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THE IMPACT OF THE DUPUIT-FORCHHEIMER APPROXIMATION ON
SALTWATER INTRUSION SIMULATION

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Abstract

Numerical analysis is used to investigate the impact of the Dupuit-Forchheimer approximation on results of saltwater intrusion simulation. It is found that for sinks of sources (drain, ditch), that are small compared to the freshwater-zone thickness, the use of the Dupuit-Forchheimer approximation may cause substantial errors. For such cases, a correction method is proposed and verified.

Introduction

The Dupuit approximation has been used in the numerical analysis of saltwater upconing for about 20 years. Only recently the development of the Boundary Element Method (BEM) resulted in a cross sectional, sharp interface model of saltwater intrusion, which does not utilize the Dupuit-Forchheimer approximation (Liu et al., 1981; Kemblowski, 1985). This solution, although more sophisticated and accurate, unfortunately requires more data-preparation time and computer time than the models which use the Dupuit-Forchheimer approximation. Thus, it is important to establish under what circumstances the use of the Dupuit-Forchheimer approximation is acceptable and how the errors related to its use may be corrected. This analysis is performed by means of numerical simulation of two problems: 1) upconing under a sink of small size (drain), compared to the thickness of the fresh-water zone, and 2) upconing under a river whose depth is of the same order as that of the fresh-water zone.

Mathematical Statement of the Problem

We are considering flow of two fluids, assuming that a sharp interface separates them such that each fluid occupies a separated zone (Fig. 1). The shapes of these zones may vary with time because of the interface motion. Assuming that the porous medium and fluids are incompressible, we can write for each zone a continuity equation in vertical plane (x,z; Bear, 1979)

$$\frac{\partial^2 \phi^F}{\partial x^2} + \frac{\partial^2 \phi^F}{\partial z^2} = 0 \quad (1)$$

$$\frac{\partial^2 \phi^S}{\partial x^2} + \frac{\partial^2 \phi^S}{\partial z^2} = 0 \quad (2)$$

where $\phi^F = z + \frac{p}{\gamma_F}$, freshwater potential

$\phi^S = z + \frac{p}{\gamma_S}$, saltwater potential

z = elevation above datum level

p = pressure

γ_F, γ_S = specific weights of freshwater and saltwater.

In order to be able to solve equations 1 and 2, initial and boundary conditions for each fluid zone must be specified. The initial conditions are given by initial location of the interface. The boundary conditions on the boundaries excluding the interface are of the same kind as for the flow of a single fluid and may be written as follows

$$\phi(\Gamma_1) = \phi_B(\Gamma_1) \text{ (Dirichlet type, given potential)} \quad (3)$$

$$\frac{\partial \phi}{\partial n}(\Gamma_2) = -q_B(\Gamma_2) \text{ (Neumann type, given flux)} \quad (4)$$

$$\frac{\partial \phi}{\partial n}(\Gamma_3) + a\phi(\Gamma_3) = b \text{ (mixed, Robbin type, semi-impervious boundary)} \quad (5)$$

where a and b are given functions, and Γ_1 , Γ_2 , and Γ_3 are portions of the boundary Γ .

The interface boundary condition requires the continuity of pressure and the fluxes normal to the interface and may be written as

$$\phi^S = \frac{\gamma_F}{\gamma_S} \phi^F + \frac{\gamma_S - \gamma_F}{\gamma_S} \zeta \quad (6)$$

$$\frac{\partial \phi^S}{\partial n_S} = - \frac{K_F}{K_S} \frac{\partial \phi^F}{\partial n_F} \quad (7)$$

where ζ is the elevation of the interface and K_F and K_S are the hydraulic conductivities in the freshwater and saltwater zones respectively.

The solution of equations 1 and 2, subject to the initial and boundary conditions and the interface relationships, in the transient case gives nonzero normal fluxes along the interface. These fluxes cause interface motion. The interface motion may be written as (Kemblowski, 1985)

$$\frac{\partial \zeta}{\partial t} = \frac{K_F}{S_0 \cos \alpha} \frac{\partial \phi^F}{\partial n_F} \quad (8)$$

where S_0 is the effective porosity.

Equations 1 and 2 along with initial and boundary conditions and the interface motion equation complete the mathematical statement of the intrusion problem, which includes the vertical component of the flow. The numerical solution of the equations used in this paper utilizes the Boundary Element Method. The solution is described in Kemblowski (1985).

This description may be substantially simplified by assuming validity of the Dupuit-Forchheimer approximation. In this case the freshwater zone and saltwater zone continuity equations may be written for the confined aquifer as (Fig. 1)

$$\frac{\partial}{\partial x} (K_F(b-\zeta) \frac{\partial \phi^F}{\partial x}) + S_O \frac{\partial \zeta}{\partial t} = Q \quad (9)$$

$$\frac{\partial}{\partial x} (K_S(\zeta-d) \frac{\partial \phi^S}{\partial x}) + \frac{K_O}{b_O} (\phi_O^S - \phi^S) = S_O \frac{\partial \zeta}{\partial t} \quad (10)$$

where K_O is the hydraulic conductivity of the semi-confining layer, Q is the unit withdrawal, and ϕ_O^S is the constant potential in the saltwater aquifer beneath the semi-confining layer. Equations 9 and 10 form a system of coupled partial differential equations with three dependent variables, namely ϕ^F , ζ , and ϕ^S . In order to eliminate one of them (ϕ^S), we use the interface relationship given by equation 6.

For the unconfined aquifer the continuity equation for the freshwater zone may be written as

$$\frac{\partial}{\partial x} (K_F(\phi^F - \zeta) \frac{\partial \phi^F}{\partial x}) = S_O \left(\frac{\partial \phi^F}{\partial t} - \frac{\partial \zeta}{\partial t} \right) + Q \quad (11)$$

Equations 9, 10, and 11 may be approximated using the finite difference method for the spatial derivatives and the implicit finite difference scheme for the time derivative. Application of this approximation to equation 11 leads to

$$\begin{aligned} & (1-w) [Y_{J,J-1}^k \{ (\phi_{J-1}^F)^k - (\phi_J^F)^k \} + Y_{J,J+1}^k \{ (\phi_{J+1}^F)^k - (\phi_J^F)^k \}] \\ & + w [Y_{J,J-1}^{k+1} \{ (\phi_{J-1}^F)^{k+1} - (\phi_J^F)^{k+1} \} + Y_{J,J+1}^{k+1} \\ & \quad \{ (\phi_{J+1}^F)^{k+1} - (\phi_J^F)^{k+1} \}] \\ & = \frac{S_O \Delta x}{\Delta t} [\{ (\phi_J^F)^{k+1} - (\phi_J^F)^k \} - \{ (\zeta_J)^{k+1} - (\zeta_J)^k \}] + Q_J \end{aligned} \quad (12)$$

where: $Y_{J,J-1}$ = transmissibility of connection between nodes J and $J-1$ (Fig.

3).

$$y_{J,J-1}^k = \frac{\{(\phi_J^F)^k + (\phi_{J-1}^F)^k - (\zeta_J)^k - (\zeta_{J-1})^k\} K_F}{2\Delta x} \quad (13)$$

w = weighting factor

k = time step index

Δx = spatial step

Δt = time step

J = index of finite-difference node.

The finite difference approximation of equations 9 and 10 leads to expressions similar to equation 12.

After formulating the finite difference equation for each node in the freshwater and saltwater zone we obtain a system of quasilinear equations. The Gauss-Seidel method was used in this paper to solve the resulting system of equations. The solution for each time step gives spatial and temporal distribution of ϕ^F and ζ (ϕ^S may be calculated from equation 6).

Upconing Under a Drain

The hydrogeologic situation of this problem is depicted in Fig. 2. A semicircular drain ($r_0 = .05$ m) is located at the top of a confined aquifer occupied by fresh ($\gamma_F = 1.00$) and salt ($\gamma_S = 1.03$) waters. Because of the symmetry of the problem we consider only half of the region. The hydraulic conductivities are $K_F = 100$ m/day and $K_S = 103$ m/day. The boundary conditions of the problem are depicted in Fig. 2; the potentials ϕ^F and ϕ^S on the right hand side are equal to zero, and the potential at the drain, ϕ_D^F , was varied from one experiment to another.

The steady-state locations of the interface obtained by the "non-Dupuit" and "Dupuit" models for two different values of the potential at the

drain ($\phi_D^F = -.175$ m and $\phi_D^F = -.36$ m) are shown in Figs. 4 and 5. Clearly, the results obtained by the two models are quite different. This difference is caused by the potential loss between the interface and the sink, which is due to the partial penetration of the sink and is not accounted for by the "Dupuit" model. According to this model the interface elevation under the drain is given by (Bear, 1979)

$$\zeta_1 = \frac{-\gamma_F}{\gamma_S - \gamma_F} \phi_D^F \quad (14)$$

whereas the "non-Dupuit" model gives

$$\zeta_2 = \frac{-\gamma_F}{\gamma_S - \gamma_F} (\phi_D^F - \Delta\phi^F) \quad (15)$$

where $\Delta\phi^F$ is the additional potential loss between the interface and the drain. Notice that in this case both ϕ_D^F and $\Delta\phi^F$ are negative, and therefore $\zeta_2 < \zeta_1$ (Figs. 4 and 5). In addition to the error in estimating the location of the interface, the "Dupuit" model gives erroneous results with regard to the drain's withdrawal rate Q_O . This rate for the first case ($\phi_D^F = -.175$) was calculated to be $Q_O = 6.3$ m²/day from the "non-Dupuit" model and $Q_O = 12.2$ m²/day from the other model.

In order to correct this discrepancy we use an analytic solution for calculating the additional potential loss due to the partial penetration. For this particular geometry the analytic solution is given by (Huisman, 1972)

$$\Delta\phi^F = \frac{Q_O}{\pi K_F} \ln \left(\frac{b-\zeta}{\pi r_O} \right) \quad (16)$$

where r_O is the drain radius. Simulation of $\Delta\phi^F$ is provided by adding an extra node (J*) to the drain node (J; Fig. 3). The transmissibility of the

connection between node J and J*, which represents the ratio of the flow through the connection (Q_O) to the difference of potential between the two nodes, may be obtained

$$Y_{J,J^*} = \frac{Q_O}{\Delta\phi_F} = \pi K_F \left\{ \ln \left(\frac{b-z}{\pi r_O} \right) \right\}^{-1}. \quad (17)$$

The contribution of this connection may be easily added to equation 12 written for the "drain node." Notice that the boundary condition ϕ_D^F is now given at the additional node.

The results obtained by this correction method are presented in Figs. 4, 5, and 6 and are in good agreement with the one obtained by the "non-Dupuit" model. A small difference between the results just under the drain may be explained by the fact that the correction method assumes that the total difference of potential between the interface and the drain is due only to the partial penetration ($\Delta\phi^F$), whereas in fact this difference is slightly bigger due to the curvature of the equipotential lines in the vicinity of the drain (Fig. 6). The best agreement between the two solutions should be expected for the location of the drain in the middle of the freshwater zone. The flow rates obtained by this method were correct.

Upconing Under a River

The second problem is concerned with saltwater upconing under a river, which flows through an unconfined freshwater ($\gamma_F = 1.00$) alluvial aquifer (Fig. 7) underlain by a semi-impervious layer of conductivity K_O and thickness b_O , which separates the alluvial aquifer from a deep saltwater aquifer whose constant saltwater potential ϕ_O^S is equal to 49.56 m (Kemblowski, 1984). The hydraulic conductivities K_F and K_S are 2.00 m/day and 2.06 m/day respectively. The saltwater ($\gamma_S = 1.03$) flows through the semi-confining layer and forms

upconing under the river. The geometry and boundary conditions of a simplified hydrologic scheme are shown in Fig. 8. The location of the interface and the free surface, obtained by the "Dupuit" and "non-Dupuit" models for the rate of effective recharge Q equal to 2.6×10^{-3} m/day, are shown in Fig. 9. The results show good agreement, and apparently no correction for the Dupuit approximation is required. For the same hydrologic conditions a transient simulation was performed. The initial conditions were those obtained for the steady-state and $Q = 2.0 \times 10^{-3}$ m/day. Then the recharge was increased to $Q = 4.0 \times 10^{-3}$ m/day, and the behavior of the system was simulated for various ratios of K_0/b_0 . The rise of the interface crest versus time, calculated by both methods, is shown in Fig. 10. The methods produced close results, which may be attributed to the fact that the river is almost fully penetrating.

Conclusions

Investigation of the impact of the Dupuit-Forchheimer approximation on salt-water upconing simulation is presented. The results indicate that as long as the sinks or sources are of a vertical size comparable with the thickness of the fresh-water zone, the location of the interface calculated by the "Dupuit" model gives satisfactory results, even through a strong vertical flow obviously occurs in some regions (above the semi-confining layer, below the free surface, along the river valley boundaries, and under the river). However, when the size of the sink is small compared to the fresh-water-zone thickness, the use of the Dupuit approximation may cause substantial errors in the estimation of the interface location.

For one case of practical interest (a drain located at the top of an aquifer), a correction method was proposed and verified. The same correction

method may be used to simulate partially penetrating rivers, ditches, and skimming wells. The analytical solutions for the additional loss of the potential $\Delta\phi^F$ (Eq. 16) may be found in Huisman (1972). This procedure may be used in two-dimensional, confined or unconfined, horizontal-flow models.

Another conclusion that may be derived from the analysis of the numerical results is that the Ghyben-Herzberg approximation which assumes constant potential in the whole saltwater zone (and therefore no flow there) would lead to erroneous results for the upconing under a river problem. If this approximation were used, then both $\phi^F(\phi_R^F)$ and $\phi^S(\phi_O^S)$ would be kept constant in the river node, and no rise of the interface crest due to increased effective recharge would occur (see eq. 6). Therefore the Ghyben-Herzberg approximation should be used only after careful analysis of the simulated system; whenever the flow in the saltwater zone may be of some importance, this approximation should not be applied.

Acknowledgments

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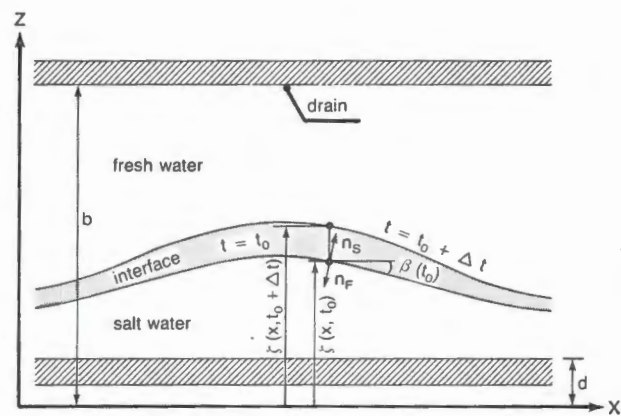
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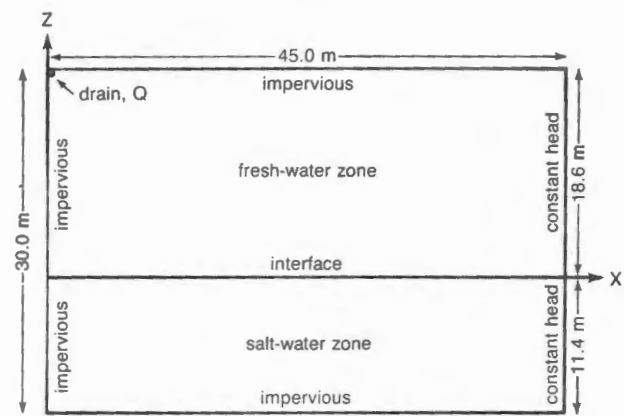
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Illustrations

1. Scheme of saltwater upconing.
2. Scheme of hypothetical model.
3. Numerical network in freshwater zone; A) without correction for partial penetration, B) with correction.
4. Location of steady-state interface, $\phi_D = -.175$ m.
5. Location of steady-state interface, $\phi_D = -.36$ m.
6. Interface location and freshwater flow net, $\phi_D = -.564$ m.
7. Hydrogeologic scheme of saltwater upconing under a river.
8. Geometry and boundary conditions of upconing under a river.
9. Steady-state location of free surface and interface.
10. Rise of interface under a river.

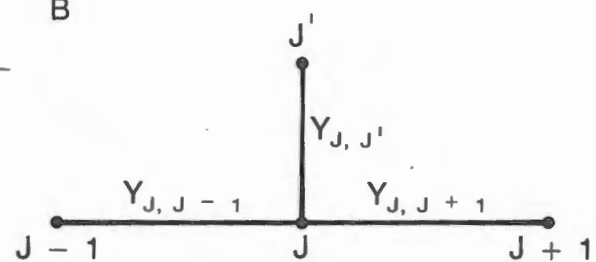




A



B



B.

