Mathematical Derivations of Semianalytical Solutions for Pumping-Induced Drawdown and Stream Depletion in a Leaky Aquifer System

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KANSAS GEOLOGICAL SURVEY OPEN-FILE REPORT #2005-10

April 26, 2005
I. Introduction

In this report, derivations of the transform-space solutions to the mathematical model describing the drawdown and stream depletion produced by a pumping well in a leaky aquifer system are presented. The back transforms of these expressions are used by Butler et al. [in review] to develop new insights into stream-aquifer interactions in a leaky aquifer system.

In Section II of this report, governing equations and auxiliary conditions for a leaky aquifer system hydraulically connected to a stream are presented. The corresponding equations in Laplace-Fourier space are derived in Sections III and IV using standard integral transform methods. The general transform-space solution is presented in Section V. In Sections VI and VII, simplified solutions for an unbounded domain and an unbounded homogeneous domain, respectively, are presented. The derivation of a solution for the stream-depletion rate is presented in Section VIII. The methods used to numerically invert the transform-space solutions are described briefly in Section IX.
II. Governing Equations

The problem of interest here is that of the drawdown and stream depletion produced by pumping from a fully penetrating well in the leaky aquifer system of Figure 1. Following the approach of Butler et al. [2001], vertical flow within the upper aquifer is neglected (Dupuit assumptions). The stream and upper aquifer are separated by a zone of relatively low hydraulic conductivity, which is represented mathematically as an incompressible layer (Hantush [1965]). Portions of the upper aquifer underneath the stream are confined, but can be confined or unconfined elsewhere. Flow in the aquitard is incorporated using the model of Hantush [1960], which includes aquitard storage but neglects lateral flow. Similar to Hantush and Jacob [1955], the underlying (lower) aquifer is assumed to be a unit of relatively high permeability so that heads within that aquifer are unaffected by pumping in the upper aquifer. Hydraulic properties are assumed to be a function of \( x \), but can be linearized into a series of zones of uniform properties that are arranged parallel to the stream. Any number of zones can be considered in this derivation but three are used in Butler et al. [in review].

Following the approach of Butler et al. [2001], governing equations and auxiliary conditions can be defined for the leaky aquifer system of Fig. 1.

Aquifer Flow

\[
\begin{align*}
\frac{\partial^2 s(x, y, t)}{\partial x^2} &+ \frac{\partial^2 s(x, y, t)}{\partial y^2} - \frac{k_{sb}}{b_{ab} T(x)} s(x, y, t) (H(x - x_{sl}) - H(x - x_{sr})) \\
- \frac{k_c(x) \partial s_c(x, y, z, t)}{T(x) \partial z} \bigg|_{z=0} &+ \frac{Q}{T(x)} \delta(x - x_p) \delta(y) \\
= \frac{S(x) \partial s(x, y, t)}{T(x) \partial t} &\quad x_{lb} < x < x_{rb}, \quad -\infty < y < \infty, \quad t > 0
\end{align*}
\]  

(1)

Note that Eq. (1) is a condensed but generalized form of Eq. (1)-(3) in Butler et al. [in review].

Initial condition,

\[ s(x, y, t = 0) = 0 \]  

(2)

Left boundary condition in \( x \) can be either constant head (Dirichlet condition),

\[ s(x, y, t) \bigg|_{x=x_{lb}} = 0 \]  

(3)

or no flow (Neummann condition),

\[ \frac{\partial s(x, y, t)}{\partial x} \bigg|_{x=x_{lb}} = 0 \]  

(4)
Similarly, right boundary condition in $x$ can be,

$$s(x, y, t) \bigg|_{x=x_{rb}} = 0$$  (5)

or

$$\frac{\partial s(x, y, t)}{\partial x} \bigg|_{x=x_{rb}} = 0$$  (6)

Drawdown at infinity in $y$ is bounded,

$$s(x, y = -\infty, t) < \infty$$  (7)

$$s(x, y = \infty, t) < \infty$$  (8)

A Cauchy boundary condition could also be readily incorporated into this development. However, for large $x_{lb}$ and $x_{rb}$, the solution is not sensitive to the form of the lateral boundary conditions.

Aquitard

$$\frac{\partial^2 s_c(x, y, z, t)}{\partial z^2} = \frac{S_{Sc}(x)}{k_c(x)} \frac{\partial s_c(x, y, z, t)}{\partial t}$$

$$x_{lb} < x < x_{rb}, \quad -\infty < y < \infty, \quad -b_c < z < 0, \quad t > 0$$  (9)

Eq. (9) is equivalent to Eq. (4) in Butler et al. [in review].

Initial condition,

$$s_c(x, y, z, t = 0) = 0$$  (10)

Continuity condition at interface of aquitard and upper aquifer,

$$s_c(x, y, z = 0, t) = s(x, y, t)$$  (11)

Constant head at base of aquitard,

$$s_c(x, y, z = -b_c, t) = 0$$  (12)

where

$x, y =$ Cartesian coordinates in lateral plane. The origin of the $x$-axis can be any arbitrary location (e.g., origin defined at right bank of stream in Butler et al. [in review]) and the values increase from left to right. The origin of the $y$ axis is at the pumping well and the values increase upward, $[L]$;

$z =$ vertical distance from bottom of upper aquifer, $[L]$;

$t =$ time, $[T]$;
\( s(x, y, t) \) = drawdown in the upper aquifer, \([L]\);
\( T(x) \) = transmissivity of the upper aquifer, \([L]\);
\( S(x) \) = specific yield or storativity of the upper aquifer, \([1]\);
\( k_{sb} \) = hydraulic conductivity of streambed, \([L/T]\);
\( b_{sb} \) = streambed thickness, \([L]\);
\( H(x - x_{sl}) \) = Heaviside function (= 0 for \( x - x_{sl} < 0 \), = 1 for \( x - x_{sl} > 0 \)), respectively;
\( x_{sl}, x_{sr} \) = left and right boundary of the stream, respectively \([L]\);
\( s_c(x, y, z, t) \) = drawdown in the aquitard, \([L]\);
\( S_{Sc}(x) \) = specific storage of the aquitard, \([L^{-1}]\);
\( k_c(x) \) = hydraulic conductivity of aquitard, \([L/T]\);
\( b_c(x) \) = thickness of aquitard, \([L]\);
\( x_{lb}, x_{rb} \) = left and right boundary of the aquifer, respectively, \([L]\);
\( x_p \) = \( x \) coordinate of pumping well, \([L]\);
\( Q \) = pumping rate from well located at \((x_p, 0)\), \([L^3/T]\).

A constant rate of pumping is assumed for this development. A variable rate of pumping or a cyclic pumping strategy could be readily incorporated using standard convolution approaches (Wallace et al. [1990])

Notation used in this report is the same as that used in the Mathematica package developed for this project and may differ from that used in Butler et al. [in review] because of the notation rules for Mathematica and the more generalized form of this development.
Figure 1: Schematic (a) cross-sectional and (b) areal views of the stream-aquifer system considered in this paper (notation explained in text; stream depletion in this configuration consists of vertical leakage across the low-permeability streambed).
III. Laplace Space Equations

Applying a Laplace transform in $t$ to the equations of the previous section yields

$$\frac{\partial^2 \bar{s}(x,y)}{\partial x^2} + \frac{\partial^2 \bar{s}(x,y)}{\partial y^2} - \frac{k_{sb}}{b_{sb} T(x)} \bar{s}(x,y) (H(x-x_{sl}) - H(x-x_{sr}))$$

$$- \frac{k_c(x) \partial \bar{s}_c(x,y,z)}{T(x) \partial z} \bigg|_{z=0} + \frac{Q}{p T(x)} \delta(x-x_p) \delta(y)$$

$$= \frac{p S(x)}{T(x)} \bar{s}(x,y) \quad x_{lb} < x < x_{rb}, \quad -\infty < y < \infty$$

(13)

$$\bar{s}(x,y) \bigg|_{x=x_{lb}} = 0 \quad \text{or} \quad \frac{\partial \bar{s}(x,y)}{\partial x} \bigg|_{x=x_{lb}} = 0$$

(14)

$$\bar{s}(x,y) \bigg|_{x=x_{rb}} = 0 \quad \text{or} \quad \frac{\partial \bar{s}(x,y)}{\partial x} \bigg|_{x=x_{rb}} = 0$$

(15)

$$\bar{s}(x,y = -\infty) < \infty \quad \text{and} \quad \bar{s}(x,y = \infty) < \infty$$

(16)

$$\frac{\partial^2 \bar{s}_c(x,y,z)}{\partial z^2} = p \frac{S_c(x)}{k_c(x)} \bar{s}_c(x,y,z)$$

$$x_{lb} < x < x_{rb}, \quad -\infty < y < \infty, \quad -b_c < z < 0$$

(17)

$$\bar{s}_c(x,y,z = 0) = \bar{s}(x,y)$$

(18)

$$\bar{s}_c(x,y,z = -b_c) = 0$$

(19)

where $\bar{s}$ and $\bar{s}_c$ are the Laplace transform of $s$ and $s_c$, respectively, and $p$ is the Laplace transform parameter. Note the overbar is used to indicate the dependence of $s$ and $s_c$ on $p$.

After applying the boundary conditions, the general solution for $s_c$ at any $z$ and its first derivative at $z = 0$ are, respectively,

$$\bar{s}_c(x,y,z) = \cosh \left[ \frac{S_c(x)p}{\sqrt{k_c(x)}} z \right] \bar{s}(x,y)$$

$$+ \coth \left[ \frac{S_c(x)p}{\sqrt{k_c(x)}} \right] \sinh \left[ \frac{S_c(x)p}{\sqrt{k_c(x)}} z \right] \bar{s}(x,y)$$

(20)

and
\[ \frac{\partial \sigma_c(x, y, z)}{\partial z} \bigg|_{z=0} = \sqrt{\frac{S_c(x)p}{k_c(x)}} \coth \left[ \sqrt{\frac{S_c(x)p}{k_c(x)}} \right] \overline{s}(x, y) \]  

Substituting the above equation into equation (13) produces,

\[
\frac{\partial^2 \overline{s}(x, y)}{\partial x^2} + \frac{\partial^2 \overline{s}(x, y)}{\partial y^2} - \frac{k_{sb}}{b_{sb} T(x)} [H(x - x_{sl}) - H(x - x_{sr})] \overline{s}(x, y)
\]

\[- \frac{1}{T(x)} \sqrt{\frac{S_c(x)p}{k_c(x)}} \coth \left[ \sqrt{\frac{S_c(x)p}{k_c(x)}} \right] \overline{s}(x, y) + \frac{Q}{pT(x)} \delta(x - x_p) \delta(y) = p \frac{S(x)}{T(x)} \overline{s}(x, y) \quad x_{lb} < x < x_{rb}, \quad -\infty < y < \infty \]  

(22)
IV. Fourier-Laplace Space Equations

Applying a Fourier transform with respect to $y$ to (22) and (14)-(16) produces:

\[
\frac{d^2 \tilde{s}(x)}{dx^2} - \omega^2 \tilde{s}(x) - \frac{k_{sb}}{b_{sb}T(x)} [H(x - x_{sl}) - H(x - x_{sr})] \tilde{s}(x)
\]
\[
- \frac{1}{T(x)} \sqrt{S_{sc}(x)p} \frac{S_{sc}(x)p}{k_c(x)} \coth \left( \sqrt{\frac{S_{sc}(x)p}{k_c(x)}} \right) \tilde{s}(x) + \frac{Q}{\sqrt{2\pi pT(x)}} \delta(x - x_p) = 0
\]

where $\tilde{s}$ is the Fourier-Laplace transform of $s$, and $\omega$ is the Fourier transform variable.

Note the double overbar is used to indicate the dependence of $s$ on $p$ and $\omega$.

By defining,

\[
\beta^2 = \omega^2 + \frac{k_{sb}}{b_{sb}T(x)} [H(x - x_{sl}) - H(x - x_{sr})]
\]
\[
+ \frac{1}{T(x)} \sqrt{S_{sc}(x)p} \frac{S_{sc}(x)p}{k_c(x)} \coth \left( \sqrt{\frac{S_{sc}(x)p}{k_c(x)}} \right) + p \frac{S(x)}{T(x)}
\]

The above equation is simplified to,

\[
\frac{d^2 \tilde{s}(x)}{dx^2} - \beta^2 \tilde{s}(x) + \frac{Q}{\sqrt{2\pi pT(x)}} \delta(x - x_p) = 0 \quad x_{lb} < x < x_{rb}
\]
V. Transform-Space Solution

Dividing the aquifer into \( n \) zones of homogeneous properties in the \( x \) direction enables Eq. (27) to be written for each zone \( i \) as

\[
\frac{d^2 \tilde{s}_i(x)}{dx^2} - \beta_i^2 \tilde{s}_i(x) + \frac{Q}{\sqrt{2\pi pT(x)}} \delta(x - x_p) = 0 \quad x_i < x < x_{i+1}, \quad i = 1, \ldots n \tag{28}
\]

Because \( \delta(x - x_p) \) vanishes when \( x \) is not equal to \( x_p \), the general solution for the above equation is,

\[
\tilde{s}_i(x) = c_{2i-1} \sinh(\beta_ix) + c_{2i} \cosh(\beta_ix) \quad x_i < x < x_{i+1}, \quad i = 1, \ldots n \tag{29}
\]

for bounded zones or

\[
\tilde{s}_i(x) = c_{2i-1} e^{\beta_ix} + c_{2i} e^{-\beta_ix} \quad x_1 \to -\infty, \quad x_{n+1} \to \infty \tag{30}
\]

for unbounded zones. In all cases, the constant coefficients \( c_i \) are determined from the boundary conditions.

For external lateral boundary conditions, Eq. (24) and (25) are applied to \( x_1 \) at the left boundary and to \( x_{n+1} \) at the right boundary, respectively. For inter-boundary conditions at \( x_i \) (\( i = 2, \ldots n \)), integrating Eq. (28) with respect to \( x \) across narrow regions containing \( x_i \) (\( i = 2, \ldots n \)) reveals that \( \tilde{s}_i(x) \) and its first derivative are continuous at all \( x_i \) (\( i = 2, \ldots n \)) except at \( x_p \) (\( i = i_p \)) where the first derivative has a discontinuity because of the singular point at the pumping well. The boundary conditions can thus be written as,

\[
\text{Dirichlet: } \tilde{s}_i(x_i) \quad \text{Neumann: } \left. \frac{d\tilde{s}_i(x)}{dx} \right|_{x=x_i} = 0, \quad i = 1
\]

\[
\left( \tilde{s}_i(x_{i+1}) - \tilde{s}_{i+1}(x_{i+1}) \right) = 0, \quad i = 1, \ldots, n - 1
\]

\[
\frac{d \left( \tilde{s}_i(x) - \tilde{s}_{i+1}(x) \right)}{dx} \bigg|_{x=x_{i+1}} = \begin{cases} \frac{Q}{\sqrt{2\pi pT(x)}}, & i = i_p - 1 \\ 0, & all \ other \ i = 1, \ldots, n - 1 \end{cases}
\]

\[
\text{Dirichlet: } \tilde{s}_i(x_{i+1}) \quad \text{Neumann: } \left. \frac{d\tilde{s}_i(x)}{dx} \right|_{x=x_{i+1}} = 0, \quad i = n \tag{31}
\]

Substituting Eq. (29) into Eq. (31) produces,
Dirichlet: \( \sinh(\beta_i x_i) c_{2i-1} + \cosh(\beta_i x_i) c_{2i} \) 
Neumann: \( \cosh(\beta_i x_i) c_{2i-1} + \sinh(\beta_i x_i) c_{2i} \)

\[ \begin{align*}
- \sinh(\beta_{i+1} x_{i+1}) c_{2i+1} - \cosh(\beta_{i+1} x_{i+1}) c_{2i+2} &= 0, \quad i = 1, \ldots, n - 1 \\
\beta_i \cosh(\beta_i x_{i+1}) c_{2i-1} + \beta_i \sinh(\beta_i x_{i+1}) c_{2i} - \beta_{i+1} \cosh(\beta_{i+1} x_{i+1}) c_{2i+1} &
\end{align*} \]

\[ - \beta_{i+1} \sinh(\beta_{i+1} x_{i+1}) c_{2i+2} = \left\{ \begin{array}{ll}
\frac{Q}{\sqrt{2\pi pT(x)}} & i = i_p - 1 \\
0 & all \ other \ i = 1, \ldots, n - 1
\end{array} \right. \]

Dirichlet: \( \sinh(\beta_i x_{i+1}) c_{2i-1} + \cosh(\beta_i x_{i+1}) c_{2i} \) 
Neumann: \( \cosh(\beta_i x_{i+1}) c_{2i-1} + \sinh(\beta_i x_{i+1}) c_{2i} \)

\[ = 0, \quad i = n \] (32)

By defining

\[ \sinh(\beta_i x_j) = a_{i,j}, \quad \cosh(\beta_i x_j) = b_{i,j}, \quad a_{i,i} = a_i, \quad b_{i,i} = b_i \] (33)

Eq. (32) can be rewritten in a \( 2n \times 2n \) matrix format, as shown in the following for which the left boundary is a Dirichlet condition and the right boundary is a Neumann condition,

\[
\begin{pmatrix}
  a_1 & b_1 \\
  a_{1,2} & b_{1,2} & -a_2 & -b_2 \\
  \beta_1 b_{1,2} & \beta_1 a_{1,2} & -\beta_2 b_2 & -\beta_2 a_2 \\
  \cdot & \cdot & \cdot & \cdot \\
  a_{i,i+1} & b_{i,i+1} & -a_{i+1} & -b_{i+1} \\
  \beta_i b_{i,i+1} & \beta_i a_{i,i+1} & -\beta_{i+1} b_{i,i+1} & -\beta_{i+1} a_{i+1} \\
  \cdot & \cdot & \cdot & \cdot \\
  a_{n-1,n} & b_{n-1,n} & -a_n & -b_n \\
  \beta_{n-1} b_{n-1,n} & \beta_{n-1} a_{n-1,n} & -\beta_n b_n & -\beta_n a_n \\
  b_{n,n+1} & a_{n,n+1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_{2i-1} \\
  c_{2i} \\
  c_{2i+1} \\
  \vdots \\
  c_{2n-1} \\
  c_{2n}
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0 \\
  \vdots \\
  0 \\
  0 \\
  \cdot \\
  \vdots \\
  0
\end{pmatrix}
\]

The expression for \( \tilde{\phi}_i(x) \) can be obtained by solving the above linear equation for \( c_i \) and substituting the solution into Eq. (29).
For \( n = 4 \) zones with \( x = \{x_1, x_2, x_3, x_4, x_5\} = \{x_{lb}, x_{sl}, x_{sr}, x_{p}, x_{rb}\} \), the above matrix equation can be simplified to,

\[
\begin{pmatrix}
  a_1 & b_1 \\
  a_{1,2} & b_{1,2} & -a_2 & -b_2 \\
  \beta_1 b_{1,2} & \beta_1 a_{1,2} & -\beta_2 b_2 & -\beta_2 a_2 \\
  a_{2,3} & b_{2,3} & -a_3 & -b_3 \\
  \beta_2 b_{2,3} & \beta_2 a_{2,3} & -\beta_3 b_3 & -\beta_3 a_3 \\
  a_{3,4} & b_{3,4} & -a_4 & -b_4 \\
  \beta_3 b_{3,4} & \beta_3 a_{3,4} & -\beta_4 b_4 & -\beta_4 a_4 \\
  b_{4,5} & a_{4,5}
\end{pmatrix}
\begin{pmatrix}
  c_1 \\
  c_2 \\
  c_3 \\
  c_4 \\
  c_5 \\
  c_6 \\
  c_7 \\
  c_8
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  \frac{Q}{\sqrt{2\pi pT(x)}} \\
  0
\end{pmatrix}
\]

Expressions for \( c_i \) and \( \tilde{s}_i \) are very lengthy and not provided here for the sake of brevity. A Mathematica notebook is available for interested readers. However, the expressions used for all examples in Butler et al. [in review] are presented in the Section VII.
VI. Transform-Space Solution for Unbounded Domain

For an unbounded domain in the x direction, \( x_{lb} = -\infty \) and \( x_{rb} = \infty \), an approach similar to that in Section V is adopted to derive the boundary conditions in which Eq. (30) is used for the unbounded (far left and far right) zones.

\[
\begin{align*}
c_{2i} & = 0, \quad i = 1 \\
e^{\beta_{i+1}x_i}c_{2i-1} - \sinh(\beta_{i+1}x_i)c_{2i+1} - \cosh(\beta_{i+1}x_i)c_{2i+2} = 0, \quad i = 1 \\
\beta_i e^{\beta_{i+1}x_i}c_{2i-1} - \beta_{i+1} \cosh(\beta_{i+1}x_i)c_{2i+1} & = 0 \\
- \beta_{i+1} \sinh(\beta_{i+1}x_i)c_{2i+2} & = \begin{cases} 
\frac{Q}{\sqrt{2\pi pT(x)}}, & i = i_p - 1 = 1 \\
0, & i = 1 \neq i_p - 1 
\end{cases} \\
\sinh(\beta_{i+1}x_i)c_{2i-1} + \cosh(\beta_{i+1}x_i)c_{2i} & = 0 \\
- \sinh(\beta_{i+1}x_i)c_{2i+1} - \cosh(\beta_{i+1}x_i)c_{2i+2} = 0, \quad i = 2, ..., n - 2 \\
\beta_i \cosh(\beta_{i+1}x_i)c_{2i-1} + \beta_i \sinh(\beta_{i+1}x_i)c_{2i} - \beta_{i+1} \cosh(\beta_{i+1}x_i)c_{2i+1} & = 0 \\
- \beta_{i+1} \sinh(\beta_{i+1}x_i)c_{2i+2} & = \begin{cases} 
\frac{Q}{\sqrt{2\pi pT(x)}}, & i = i_p - 1 \\
0, & \text{all other } i = 2, ..., n - 2 
\end{cases} \\
sinh(\beta_{i+1}x_i)c_{2i-1} + \cosh(\beta_{i+1}x_i)c_{2i} & = 0 \\
- \beta_{i+1} e^{-\beta_{i+1}x_i}c_{2i+2} & = \begin{cases} 
\frac{Q}{\sqrt{2\pi pT(x)}}, & i = i_p - 1 = n - 1 \\
0, & \text{all other } i = n - 1 \neq i_p - 1 
\end{cases} \\
c_{2i-1} & = 0, \quad i = n 
\end{align*}
\]

(34)

By applying Eq. (33) and redefining

\[
e^{\beta_1x_2} = a_{1,2} = b_{1,2} \quad e^{-\beta_n x_n} = a_n = b_n
\]

Eq. (34) can be rewritten in a \( 2n \times 2n \) matrix format, as shown in the following for which left boundary is a Dirichlet condition and the right boundary is a Neumann condition,

\[
\begin{pmatrix}
0 & 1 \\
an_{1,2} & 0 & -a_2 & -b_2 \\
\beta_1 b_{1,2} & 0 & -\beta_2 b_2 & -\beta_2 a_2 \\
\vdots & \vdots & \vdots & \vdots \\
\beta_{i+1} b_{i+1} & a_{i+1} & -\beta_{i+1} b_{i+1} & -a_{i+1} \\
\beta_i b_{i+1} & 0 & -\beta_i b_2 & -\beta_i a_2 \\
\ddots & \ddots & \ddots & \ddots \\
\beta_{n-1} b_{n-1} & a_{n-1} & 0 & -b_n \\
\beta_{n-1} b_{n-1} & 0 & -\beta_{n-1} a_{n-1} & 0 \\
\beta_{n-1} b_{n-1} & \beta_{n-1} a_{n-1} & 0 & -\beta_n a_n \\
\beta_{n-1} b_{n-1} & \beta_{n-1} a_{n-1} & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_{2i-1} \\
c_{2i} \\
c_{2i+1} \\
c_{2i+2} \\
\vdots \\
c_{2n-1} \\
c_2n
\end{pmatrix}
\]

(35)
The expression for \( \tilde{s}_i(x) \) can be obtained by solving the above linear equation for \( c_i \) and substituting the solution into Eq. (29) or (30).

For \( n = 4 \) zones with \( x = \{x_1, x_2, x_3, x_4, x_5\} = \{x_l = -\infty, x_{sl}, x_{sr}, x_p, x_{rb} = \infty\} \), the above matrix equation is written as,

\[
\begin{pmatrix}
0 & 1 & -a_2 & -b_2 \\ -\beta_2 b_2 & -\beta a_2 & -a_3 & -b_3 \\ a_2 b & b_3 & -\beta_3 b_3 & -\beta_3 a_3 \\ -\beta_2 b_3 & -\beta a_3 & a_4 & b_4 \\ \beta_3 b_3 & \beta a_3 & 0 & -\beta_4 a_4 \\
\end{pmatrix}
\begin{pmatrix}
c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 
\end{pmatrix}
= 
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ Q \\ 0 
\end{pmatrix}
\]

A Mathematica notebook is available for interested readers with expressions for \( c_i \) and \( \tilde{s}_i \). Expressions for the simple unbounded and homogeneous system used for all examples in Butler et al. [in review] are presented in the next section.
VII. Transform-Space Solution for Unbounded Homogeneous Aquifer

For a homogeneous domain and the zone underneath the stream \((x_{sl} < x < x_{sr})\), Eq. (26) can be simplified to

\[
\beta_s^2 = \omega^2 + \frac{k_{sb}}{b_{sb} T} + \frac{1}{T} \sqrt{\frac{S_{ScP}}{k_c}} \coth \left[ \sqrt{\frac{S_{ScP}}{k_c}} \right] + \frac{p S}{T} \tag{36}
\]

Outside that zone, the equation can be further simplified to

\[
\beta^2 = \omega^2 + \frac{1}{T} \sqrt{\frac{S_{ScP}}{k_c}} \coth \left[ \sqrt{\frac{S_{ScP}}{k_c}} \right] + \frac{p S}{T} \tag{37}
\]

Given the homogeneous domain, the problem can be reduced to solving a group of linear equations for four zones, the boundaries of which are defined by the pumping well and by the two sides of the stream:

\[
\tilde{s}(x) = \begin{cases} 
\sqrt{\frac{2}{\pi}} Q \beta_s e^{-(x_p - x_{sr} + x) \beta_s} (x_{sr} - x) \beta_s \frac{1}{D}, & -\infty < x < x_{sl} \\
\frac{1}{\sqrt{2\pi}} Q e^{-(x_p - x_{sr}) \beta_s} (x_{sr} - x) \beta_s \frac{1}{D}, & x_{sl} < x < x_{sr} \\
-\frac{1}{\sqrt{2\pi}} Q e^{-(x_p + x) \beta - 2x_{sr} \beta_s} \left[(e^{2x_{sr} \beta} + e^{2x_{sr} \beta_s}) (e^{2x_{sr} \beta_s} - e^{2x_{sr} \beta}) \beta^2 \right] \\
2e^{2x_p \beta_s} e^{2x_{sr} \beta} (e^{2x_{sr} \beta_s} + e^{2x_{sr} \beta}) \beta_s \frac{1}{D}, & x_{sr} < x < x_p \\
-\frac{1}{\sqrt{2\pi}} Q e^{-(x_p + x) \beta - 2x_{sr} \beta_s} \\
\left[(e^{2x_p \beta} + e^{2x_{sr} \beta}) (e^{2x_{sr} \beta_s} - e^{2x_{sr} \beta}) \beta^2 \right] \\
2e^{2x_p \beta_s} (e^{2x_{sr} \beta_s} + e^{2x_{sr} \beta}) \beta_s \frac{1}{D}, & x_p < x < \infty
\end{cases} \tag{38}
\]

in which

\[D = p T (e^{2(x_{sr} - x_{sl}) \beta_s} (\beta - \beta_s)^2 - (\beta + \beta_s)^2).\]

For the case of an impermeable formation underlying the shallow aquifer of Fig. 1, the leakage \((\coth)\) term in Eq. (36) and (37) is negligible, resulting in:

\[
\beta_s^2 = \omega^2 + \frac{k_{sb}}{b_{sb} T} + \frac{p S}{T} \tag{39}
\]
for the zone underneath the stream \((x_{sl} < x < x_{sr})\) and

\[
\beta^2 = \omega^2 + p \frac{S}{T}
\]  

(40)

for all other zones. The solution is thus the same as Eq. (38) except for the definitions of \(\beta\) and \(\beta_s\).
VIII. Transform-Space Solution for Stream Depletion

The rate of stream depletion $\Delta q$ is defined as the total volumetric discharge across the incompressible streambed at any given time (see Fig. 1 and caption). For the simplified scenario of Fig. 1 and the previous section, $\Delta q$ can be expressed as:

$$
\Delta q(t) = \frac{k_{sb}}{b_{sb}} \int_{-\infty}^{\infty} \int_{x_{sl}}^{x_{sr}} s_s dx dy
$$

(41)

in which $s_s$ is the drawdown beneath the stream.

Application of the Laplace transform to Eq. (41) and switching the $x$ and $y$ integrals yields:

$$
\Delta \bar{q}(p) = \frac{k_{sb}}{b_{sb}} \int_{x_{sl}}^{x_{sr}} \int_{-\infty}^{\infty} \bar{s}_s dy dx = \frac{k_{sb}}{b_{sb}} \int_{x_{sl}}^{x_{sr}} \bar{\tilde{s}}_s dx
$$

(42)

in which $\Delta \bar{q}(p)$ is the Laplace transform of $\Delta q$ and $\bar{\tilde{s}}_s$ is the Fourier-Laplace transform of $s_s$ for $\omega = 0$.

Substitution of Eq. (38) into Eq. (42) and performing the integration results in:

$$
\Delta \bar{q}(p) = -\frac{1}{\sqrt{2\pi}} Q e^{-(x_p-x_{sr})\beta} (e^{x_{sr}\beta_s} - e^{x_{sl}\beta_s}) \frac{1}{D_s}
$$

(43)

in which

$$
D_s = pT((e^{x_{sr}\beta_s} + e^{x_{sl}\beta_s})\beta \beta_s + (e^{x_{sr}\beta_s} - e^{x_{sl}\beta_s})\beta_s^2).
$$
IX. Numerical Inversion of Laplace-Fourier Space Solutions

The solutions in Laplace-Fourier space given in the previous sections are most readily evaluated using a numerical inversion scheme. A Mathematica Add-On package prepared by Mallet [2000] is used for the joint Laplace-Fourier numerical inversion. This package provides five inversion methods to invert Laplace transforms and joint Fourier/Hankel-Laplace transforms. The inversion techniques are those of Durbin [1974], Stehfest [1970], Weeks [1966], Piessens [1971], and Crump [1976]. The Stehfest [1970] algorithm, the most commonly used inversion algorithm for well-hydraulics applications, is selected for the inversion of head responses and stream depletion from Laplace space. The Fourier inversion uses the symbolic inverse Fourier transform provided in Mathematica.
References


