Carl McElwee

KGS OF 72-10

Theory, Sensitivity and Application of a Regional Unconfined Ground-Water Model

By

John C. Halepaska
Larry Keearns
Fred W. Hartman

Prepared by the State Geological Survey of Kansas in cooperation with the U.S. Geological Survey and the Kansas Water Resources Board.

THEORY, SENSITIVITY AND APPLICATION OF REGIONAL GROUND-WATER MODELS

J. C. Halepaska*, L. Keearns**, and F. W. Hartman**

ABSTRACT

The need to model unconfined aquifers is a common problem. Numerical forms have been developed that solve a special form of the classical differential equation of ground-water flow suitable for regional unconfined-aquifer analysis. Under constraints, adequate interpretation can be made for general application. Use of the technique is demonstrated by application of a numerical form of the partial differential equation to the Wichita well field in central Kansas. Application of the numerical form shows that the aquifer system is capable of handling projected ground-water withdrawals through 1980.

* Chief, Ground Water Section, Kansas Geological Survey

** Research Assistant, Kansas Geological Survey

INTRODUCTION

Emphasis on understanding flow through porous media began to receive intensive attention, primarily in the oil and gas industries, in the middle and late 1930's. Investigators began to rediscover and to solve special cases of the Navier-Stokes equations. A wealth of information was drawn from such classics as those by Lamb (1932), Kellogg (1953) and Carslaw and Jaeger (1959). The state of understanding fluid flow in porous media was detailed by Muskat (1937). Before the digital computer became available in about 1950, emphasis was placed on setting up and solving special tractable cases using transform calculus (e.g., Hantush, 1964). The oil and gas industry and many other sectors of industry quickly utilized the computer. as evidenced by the large number of theoretical and applied papers (e.g., Giese, 1971). Ground-water theory and application, due primarily to economics, remained somewhat rooted in the area of analytic solutions and analogue computer techniques.

Digital-computer techniques began to be used as a ground-water tool in the middle and late 1960's when computers became generally available. Numerous books appeared during this period that outlined, in a general fashion, numerical techniques for solving partial differential equations (e.g., Smith, 1965; Zienkiewicz, 1967; Carnahan, et. al., 1969; Von Rosenberg, 1969).

The rigors involved in solving a boundary-value problem must necessarily be relaxed when applying a digital model to a field case. However, all too often models developed for specific conditions are applied to markedly different field situations with little or no modification.

The objectives of this paper are to document efforts by the authors to stepwise examine the various elements of a groundwater modeling project. To this end the groundwater theory has been examined and modified to satisfy more rigorously field conditions. In addition, a variety of numerical experiments were conducted to qualatively assess the sensitivity of the theory to representative variations in aquifer parameters. Finally, the theory has been applied to a field case and interpreting projections have been performed.

THEORY

In hydrodynamics, special attention is given to conserving mass, energy and momentum. In groundwater flow it is generally assumed that energy and momentum are negligible considerations compared to conservation of mass (Hubbert, 1940). The continuity equation and Darcy's law yield the classical equation of flow through porous media (e.g., Jacob, 1950) as follows:

$$\nabla \cdot K\nabla h = S_s \frac{\partial h}{\partial t} \qquad (1)$$

where

h	$= (\frac{P}{\gamma} + z)$, hydraulic head	(L)
K	= hydraulic conductivity	$(\frac{L}{T})$
P	= pressure	$(\frac{M}{LT^2})$
S_s	= specific storage	$(\frac{1}{L})$
t	= time	(t)
z	= reference height	(L)
γ	= specific weight	$(\frac{M}{T^2L^2})$
∇	= gradient $= \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$	$(\frac{1}{L})$
$\nabla\cdot$	= divergence $= \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$	$(\frac{1}{L})$

Except for simple boundary conditions, equation (1) cannot be solved without the use of computer techniques.

Equation (1) can be written in 2-dimensional form as follows:

$$\frac{\partial}{\partial x}\left(T\,\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\,\frac{\partial h}{\partial y}\right) + \frac{Q}{\Delta x^2} - \frac{QQ}{\Delta x^2} = S\,\frac{\partial h}{\partial t} \qquad (2)$$

where

T = Kb = transmissivity $\left(\frac{L^2}{T}\right)$

b = saturated thickness (L)

$\frac{Q}{\Delta x^2}$ = specific flux $\left(\frac{L}{T}\right)$

$\frac{QQ}{\Delta x^2}$ = composite natural flux $\left(\frac{L}{T}\right)$

S = storage coefficient

When the composite natural flux is zero, equation (2) is valid for horizontal flow in a single horizontal confined aquifer if the following conditions are met: (1) transmissivity can be approximated by the major X and Y components of a space dependent tensor, (2) storativity is independent of space and time, and (3) the initial head is constant in space.

A common approach to applying equation (2) when transmissivity and storage are unknown is to use the history match approach. The history match approach entails repetitively adjusting the aquifer coefficients until the model matches some measured history of hydrologic change.

When a field case violates the assumptions for applying equation (2) outlined above, a different or modified equation should be used. The approach taken here was to pursue the problem from a generalized inverse problem concept.

Emphasis upon understanding and solving the inverse problem, that is, satisfying some potential surface by resistance terms, has received attenti in the last ten years. Several investigators have attempted to generate aquifer or reservoir coefficients from either static- or dynamic-flow condit Stallman (1963, a, 1963, b) derived conductivity terms using an analog mode Nelson and McCollum (1969) defined the coefficients in terms of flow and equipotential lines. Emsellum and De Marcily (1971) developed a technique that guides the development of the "smoothest T surface" to satisfy the hea configuration. Kleinecke (1971) made use of linear programing techniques t estimate aquifer coefficients. Johns (1966), Coats et. al. (1970), and Nan (1970) used an automatic history match for a dynamic system. All of these methods attempt to solve the problem by satisfying some potential surface i terms of resistance in the system. However, in semi-confined or unconfined systems, natural flux rates often play a significant part in defining a gro water surface. Reddell (1967) used a numerical form to generate flux terms with known transmissivity values.

The problem in an unconfined system is to approximately separate the effects of aquifer coefficients from the effects of recharge upon a given potential surface.

The problem of separating flux conditions from aquifer coefficients can be handled in an approximate lumped fashion. Reddell (1967) defined recharge rates as

$$\nabla \cdot T\nabla h = QQ/\Delta x^2 \tag{3}$$

where

$\frac{QQ}{\Delta x^2}$ = Composite natural flux

Equation 3 will yield correct composite and natural flux terms only if the areal distribution of flux and transmissivity are the only contributing parts to the shape of the potential surface and the transmissivity and head values are correct.

A more complete definition of equation (3) would be

$$\nabla \cdot T\nabla h = [Q_R + Q_{ET} + Q_L + Q_E + Q_N] / \Delta x^2 = \frac{QQ}{\Delta x^2} \tag{4}$$

Where

Q_N = natural discharge

Q_R = natural recharge

Q_{ET} = evapotranspiration

Q_E = total composite-flux error

Q_L = leakage

Equation (4) then details the composite flux term with an error component. The error component would contain both data-input errors and errors resulting from an approximation of an actual system by a 2-dimensional partial-differential equation, which has been further approximated by a numerical analog. Q_E may be defined as the sum of the errors involved in applying equation (4).

Unfortunately, an analysis based on equation (4) yields an approximate lumped flux and the numerical values for individual components cannot be found without other sources of information. Such information may include empirical relationships between rainfall and recharge, evapotranspiration and temperature and/or depth to water.

Equation (4) combined with equation (2) can generate information of value to the analysis. If we assume that the modeled area is initially a steady-state system, substitution of equation (4) into equation (2) yields

$$\frac{Q}{\Delta x^2} + \nabla \cdot T\nabla h - [(\nabla \cdot T\nabla h)]_{t = t_o} = S \frac{\partial h}{\partial t} \tag{5}$$

Where

t_o = initial time

If the specific flux terms are allowed to be zero, application of equation (5) to any data set will exactly preserve the initial potentiometric surface. It is thus noted that the potential change calculated by using equation (5) with non-zero specific flux is independent of the initial potentiometric surface.

In practice, equation (5) is implemented by simply solving equation (4) at the start of the problem and maintaining the calculated values as a constant throughout the transient phase of the problem (see appendix A for a numerical form of equation 5).

Although the history-match procedure has inherent weaknesses, it is the best procedure available to support an investigator's interpretation of regional model projections. A comparison of best history matches can help to separate the effect of flux from the effect of aquifer coefficients in satisfying a potentiometric surface. The following procedure is suggested as a guide:

1) Assume QQ=0 and apply equation (2) to obtain the best history-match by adjusting the aquifer coefficients.
2) Determine how well step 1 above satisfies the problem, assume the aquifer coefficients from step 1 are correct, set the specific flux and natural specific flux terms to zero and apply equation (2) for a span of time at least equal to the time used for the history match. This application will indicate the location and magnitude of erroneous potential change.
3) Apply equation (5) to the data set obtained from steps 1 and 2. This generally markedly improves the match, but can still contain considerable error.
4) If the degree of match is not sufficient, adjust the aquifer coefficients and apply equation (5) until a sufficient degree of accuracy is realized.

If the problem involves modeling an area that has already undergone development, equation (4) is slightly modified to the form

$$[(\nabla \cdot T\nabla h)]_{t = t_1} = [Q_R + Q_{ET} + Q_L + Q_E + Q_D + Q_N] / \Delta x^2 \quad (6)$$

where Q_D represents pumpage rates at time t_1, and is reflected in a steady-state fashion in the input-head values. This basically means that only new specific flux terms can be added, as all specific flux terms prior to t_1 will be included by the addition of equation (6) to equation (2). Inherent in equation (4) or (6) is the assumption that input values of head represent average regional steady-state values. It is important to understand that equation (5) generates source functions, which completely arrest the decline of the system. Because a given potential surface representing a water table can be satisfied by an infinite number of overlapping resistance and flux arrays, repetitive matches over some transient period of history using equation (5) appear to be a reasonable approach.

Sensitivity Analysis

The basic equations and the corresponding numerical forms used in this study were subjected to variations in aquifer coefficients to qualitatively assess degrees of sensitivity. Included in this basic analysis were variations in transmissivity, specific yield, and specific flux. In addition, estimates are made on the degree of error involved in violating a constant-head boundary condition. An example comparing the analytic solution for a single well to the numerical form is also included.

In this study qualitative numerical experiments were used as judgement guides in satisfying incomplete model data sets. The results that follow summarize the set of experiments performed.

A comparison of results derived by analytical solutions and by numerical solutions is shown in figure 1. The numerical form is assumed to have a

---

Figure 1 near here.

---

square grid of 1-mile spacing, a specific yield of 0.1, a transmissivity of 150,000 gal/day-ft (gallons per day per foot), and an initial surface that is flat. The well is pumped at a constant 5 cfs (cubic feet per second) for 1 year, 6 cfs the second year, 7 cfs the third year, 8 cfs the fourth year and 10 cfs the fifth year. This numerical form, which gives a poor fit of the data close to the well and an excellent fit at one mile beyond, is a good example of discretization error. This type of error is unavoidable if the solution objective is to predict regional trends when the investigator is limited by computer storage and data availability. Obviously, if the solution object is local in scope, a finer mesh will give better results.

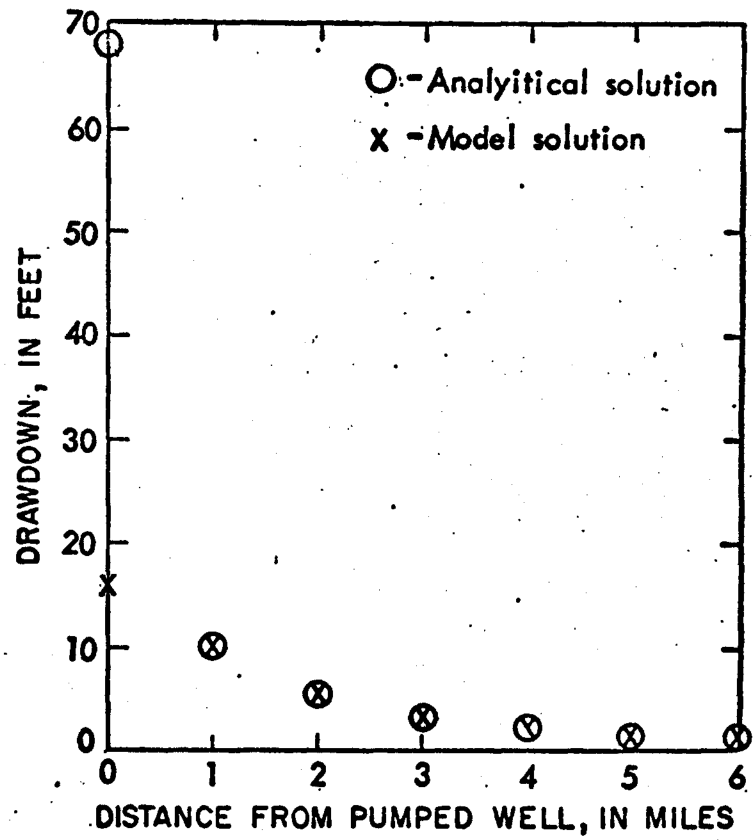

Figure 1.--Comparison between analytical and numerical solutions.

Node spacing is an important function of the particular problem and the capability of the machine available. Spacing generally should be small enough to allow good representation of a surface by a series of points. The limit of large spacing generally is indicated when the accuracy of the answer begins to diminish as numerical experiments are compared using larger and larger spacing.

A rather common problem in modeling is the determination of effects from a boundary approximated as a constant-head boundary. If the effects of pumping reach a constant-head boundary, error will be introduced back into the system under investigation. An obvious solution is to move the boundary as far away as the additional mesh of machine storage will permit. This may not be necessary in cases where the error would be negligible or in cases where the error at the nodes nearest the boundary can be ignored. An analysis of boundary effects also is helpful in determining how a large area can be broken into smaller areas for modeling. An indication of the overlap necessary for a good splice of areas can be determined.

Moving the effects of pumping from 5 wells progressively closer to a constant-head boundary is shown in figure 2. The plotting in figure 2

Figure 2 near here.

---

corresponds to the drawdown along the dashed line shown in the schematic grid. The node number at the top of each peak corresponds to the node point shown as a solid dot in the grid. It is assumed that each of the 5 wells is pumping 2 cfs from a square grid of 1 mile, which has a specific yield of 0.15 and a transmissivity of 100,000 gal/day-ft. The results shown in figu 2 are at the end of 2,089 days.

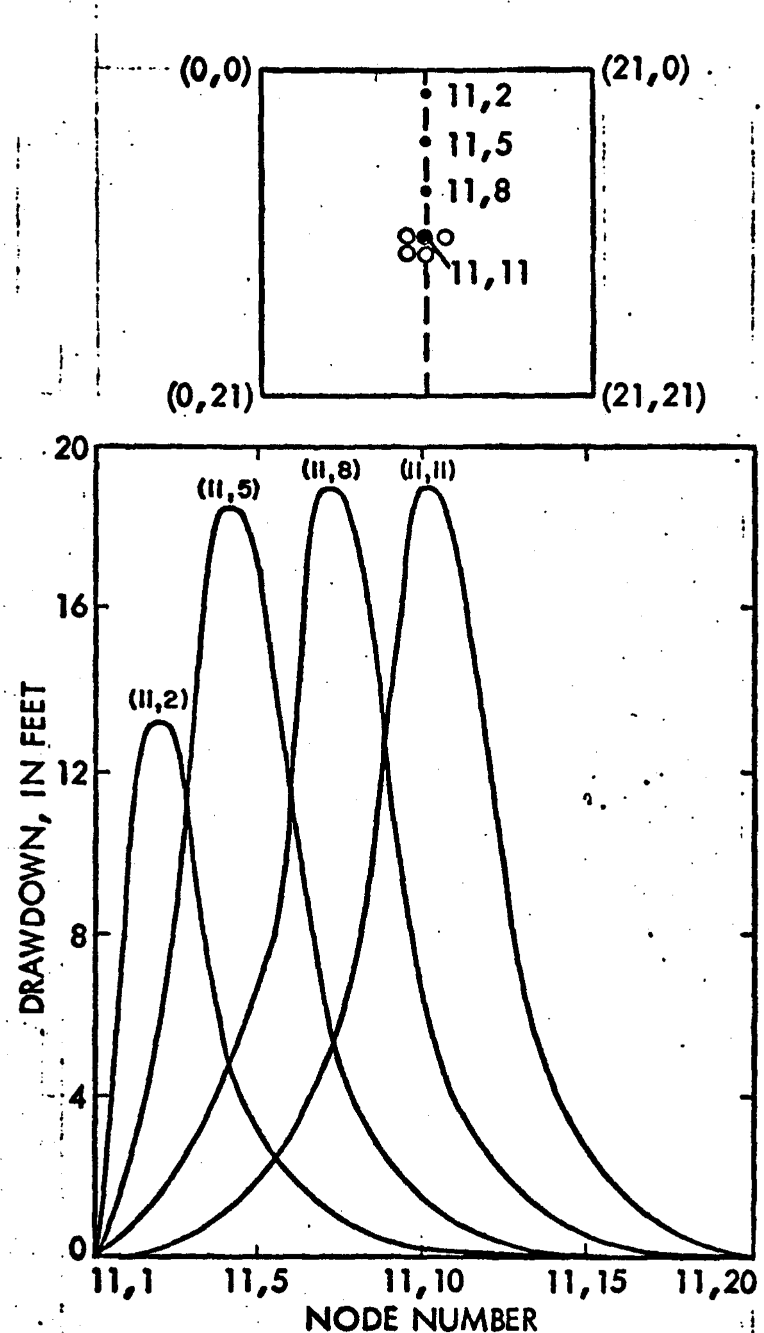

Figure 2.--Influence of constant-head boundary on numerical solution.

When wells are located near the center of the field, the solution is only slightly affected by the boundary condition. As wells are moved closer to the boundary, the assumption of negligible effect from a constant-head boundary becomes increasingly invalid.

A very important part of modeling is the derivation of correct storage coefficient or specific yield. This is ordinarily done by going through a history match, which basically entails inputting historical data to match trends in the system up to some present time by varying transmissivity and storage coefficient. If the match is reasonably good, then the investigator can claim verification. A blind application of a history match, however, can lead to difficulties for the following reasons:

1) the mathematics never apply to real case;
2) parts of the problem that generally are assumed can tend to balance or offset one another (see assumptions outlined in theory).
3) possibility of error in the projection of future behavior increases as the projected flux conditions change from those used in history match. This is always true because a history match does not constitute a unique solution. Since a prime objective of modeling is to view the system under different conditions of development, care must be exercised.
4) matching a period of record for a system that has experienced little or no development can be misleading because aquifer characteristics may not be adequately reflected in the record.

The effects on drawdown at different distances from a well for various storage coefficients or specific yields is shown in figure 3. The aquifer

---

Figure 3 near here.

---

coefficients used for computing values in figure 3 are the same as those used for figure 2 except that the time period is 5 years and the storage coefficient is varied.

As the storage coefficient or specific yield can be time and space dependent, the investigator must consider this factor for proper interpretation. A clue to the variability of the term often can be found in the historical record where man has significantly interfered with the natural system.

Transmissivity, which is the product of hydraulic conductivity times thickness, is a term used in modeling aquifer systems. The term came into being as theoreticians struggled to derive coefficients that were representative of 3-dimensional aquifers using 2- or 1-dimensional theory. Transmissivity in an unconfined system is no longer a constant, and should be varied in many cases.

Ignoring the variation of aquifer thickness in a large, very prolific system is not likely to cause serious problems. In a relatively thin system, however, many other factors come into play. Spacing and strength of the projected source terms need to be examined very carefully. If the source terms are large (say in an aquifer 75-feet thick and a grid spacing of about 1 mile), the investigator may find that an analytic solution for the problem has the wate depleted in a well long before the model indicates a problem (see fig. 2). Therefore, transmissivity should be varied in many cases. However, any variation of transmissivity as a function of drawdown without considering spacing and strength of the source terms or actual drawdown at the well can cause an investigator to arrive at completely erroneous answers in relatively thin systems.

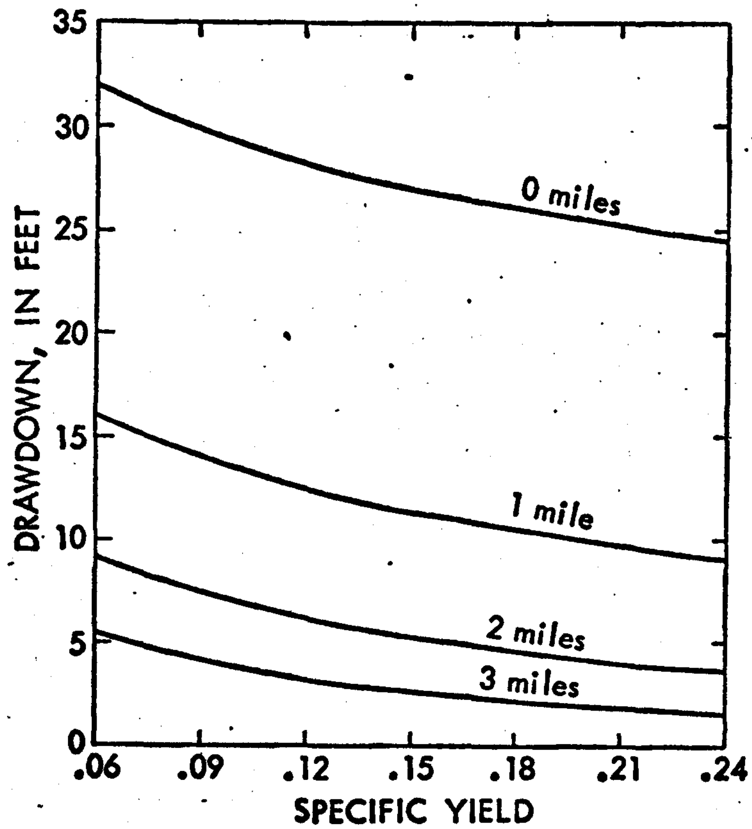

Figure 3.--Sensitivity of model to specific yield.

Drawdown plotted against transmissivity for various distances from the well is shown in figure 4. The aquifer coefficients used for computing

---

Figure 4 near here.

---

values in figures 4 and 5 are the same as those used in figure 3 except specific yield, which is 0.15.

It is difficult to conceive of any aquifer system that would respond according to the concepts of a model with any degree of accuracy unless good historical pumpage records are available. And yet, the investigator is often expected to model systems where little or nothing is known about the time and rate of pumpage, which are the few parameters of ground water that can be measured directly.

The relation of discharge to drawdown at various distances from a well is shown in figure 5. From this relation, it can be concluded that good

---

Figure 5 near here.

---

pumpage records are necessary. Also, it can be expected that an erroneous storage-coefficient value will complicate any projection attempt where historical-pumpage records are poor.

In summary, a groundwater modeling project involves judgement or best guess decisions about mating the mathematical and the real system. Many decisions involved in a modeling project can and should be guided by a system of computer experiments.

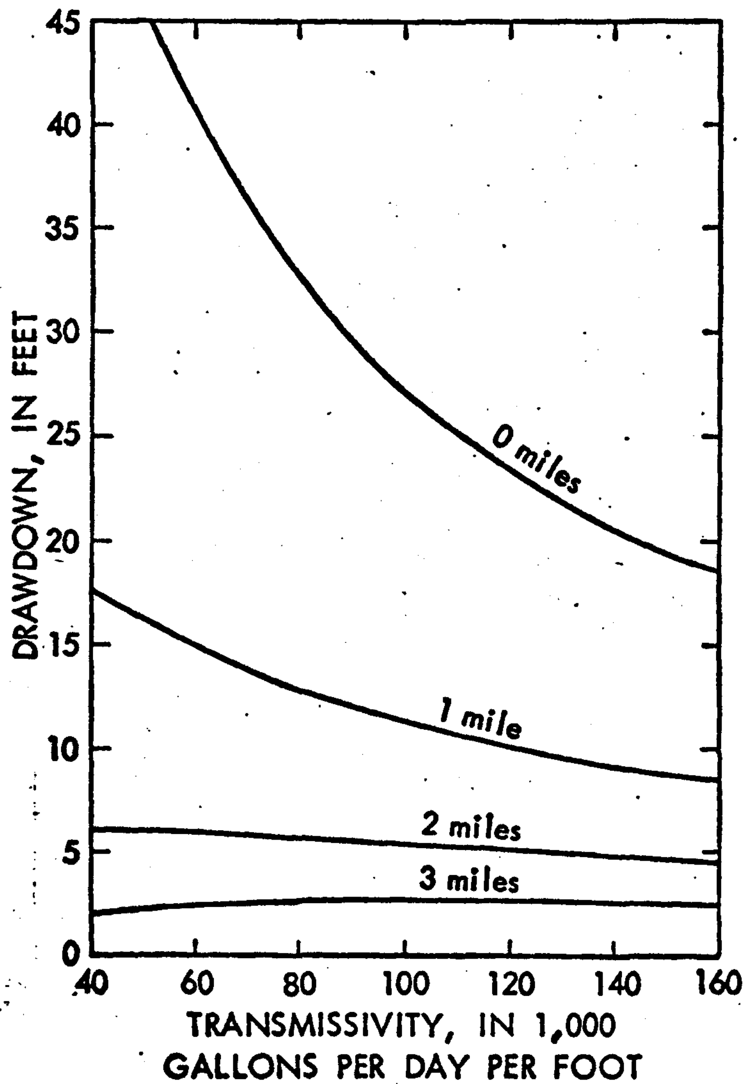

Figure 4.--Sensitivity of model to transmissivity.

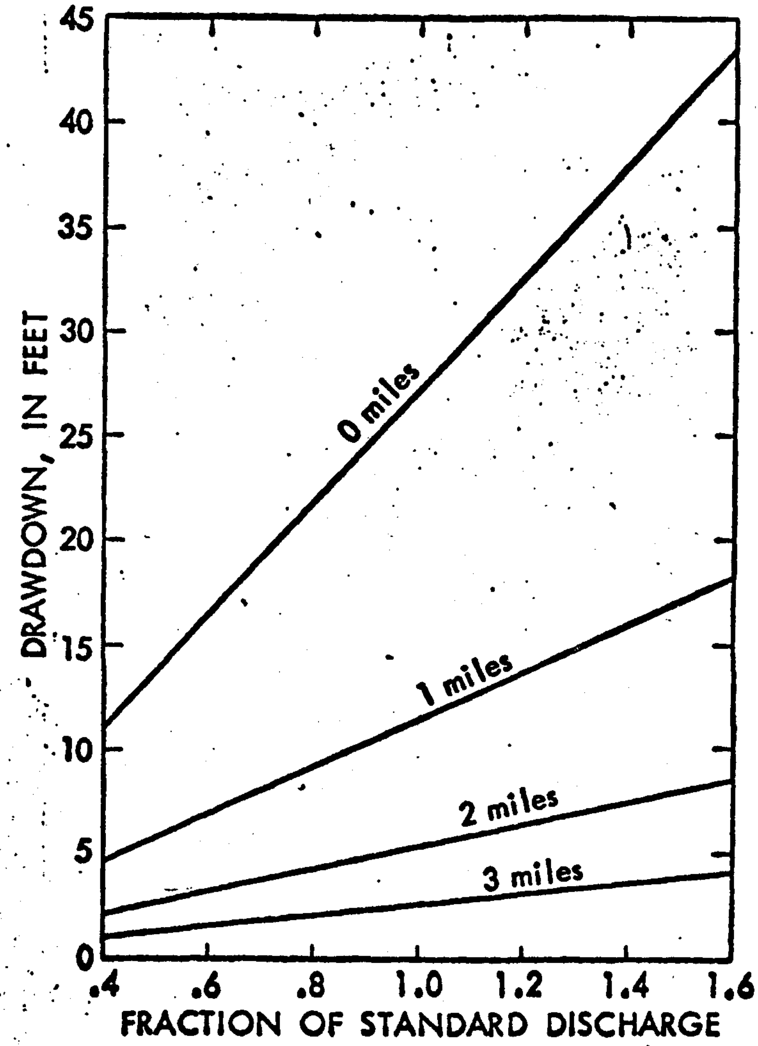

Figure 5.--Sensitivity of model to discharge.

Application

The city of Wichita, in south-central Kansas, has depended upon ground water for large supplies of high quality water since 1940. Figure 6 is a

---

Figure 6 near here.

---

map of Kansas that shows the geographic location of the city of Wichita with respect to the well field. The aquifer, commonly called the Equus beds, is composed of sand and gravel containing many clay lenses, which vary in thickness and areal distribution (see Williams and Lohman, 1949, and Petri, Lane and Furness, 1964 for a complete geologic description). The 1940-44 head configuration is taken from the 1940 water-level map by Williams and Lohman as shown in figure 7. The location of the present municipal supply

---

Figure 7 near here.

---

wells is also shown on figure 7.

The gross pumpage by year from the well field is shown in figure 8.

Figure 8 near here.

The city of Wichita, which operates this well field, attempts to minimize drawdown in the well field and thus creates a very complex set of pumpage records. The historical and projected pumpage records and the aquifer coefficients used in this study are based on records from the city of Wichita, the Kansas Geological Survey, the U.S. Geological Survey and the Kansas Water Resources Board. The compilation of these records represents a major effort, is available elsewhere (Richards and Dunaway, 1972), and will not be presented in this text.

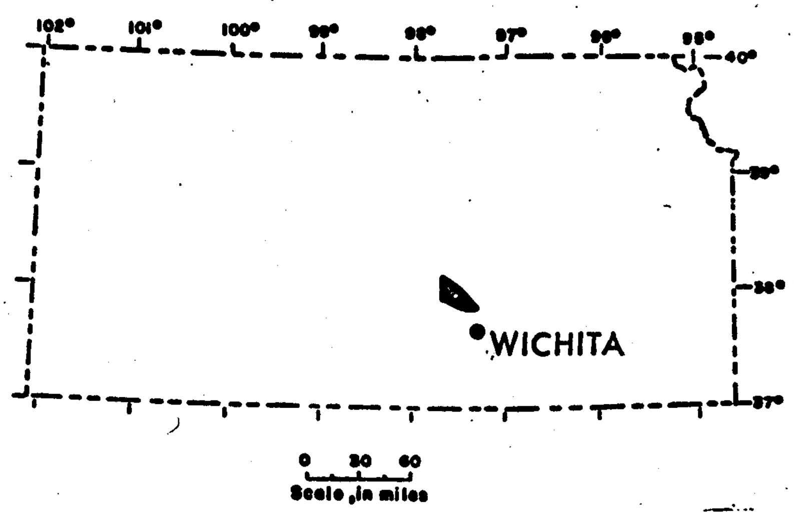

Figure 6.--Map of Kansas showing location of the city of Wichita in relation to the well field.

Harvey; Reno; Sedgwick

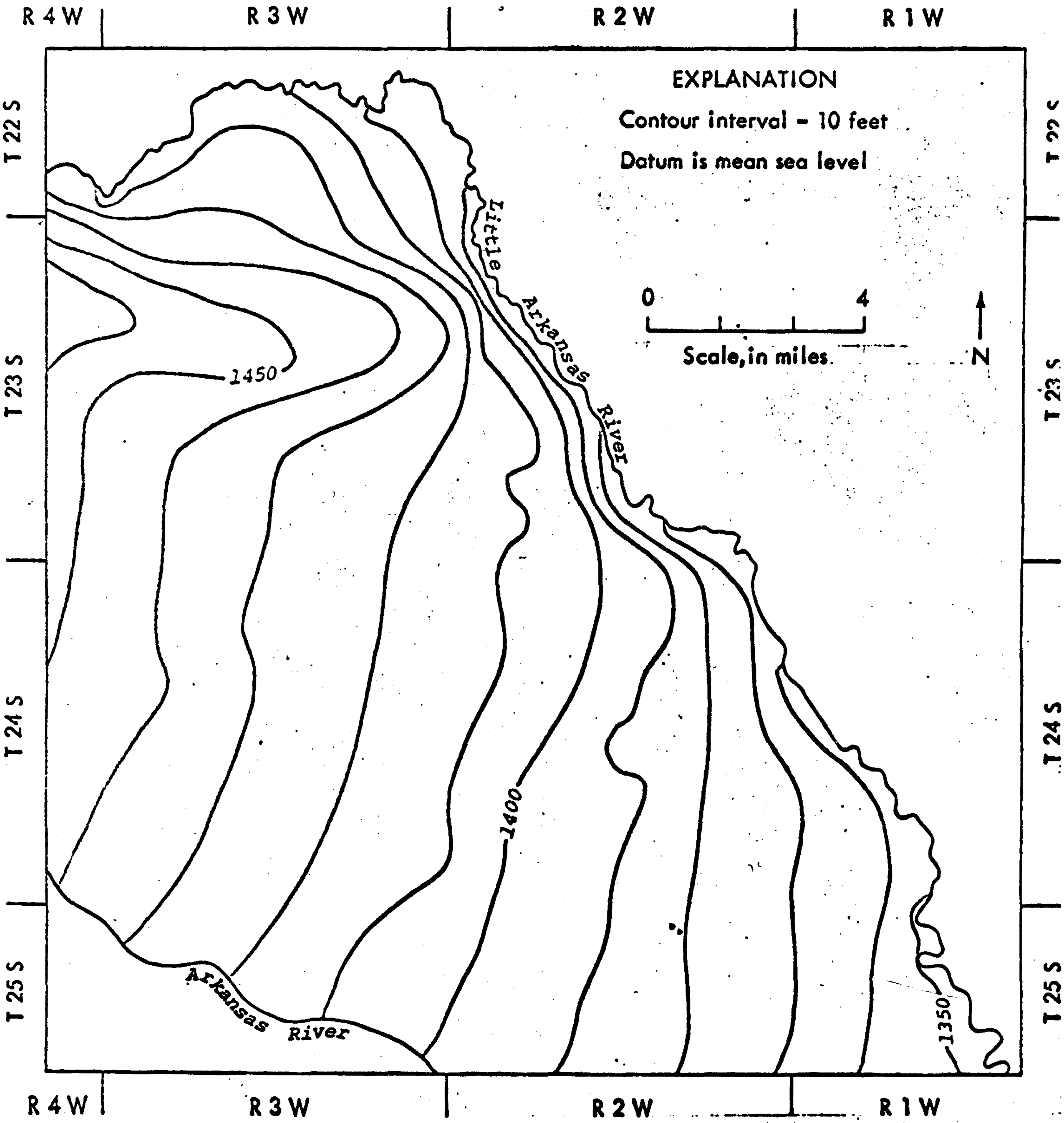

Figure 7.--Measured 1940 water levels.

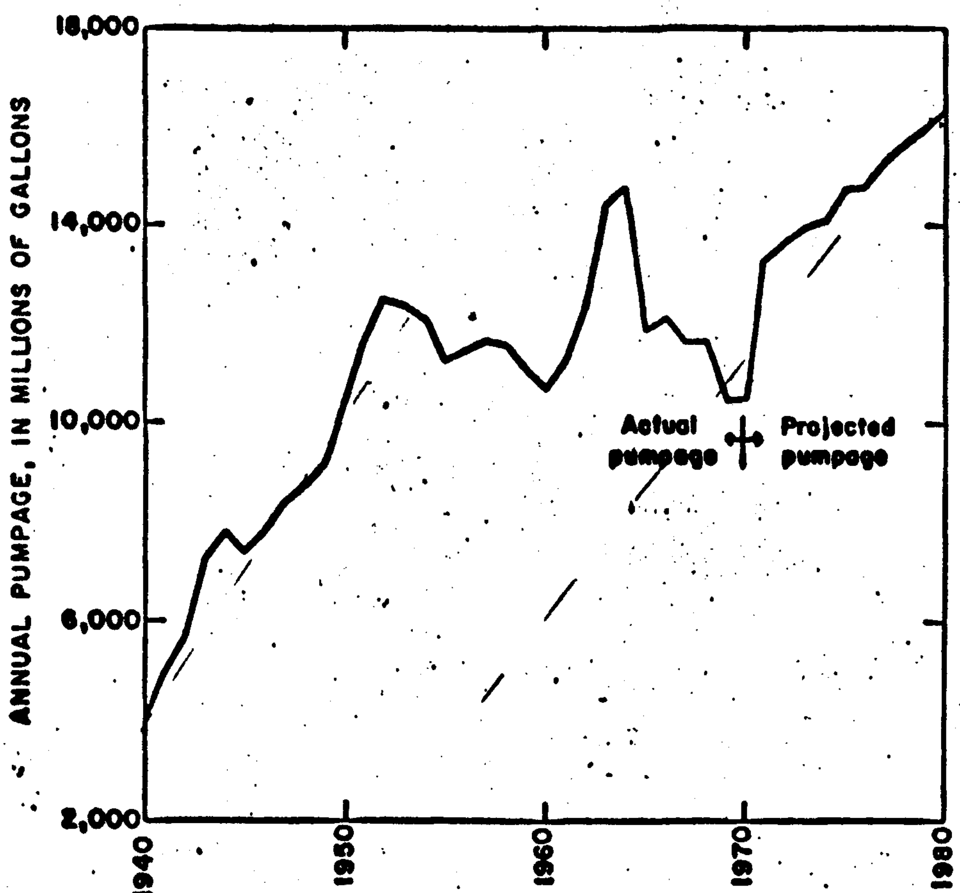

Figure 8.--Annual pumpage from Wichita well field.

Following the suggested guide defined earlier in the text, an effort was made to history match the Wichita well field using equation (2). After the best history match was obtained, all specific flux terms were set to zero and a model run equal in length of time to the history match was made. The results displayed in figure 9 indicate large areas of erroneous potential

---

Figure 9 near here.

---

change. Application of a model using equation (5) with zero specific flux terms maintain the original potential surface after a length of time equal to the history match period.

A simple plot of flux direction, that is recharge or discharge, is plotted using equation (4) (fig. 10). While this plot is highly qualative,

---

Figure 10 near here.

---

the results do check with seepage runs on the Little Arkansas River.

The configuration of the calculated 1970 head is shown in figure 11 and the measured 1970 head is shown in figure 12. It is evident by comparison

---

Figures 11 and 12 near here.

---

that calculated-head values are less than measured-head values. It can thus be expected that projections will also be on the conservative side.

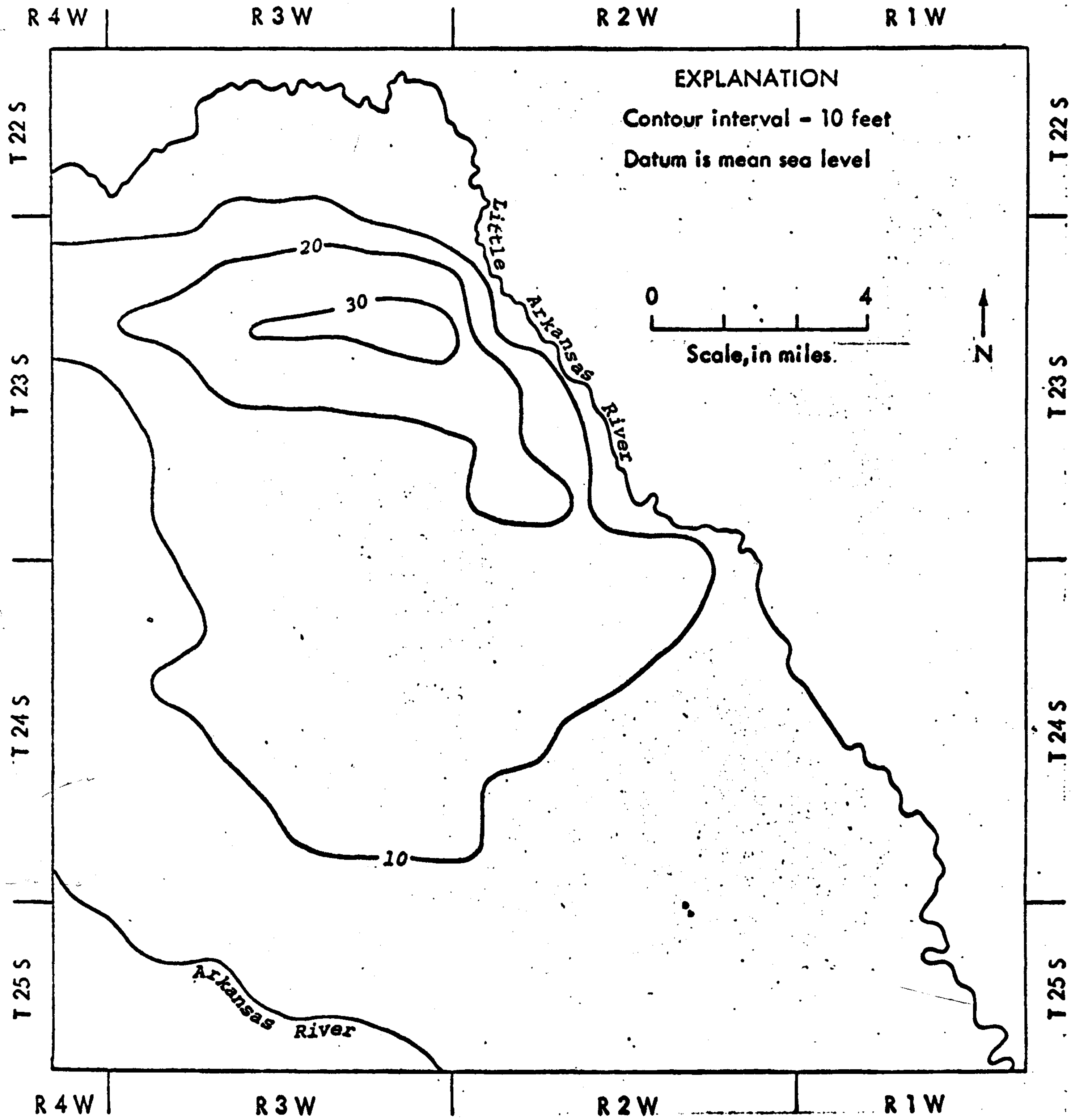

Figure 9.--Anomalous drawdown.

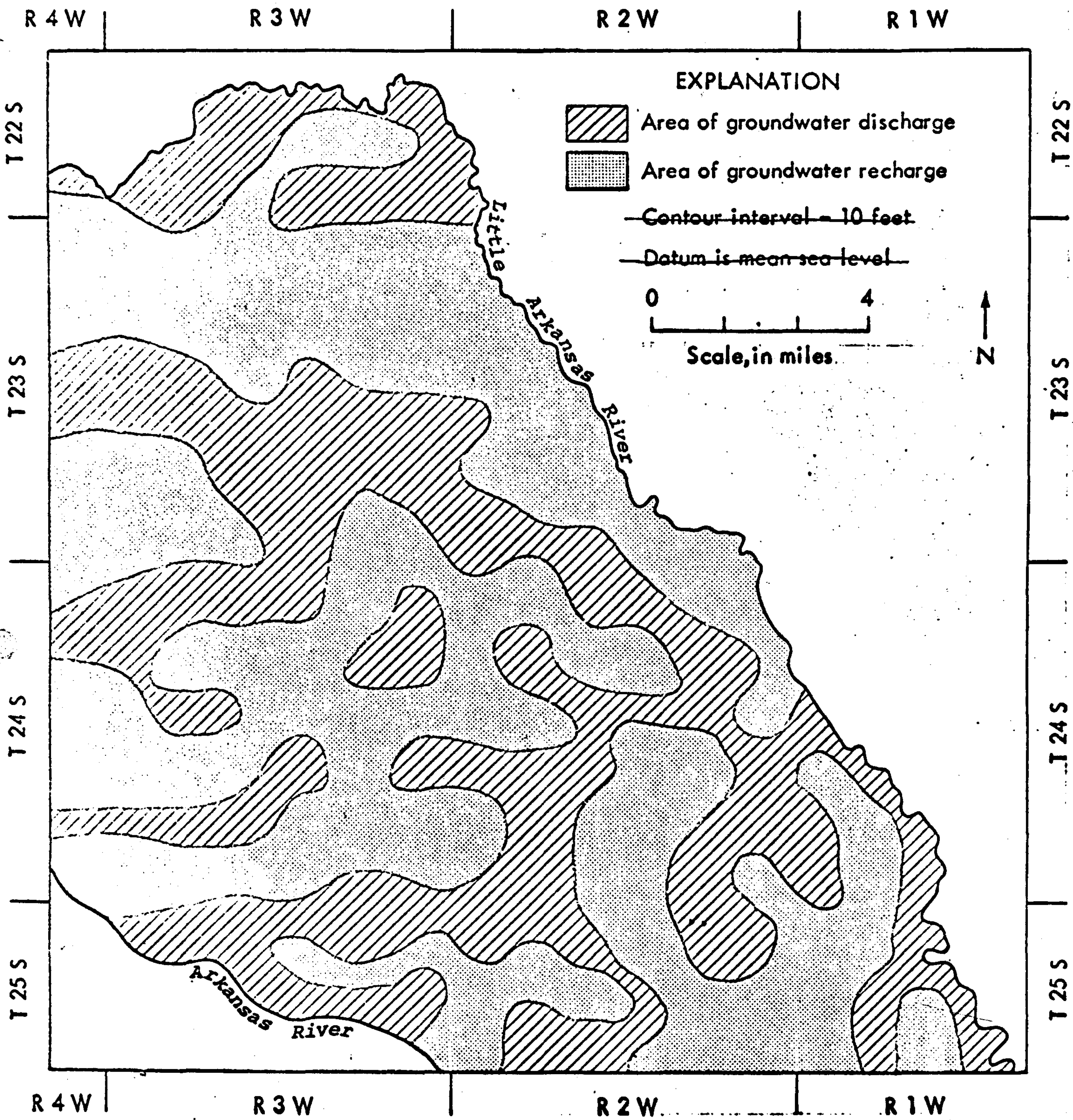

Figure ~~8~~ 10.--Calculated areas of recharge and discharge.

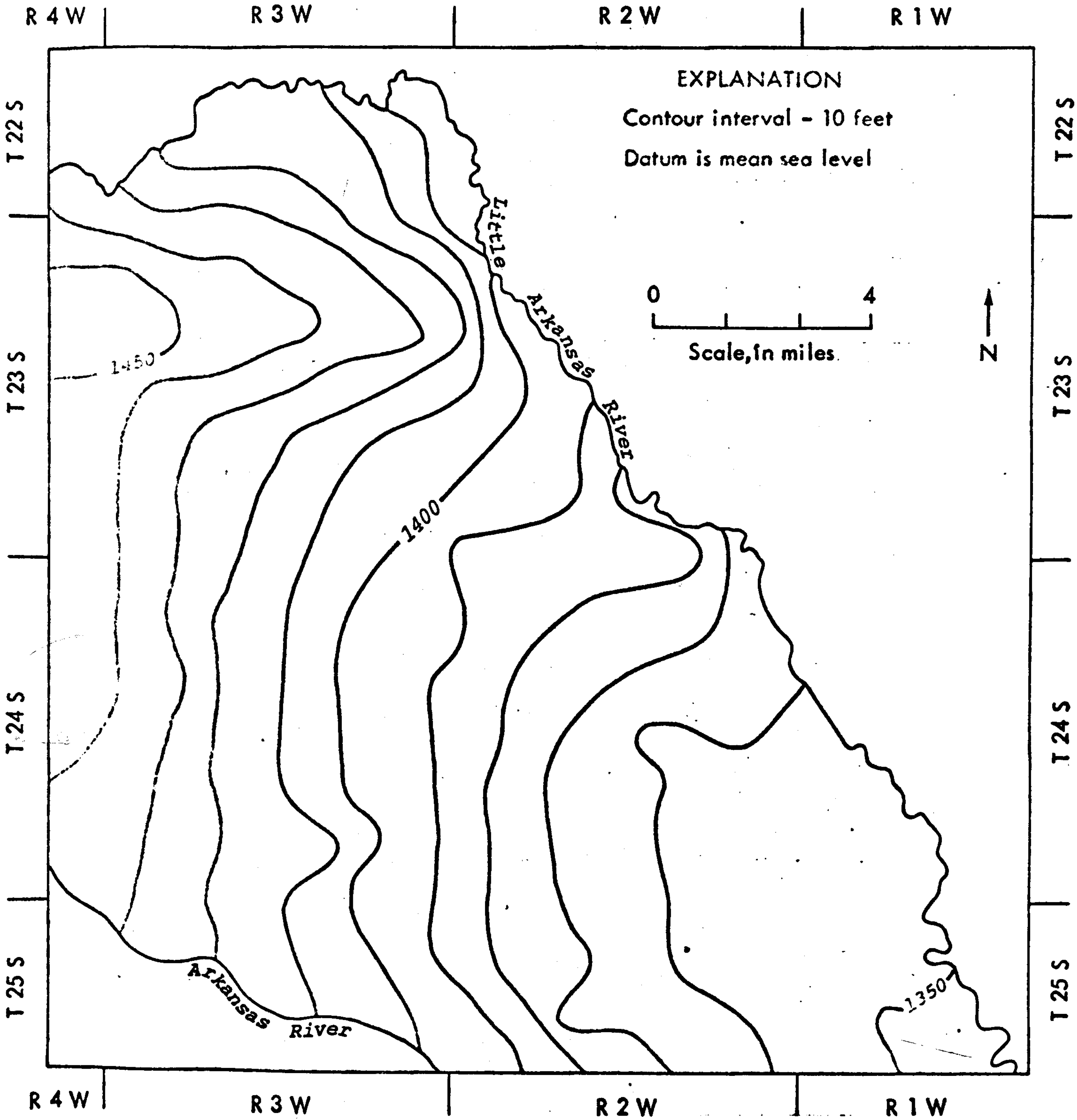

Figure 11.--Calculated 1970 head.

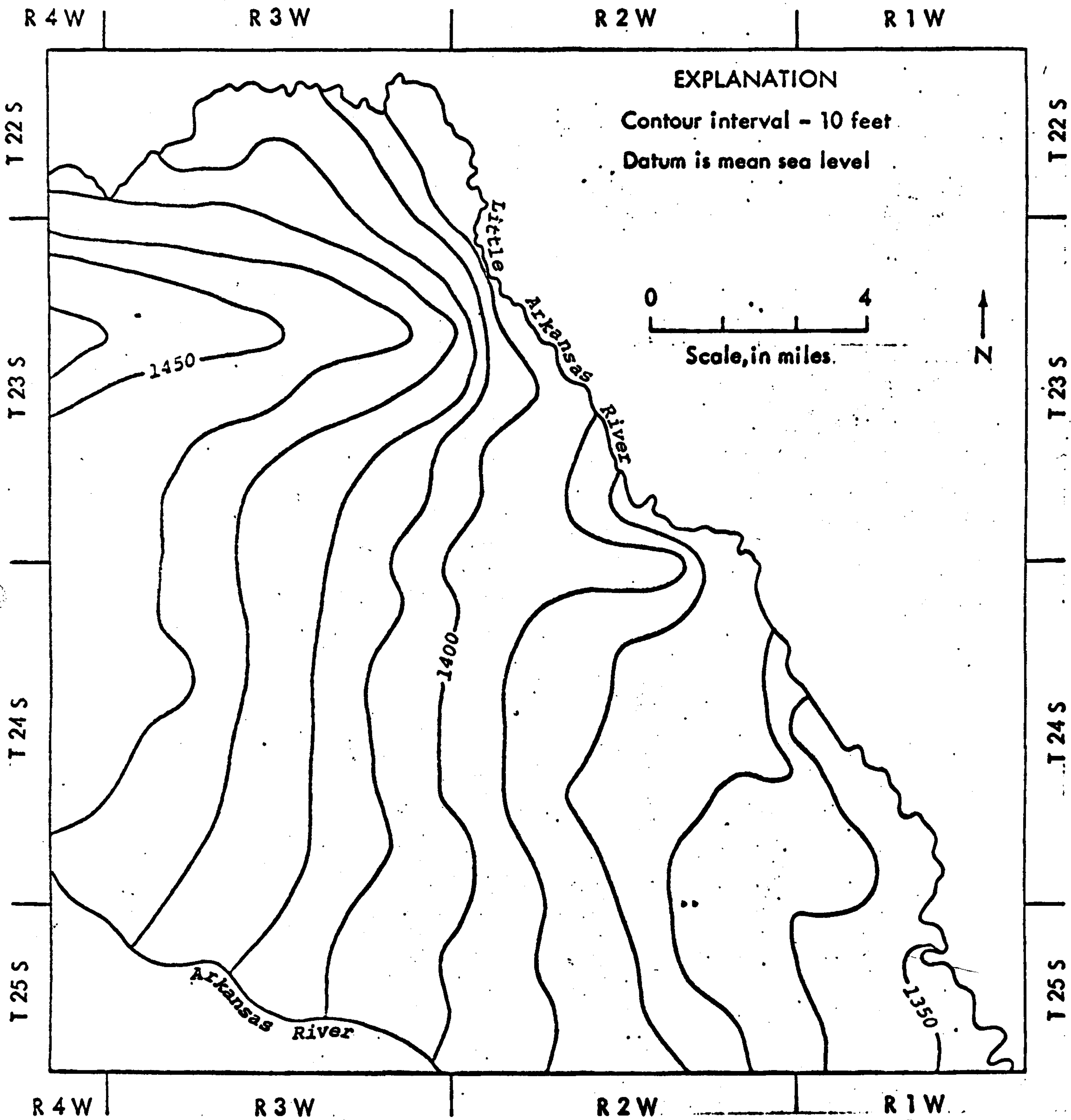

Figure ~~11~~12.--Measured 1970 head.

The head configuration projected to the year 1980 is shown in figure 13.

Figure 13 near here.

Assuming there is no radical departure from the generated discharge-recharge relationships used in the projection, the system shows only minor adverse effects through 1980.

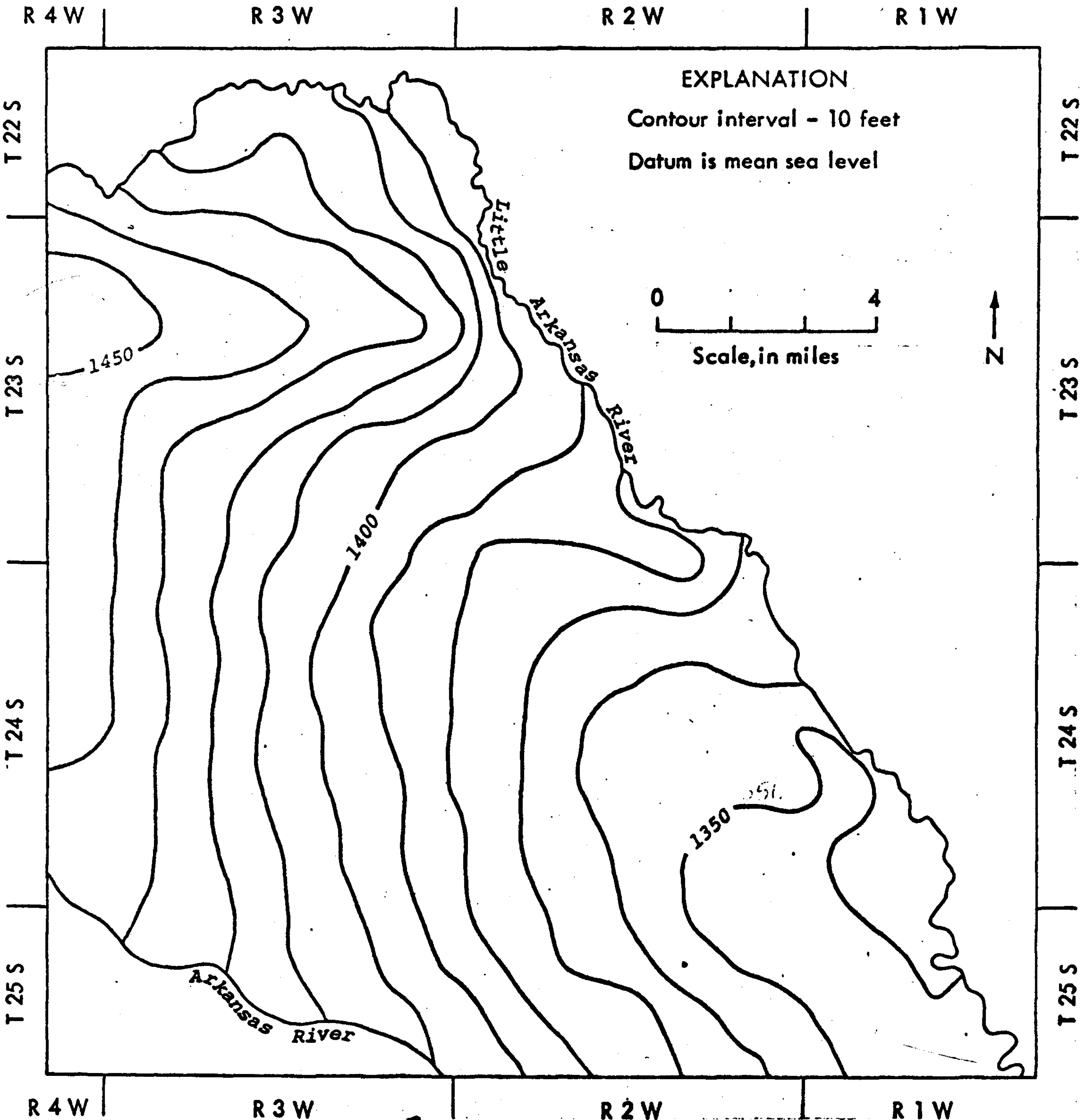

Figure 13.--Projected 1980 head.

Discussion

It appears prudent for investigators to periodically review the assumptions made in developing the two-dimensional partial-differential equation (equation 2) and the necessary constraints placed on application of said equation. A history match using equation (2), for example, implies that transmissivity, storativity, and specific flux are the only geohydraulic parameters that affect a particular potential surface. In a strict sense, if the geohydraulic conditions do not meet the constraints and assumptions of equation (2), a strict history match procedure must introduce error into the geohydraulic parameters. The error involved will appear spacially in the geohydraulic parameters, as the real system fails to meet the assumptions of equation (2) spacially. Making valid projections from a spacially erroneous data base is compounded when natural specific flux is a component in the shape of the potential surface.

The use of equation (5) does not eliminate error or natural specific flux problems from a history match situation. It does, however, allow an approximate separation which appears to enhance the ability to interpret a given projection.

Summary and Conclusions

A composite partial-differential equation is introduced that can be applied with constraints to interpret virgin or developed regional ground-water problems.

The results of ignoring natural discharge and recharge are shown by an applied example. The example of Wichita's well field is shown to be capable of delivering projected demands through 1980 without adverse effects.

Acknowledgements - The authors wish to thank the U.S. Geological Survey, the Kansas Water Resources Board and the city of Wichita for their data contributions to this study.

REFERENCES

Carnahan, B., Luther, H. A., and Wilkes, J. O., Applied numerical methods: John Wiley & Sons, Inc., New York, 1969.

Carslaw, H. S. and Jaeger, J. C., Conduction of heat in solids: Second edition, Onford Press, 1959.

Coats, K. H., Dempsey, J. R., and Henderson, J. H., A new technique for determining reservoir description from field performance data: Soc. Pet. Eng. J., vol. 10, no. 2, p. 66-74, 1970.

Emsellem, Y. and De Marsily G., An automatic solution for the inverse problem: Water Resour. Res., vol. 7, no. 5, 1971.

Giese, J. H., A bibliography for the numerical solution of partial differential equations: Ballistic Research Lab., Memorandum Report No. 2114, Aberdeen Proving Ground, Maryland, 1971.

Halepaska, J. C., Numerical analysis of air injection into an aquifer: P.H.D. Dissertation, New Mexico Institute of Mining and Technology, 1970.

Halepaska, J. C. and Hartman, F. W., Numerical solution of the 3-Dimensional heat flow equation, in Short Papers on Research in 1971: Bull. 204, Part 1.

Hantush, M. S., Hydraulics of wells in Advances in Hydroscience (Ven Te Chow, ed.): p. 281-96, Academic Press, 1964.

Hubbert, M. K., Journal of Geology: The theory of groundwater motion, vol. XLVIII, no. 8, 1940.

Jacob, C. E., Flow of ground water: Engineering Hydraulics (H. Rouse, ed.), p. 321-38, Wiley, New York, 1950.

Johns, H. O., Rapid method for obtaining a two-dimensional reservoir description from field pressure data: Soc. Pet. Eng. J., vol. 6, no. 4, p. 315-27, 1966.

Kellog, O. D., Foundations of potential theory: Dover Publications, 1953.

Kleinecke, D., Use of linear programming for estimating geohydrologic parameters for groundwater basins: Water Resour. Res., vol. 7, no. 2, 1971.

Lamb, Sir Horance, Hydrodynamics: Sixth edition, Dover Publications, 1932.

Lane, C. W. and Miller, D. E., Geohydrology of Sedgwick County, Kansas: Kansas Geol. Survey Bull. 176, 1965.

Muskat, M., The flow of homogeneous fluids through porous media: McGraw-Hill, 1937.

Nondan, B., An automatic history matching procedure for use in mathematical modeling of hydrocarbon reservoirs, M. S. Thesis, University of Kansas, 1970.

Nelson, W. R. and McCollum, W. L., Transient energy dissipation methods of measuring permeability distributions in heterogeneous porous materials, for Water Resources Division, U.S. Geological Survey, U.S.D.I. Contract no. 14-08-0001-11994, 1969.

Peaceman, D. W. and Rachford, Jr., H. H., The numerical solution of parobolic and elliptic differential equations: Jour. Soc. Indust. Appl. Math., vol. 3, no. 1, 1955.

Petri, L. R., Lane, C. W., and Furness, L. W., Water Resources of the Wichita area, Kansas: U. S. Geol. Survey Water-Supply Paper 1499-I, 1964.

Pinder, G. F. and Bredehoeft, J. D., 1968, Application of the digital computer for aquifer evaluation: Water Resour. Res., vol. 4, no. 5.

Richards, D. B. and Dunaway, T. W., Geohydrologic data for numerical modeling of ground-water-withdrawals in the Little Arkansas River Basin area, south-central Kansas, U.S. Geol. Survey open-file Report, 426 p. 1972.

Reddell, D. L., Distribution of groundwater recharge: M.S. Thesis, Colorado State University, 1967.

Smith, G. D., Numerical solution of partial differential equations: Oxford University Press, New York, 1965.

Stallman, R. W., Orientation of axes for calculating distribution of transmissibility from water-level altitudes, in Geological Research (1962) U.S. Geol. Survey Prof. Paper 450-E, p. 165-66, 1963a.

Stallman, R. W., Calculation of resistance and error in an electric analog of steady flow through nonhomogeneous aquifers: U.S. Geol. Survey Water-Supply Paper, 1544-6, p. 20, 1963b.

Von Rosenberg, D. U., Methods for the numerical solution of partial differential equations: American Elsevier Publishing Co., New York, 1969.

Westlake, J. A., A handbook of numerical matrix inversion and solution of linear equations: John Wiley, New York, 1968.

Williams, C. C. and Lohman, S. W., Geology and ground-water resources of a part of south-central Kansas: Kansas Geol. Survey Bull. 79, 1949.

Zienkiewicz, O. C., The finite element method in structural and continuum mechanics: McGraw-Hill Book Co., New York, 1967.

APPENDIX A

Numerical forms are available to solve equation (3) implicitly, explicitly, or by Crank-Nicholson differencing. Crank-Nicholson differencing is chosen here because it is unconditionally stable, fast, and second order correct in time and space.

Recalling equation (2)

$$\frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial h}{\partial y}\right) + \frac{Q}{\partial x^2} = S\frac{\partial h}{\partial t} + QQ$$

Where QQ is equation (5), equation (3) in difference form is

$$\frac{1}{\Delta x_{i,j}}\left[T_{i+\frac{1}{2},j}\frac{(h^{n+\frac{1}{2}}_{i+1,j} - h^{n+\frac{1}{2}}_{i,j})}{\Delta x_{i+\frac{1}{2},j}} - T_{i-\frac{1}{2},j}\frac{(h^{n+\frac{1}{2}}_{i,j} - h^{n+\frac{1}{2}}_{i-1,j})}{\Delta x_{i-\frac{1}{2},j}}\right] +$$

$$\frac{1}{\Delta y_{i,j}}\left[T_{i,j+\frac{1}{2}}\frac{(h^{n+\frac{1}{2}}_{i,j+1} - h^{n+\frac{1}{2}}_{i,j})}{\Delta y_{i,j+\frac{1}{2}}} - T_{i,j-\frac{1}{2}}\frac{(h^{n+\frac{1}{2}}_{i,j} - h^{n+\frac{1}{2}}_{i,j-1})}{\Delta y_{i,j-\frac{1}{2}}}\right] +$$

$$\frac{Q}{(\Delta x_{i,j})^2} = S\frac{h^{n+1}_{i,j} - h^{n}_{i,j}}{\Delta t^{n+\frac{1}{2}}} + \frac{QQ}{\Delta x^2} \tag{4a}$$

Let $h^{n+\frac{1}{2}}_{i,j} = \frac{1}{2}(h^{n+1}_{i,j} + h^{n}_{i,j})$

Equation (4) takes the form

$$\frac{1}{\Delta x_{i,j}}\left[T_{i+\frac{1}{2},j}\frac{(h^{n+1}_{i+1,j} - h^{n+1}_{i,j})}{\Delta x_{i+\frac{1}{2},j}} - T_{i-\frac{1}{2},j}\frac{(h^{n+1}_{i,j} - h^{n+1}_{i-1,j})}{\Delta x_{i-\frac{1}{2},j}}\right] +$$

$$\frac{1}{\Delta y_{i,j}}\left[T_{i,j+\frac{1}{2}}\frac{(h^{n+1}_{i,j+1} - h^{n+1}_{i,j})}{\Delta y_{i,j+\frac{1}{2}}} - T_{i,j-\frac{1}{2}}\frac{(h^{n+1}_{i,j} - h^{n+1}_{i,j-1})}{\Delta y_{i,j-\frac{1}{2}}}\right] +$$

$$G_{i,j} + \frac{2Q}{(\Delta x_{i,j})^2} = S\frac{2(h^{n+1}_{i,j} - h^{n}_{i,j})}{\Delta t} \tag{5a}$$

where

$$G_{i,j} = \frac{1}{\Delta x_{i,j}} \left[T_{i+\frac{1}{2},j} \frac{(h^n_{i+1,j} - h^n_{i,j})}{\Delta x_{i+\frac{1}{2},j}} - T_{i-\frac{1}{2},j} \frac{(h^n_{i,j} - h^n_{i-1,j})}{\Delta x_{i-\frac{1}{2},j}} \right] +$$

$$\frac{1}{\Delta y_{i,j}} \left[T_{i,j+\frac{1}{2}} \frac{(h^n_{i,j+1} - h^n_{i,j})}{\Delta y_{i,j+\frac{1}{2}}} - T_{i,j-\frac{1}{2}} \frac{(h^n_{i,j} - h^n_{i,j-1})}{\Delta y_{i,j-\frac{1}{2}}} \right] + \frac{2Q}{\Delta x^2}$$

If the system has uniform spacing

$$\Delta x_{i,j} = \Delta y_{i,j}$$

and the following coefficients can be defined

$$A_i = \frac{1}{(\Delta x_{i,j})^2} = \frac{1}{(\Delta y_{i,j})^2} \tag{6a}$$

$$TI_{i,j} = T_{i+\frac{1}{2},j} = \tfrac{1}{2}\,(T_{i,j} + T_{i+1,j}) \tag{7a}$$

$$TI_{i-1,j} = T_{i-\frac{1}{2},j} = \tfrac{1}{2}\,(T_{i,j} + T_{i-1,j}) \tag{8a}$$

$$TJ_{i,j} = T_{i,j+\frac{1}{2}} = \tfrac{1}{2}\,(T_{i,j} + T_{i,j+1}) \tag{9a}$$

$$TJ_{i,j-1} = T_{i,j-\frac{1}{2}} = \tfrac{1}{2}\,(T_{i,j} + T_{i,j-1}) \tag{10a}$$

$$E_{i,j} = \frac{2A_i}{\Delta t} \tag{11a}$$

Introducing 6a through 11a into equation 5a yields

$$TI_{i,j}\,(h^{n+1}_{i+1,j} - h^{n+1}_{i,j}) - TI_{i-1,j}\,(h^{n+1}_{i,j} - h^{n+1}_{i-1,j}) +$$

$$TJ_{i,j}\,(h^{n+1}_{i,j+1} - h^{n+1}_{i,j}) - TJ_{i,j-1}\,(h^{n+1}_{i,j} - h^{n+1}_{i,j-1}) +$$

$$A_i\,G_{i,j} + 2Q = E_i S(h^{n+1}_{i,j} - h^n_{i,j}) \tag{12a}$$

Referring to figure 14 the objective with equation (12a) is to get into a form suitable for tri-diagonal matrix solution. This is achieved by first defining the system of solution in figure 13 as left to right,

---

Figure 14 near here.

---

top to bottom. Calculating any $h^{n+1}_{i,j}$, $h^{n+1}_{i,j-1}$ is known as it was just calculated and $h^{n+1}_{i,j+1}$ is redefined as $hK^{n+1}_{i,j+1}$ and is an extrapolation value. $hK^{n+1}_{i,j+1}$ is extrapolated using the following routine

$$hK^{n+1}_{i,j+1} = h^{n}_{i,j+1} + \frac{\Delta t^{n+\frac{1}{2}}}{\Delta t^{n-\frac{1}{2}}} (h^{n}_{i,j+1} - h^{n-1}_{i,j+1}) \tag{13a}$$

Equation (12a) can then be reduced to the form

$$Ah^{n+1}_{i+1,j} + Bh^{n+1}_{i,j} + Ch^{n+1}_{i-1,j} = D \tag{14a}$$

This form is tractable via the Thomas algorithm (e.g., Westlake, 1968). Equation (12a) can be reduced to the form

$$\begin{aligned} &-h^{n+1}_{i,j} (TI_{i,j} + TI_{i-1,j} + TJ_{i,j} + TJ_{i,j-1} + E_i) + \\ &TI_{i,j}\, h^{n+1}_{i+1,j} + TI_{i-1,j}\, h^{n+1}_{i-1,j} = \\ &-TJ_{i,j}\, hK^{n+1}_{i,j+1} - TJ_{i,j-1} - A_i\, G_{i,j} - 2Q - E_i\, h^{n}_{i,j} \end{aligned} \tag{15a}$$

For simplification let all values to the right of the equals sign be $HH_{i,j}$ and introducing the Thomas algorithm that is

$$h^{n+1}_{i-1,j} = U_{i-1}\, h^{n+1}_{i,j} + V_{i-1} \tag{16a}$$

Equation (15a) reduces to the form

$$-h_{i,j}^{n+1}\left(TI_{i,j} + TI_{i-1,j} + TJ_{i,j} + TJ_{i,j-1} + E_i - TI_{i-1,j}\,U_{i-1}\right) =$$

$$H_{i,j} - TI_{i-1,j}\,V_{i-1} - TI_{i,j}\,h_{i+1,j}^{n+1} \tag{17a}$$

Allowing the coefficient of $h_{i,j}^{n+1}$ to equal DEL, equation (17a) reduces to

$$h_{i,j}^{n+1} = \frac{TI_{i-1,j}\,V_{i-1} + TI_{i,j}\,h_{i+1,j}^{n+1} - H_{i,j}}{DEL} \tag{18a}$$

Since

$$h_{i,j}^{n+1} = U_i\,h_{i+1}^{n+1} + V_i \tag{19a}$$

$$U_i = \frac{TI_{i,j}}{DEL} \tag{20a}$$

$$V_i = \frac{TI_{i-1,j}\,V_{i-1} - H_{i,j}}{DEL} \tag{21a}$$

with

$$U_1 = 0 \tag{22a}$$

$$V_1 = h_{i-1,j}^{n} \tag{23a}$$

The preceding development is unconditionally stable, fast, and has a high order of accuracy (e.g., Halepaska, 1970, 1971).