, 2\pi (r+dr)\, dz \;+\; \int_{0}^{h(r,t)} V_r\, 2\pi r\, dz \;=\; S_y\, 2\pi r\, dr\, \frac{\partial h(r,t)}{\partial t}$$

$$-\frac{1}{r\, dr}\int_{h(r,t)}^{h(r+dr,t)} V_{r+dr}\, (r+dr)\, dz \;-\; \frac{1}{r}\int_{0}^{h(r,t)} \frac{\partial (r V_r)}{\partial r}\, dz \;=\; S_y\, \frac{\partial h(r,t)}{\partial t}$$

$$h(r+dr,t) - h(r,t) = \frac{\partial h}{\partial r}\, dr$$

In the limit as $dr \longrightarrow 0$, we have

$$-\Big[V_r\Big]_{z=h(r,t)}\frac{\partial h(r,t)}{\partial r} \;-\; \frac{1}{r}\int_{0}^{h(r,t)} \frac{\partial (r V_r)}{\partial r}\, dz \;=\; S_y\, \frac{\partial h(r,t)}{\partial t}$$

$$K\left[\frac{\partial \phi(r,z,t)}{\partial r}\right]_{z=h(r,t)}\frac{\partial h(r,t)}{\partial r} \;+\; \frac{K}{r}\int_{0}^{h(r,t)} \frac{\partial}{\partial r}\left(r\,\frac{\partial \phi(r,z,t)}{\partial r}\right) dz \;=\; S_y\, \frac{\partial h(r,t)}{\partial t}$$

In Addition To The Boundary Conditions We Have

Three Equations Describing The Flow.

(1). $\nabla^2 \phi = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\,\dfrac{\partial \phi(r,z,t)}{\partial r}\right) + \dfrac{\partial^2 \phi(r,z,t)}{\partial z^2} = 0$

(2). $S_y \dfrac{\partial h(r,t)}{\partial t} = -K\left[\dfrac{\partial \phi(r,z,t)}{\partial z}\right]_{z=h(r,t)} + K\,\dfrac{\partial h(r,t)}{\partial r}\left[\dfrac{\partial \phi(r,z,t)}{\partial r}\right]_{z=h(r,t)}$

(3). $K\left[\dfrac{\partial \phi(r,z,t)}{\partial r}\right]_{z=h(r,t)}\dfrac{\partial h(r,t)}{\partial r} + \dfrac{K}{r}\displaystyle\int_{0}^{h(r,t)}\dfrac{\partial}{\partial r}\left(r\,\dfrac{\partial \phi(r,z,t)}{\partial r}\right)dz = S_y\,\dfrac{\partial h(r,t)}{\partial t}$

They Are Not Independent Any Two Can Be

Used To Obtain The Third.

The Usual Equation Describing Unconfined Flow
Can be Obtained Easily From

$$K \frac{\partial h(r,t)}{\partial r} \left[\frac{\partial \phi(r,z,t)}{\partial r} \right]_{z=h(r,t)} + \frac{K}{r} \int_0^{h(r,t)} \frac{\partial}{\partial r} \left(r \frac{\partial \phi(r,z,t)}{\partial r} \right) dz = S_y \frac{\partial h(r,t)}{\partial t}$$

by using Leibnitz's rule twice

$$\frac{\partial}{\partial r} \int_0^{B(r)} f(r,z) \, dz = \int_0^{B(r)} \frac{\partial f(r,z)}{\partial r} \, dz + f(r, B(r)) \frac{\partial B(r)}{\partial r}$$

to obtain

$$\frac{K}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \int_0^{h(r,t)} \phi(r,z,t) \, dz \right) - \frac{K}{r} \frac{\partial}{\partial r} \left(r h(r,t) \frac{\partial h(r,t)}{\partial r} \right) = S_y \frac{\partial h(r,t)}{\partial t}$$
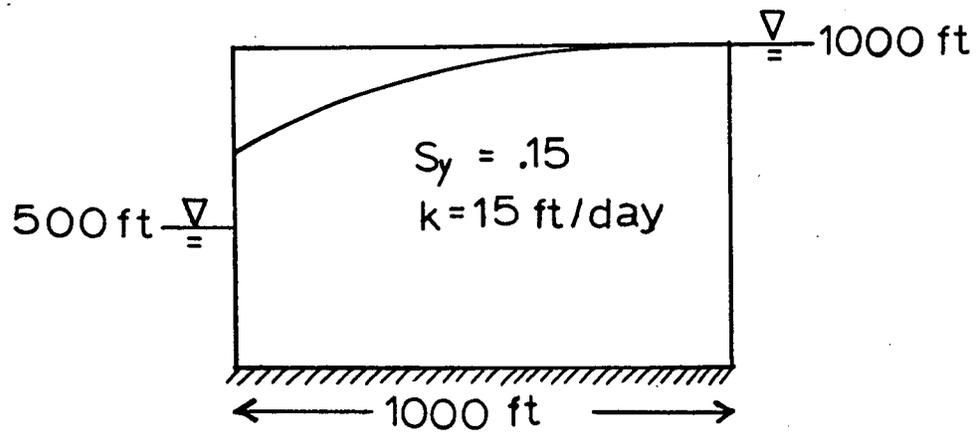
and then assume

$$\int_0^{h(r,t)} \phi(r,z,t) \, dz \cong h^2(r,t)$$

giving

$$\frac{K}{r} \frac{\partial}{\partial r} \left(r h(r,t) \frac{\partial h(r,t)}{\partial r} \right) = S_y \frac{\partial h(r,t)}{\partial t}$$

$$K \vec{\nabla} \cdot \left(h(r,t) \vec{\nabla} h(r,t) \right) = S_y \frac{\partial h(r,t)}{\partial t} \qquad \text{Boussinesq Eq.}$$

Example With Flow Only In X-Z Plane
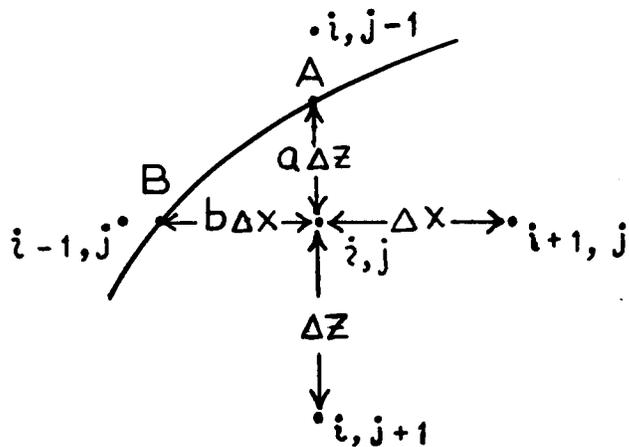

Initial Head Is 1000 ft.

At t = 0 Water On Left Suddenly Drops

To 500 ft And Remains Constant.

For Numerical Model $\Delta x = \Delta z = 50$ ft.

Numerical Solution Of Laplace's Equation

In An Irregular Region

$$\nabla^2 \phi \cong \frac{2}{(\Delta x)^2} \left[\frac{\phi_{i+1,j}}{b+1} - \frac{\phi_{i,j}}{b} + \frac{\phi_B}{b(b+1)} \right]$$

$$+ \frac{2}{(\Delta z)^2} \left[\frac{\phi_{i,j+1}}{a+1} - \frac{\phi_{i,j}}{a} + \frac{\phi_A}{a(a+1)} \right] = 0$$

If $\quad a = b = 1 \quad$ and $\quad \Delta x = \Delta z$

$$\nabla^2 \phi = \frac{1}{(\Delta x)^2} \left[\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} \right] = 0$$

Numerical Procedure

(1). Solve Laplace's equation for some initial configuration of the free surface.

(2). Use $\left[\dfrac{\partial \phi(r,z,t)}{\partial z}\right]_{z=h(r,t)}$ and $\left[\dfrac{\partial \phi(r,z,t)}{\partial r}\right]_{z=h(r,t)}$

from the previous time to calculate a new position of the free surface after time step Δt.

$$S_y \frac{\partial h(r,t)}{\partial t} = -K\left[\frac{\partial \phi(r,z,t)}{\partial z}\right]_{z=h(r,t)} + K\frac{\partial h(r,t)}{\partial r}\left[\frac{\partial \phi(r,z,t)}{\partial r}\right]_{z=h(r,}$$

(3). Calculate a better estimate of

$$\left[\frac{\partial \phi(r,z,t)}{\partial z}\right]_{z=h(r,t)} \quad \text{and} \quad \left[\frac{\partial \phi(r,z,t)}{\partial r}\right]_{z=h(r,t)}$$

by solving Laplace's Equation for new position of free surface obtained in previous step.

(4). Calculate a better free surface position $h(r,t)$.

(5). Increment time by Δt and repeat starting at step (2).

Numerical Solution Of Example