Mathematical Derivation of Drawdown and Stream Depletion Produced by Pumping in the Vicinity of a Finite-Width Stream of Shallow Penetration

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I. Introduction

In this report, the mathematical derivation of the solution of Butler et al. (in review) for drawdown and stream depletion produced by pumping from a fully penetrating well in the vicinity of a finite-width stream of shallow penetration is presented. For the sake of generality, the solution is obtained in a dimensionless form. See Butler et al. (in review) for notation definitions that are not given in this report.

II. Drawdown Solution

The drawdown solution was obtained using a straightforward extension of the approach described in Butler and Liu (1991). Equations (1)-(10) of Butler et al. (in review) describe the flow conditions of interest here. Dimensionless forms of these equations are as follows:

$$\frac{\partial^2 \Phi_1}{\partial \xi^2} + \frac{\partial^2 \Phi_1}{\partial^2 \eta} = P_1 \frac{\partial \Phi_1}{\partial \tau}, \quad -X_{LB} \le \xi \le -1, \quad -\infty < \eta < \infty, \ \tau > 0 \tag{1}$$

$$\frac{\partial^2 \Phi_2}{\partial \xi^2} + \frac{\partial^2 \Phi_2}{\partial^2 \eta} - \mathrm{B}\Phi_2 = P_2 \frac{\partial \Phi_2}{\partial \tau}, \quad -1 \le \xi \le 0, \quad -\infty < \eta < \infty, \ \tau > 0 \tag{2}$$

$$\frac{\partial^2 \Phi_3}{\partial \xi^2} + \frac{\partial^2 \Phi_3}{\partial^2 \eta} + \delta(\xi - \alpha)\delta(\eta - \beta) = \frac{\partial \Phi_3}{\partial \tau}, \quad 0 \le \xi \le X_{RB}, \ -\infty < \eta < \infty, \ \tau > 0 \tag{3}$$

$$\Phi_i(\xi, \eta, 0) = 0, -X_{LB} \le \xi \le X_{RB,} \ -\infty < \eta < \infty, \mathrm{i} = 1,3 \tag{4}$$

$$\frac{\partial \Phi_1}{\partial \xi}(-X_{LB}, \eta, \tau) = \frac{\partial \Phi_3}{\partial \xi}(X_{RB}, \eta, \tau) = 0, -\infty < \eta < \infty, \quad \tau > 0 \tag{5}$$

$$\Phi_i(\xi, \pm\infty, \tau) = 0, -X_{LB} \le \xi \le X_{RB}, \tau > 0 \qquad (6)$$

$$\Phi_1(-1, \eta, \tau) = \Phi_2(-1, \eta, \tau), -\infty < \eta < \infty, \;\; \tau > 0 \qquad (7)$$

$$\frac{\partial \Phi_1}{\partial \xi}(-1, \eta, \tau) = \gamma_1 \frac{\partial \Phi_2}{\partial \xi}(-1, \eta, \tau), -\infty < \eta < \infty, \tau > 0 \qquad (8)$$

$$\Phi_2(0, \eta, \tau) = \Phi_3(0, \eta, \tau), -\infty < \eta < \infty, \;\; \tau > 0 \qquad (9)$$

$$\frac{\partial \Phi_2}{\partial \xi}(0, \eta, \tau) = \gamma_2 \frac{\partial \Phi_3}{\partial \xi}(0, \eta, \tau), -\infty < \eta < \infty, \tau > 0 \qquad (10)$$

where

Φ_i (dimensionless drawdown) $=s_i T_3/Q$, i=1,3;
_ (dimensionless time)=$(T_3 t)/(w^2 S_3)$;
_=x / w; η=y / w; _ = a / w; β=b/w;
_ (stream leakance) = $(k' w^2)/(b' T_2)$;
$X_{RB}=x_{rb}/w$; $X_{LB}=x_{lb}/w$; $\gamma_i=T_{i+1}/T_i$, i=1,2;
$\mu_i=S_i/T_i$, i=1,3; $P_i= \mu_i/\mu_3$, i=1,2.

A solution can be obtained for equations (1)-(10) through use of integral transforms (Robinson, 1968; Churchill, 1972). A Laplace transform in time followed by a Fourier exponential transform in the η direction produces Fourier-Laplace space analogues to (1)-(3) of the following form:

$$\frac{\partial^2 \overline{\overline{\Phi}}_1}{\partial \xi^2} - \lambda_1^2 \overline{\overline{\Phi}}_1 = 0, \qquad - X_{LB} \le \xi \le -1, \qquad (11)$$

$$\frac{\partial^2 \overline{\overline{\Phi}}_2}{\partial \xi^2} - \lambda_2^2 \overline{\overline{\Phi_2}} = 0, \quad -1 \le \xi \le 0 \tag{12}$$

$$\frac{\partial^2 \overline{\overline{\Phi_3}}}{\partial \xi^2} - \lambda_3^2 \overline{\overline{\Phi_3}} = \frac{-e^{i\omega\beta}\delta(\xi - \alpha)}{p\sqrt{2\pi}}, \; -0 \le \xi \le X_{RB}, \tag{13}$$

where

$\overline{\overline{\Phi}}_i$ =the Fourier-Laplace transform of Φ_i, i=1,3;
p=Laplace transform variable;
ω=Fourier transform variable;
$\lambda_1 = (\omega^2+P_1p)^{0.5}$;
$\lambda_2 = (\omega^2+B+P_2p)^{0.5}$;
$\lambda_3 = (\omega^2+p)^{0.5}$.

The Fourier-Laplace space solutions to (11) and (12) are quite straightforward:

$$\overline{\overline{\Phi_1}} = C_1 e^{\lambda_1 \xi} + C_2 e^{-\lambda_1 \xi} \tag{14}$$

$$\overline{\overline{\Phi 2}} = C3 e^{\lambda_2 \xi} + C_4 e^{-\lambda_2 \xi} \tag{15}$$

The Fourier-Laplace space solution to (13) cannot be found as easily owing to the non-homogeneous delta function term in that expression. The approach used for obtaining a solution to (13) was to divide zone 3 into two subregions, subregion 31 ($0 \le \xi \le \alpha$) and subregion 32 ($\alpha < \xi \le X_{RB}$). The solution for subregion 31 consists of a homogeneous part and a particular solution ($\overline{\overline{\Phi}}_{3p}$):

$$\overline{\overline{\Phi}}_{31} = C_5 e^{\lambda_3 \xi} + C_6 e^{-\lambda_3 \xi} + \overline{\overline{\Phi}}_{3p} \tag{16}$$

Using Theorem 3.13 of Boyce and DiPrima (1986), the particular solution can be written as:

$$\begin{aligned} \overline{\overline{\Phi}}_{3p} &= \frac{-e^{i\omega\beta}}{2\lambda_3 p\sqrt{2\pi}}[-e^{\lambda_3(\alpha-\xi)} + e^{\lambda_3(\xi-\alpha)}], \; \xi = \alpha \\ &= 0, \; \text{elsewhere} \end{aligned} \tag{17}$$

The solution for subregion 32 consists solely of a homogeneous part:

$$\overline{\overline{\Phi}}_{32} = C_7 e^{\lambda_3 \xi} + C_8 e^{-\lambda_3 \xi} \tag{18}$$

The division of region 3 into two subregions requires two additional boundary conditions to ensure continuity across the division:

$$\overline{\overline{\Phi}}_{31}(\alpha,\omega,p) = \overline{\overline{\Phi}}_{32}(\alpha,\omega,p) \tag{19}$$

$$\frac{\partial \overline{\overline{\Phi}}_{31}}{\partial \xi}(\alpha,\omega,p) = \frac{\partial \overline{\overline{\Phi}}_{32}}{\partial \xi}(\alpha,\omega,p) \tag{20}$$

The constants in equations (14)-(16) and (18) can be evaluated by substituting these expressions into (19)-(20) and the following Fourier-Laplace space analogues of (5) and (7)-(10):

$$\frac{\partial \overline{\overline{\Phi}}_{1}}{\partial \xi}(-X_{LB},\omega,p) = \frac{\partial \overline{\overline{\Phi}}_{3}}{\partial \xi}(X_{RB},\omega,p) = 0 \tag{21}$$

$$\overline{\overline{\Phi}}_{1}(-1,\omega,p) = \overline{\overline{\Phi}}_{2}(-1,\omega,p) \tag{22}$$

$$\frac{\partial \overline{\overline{\Phi}}_{1}}{\partial \xi}(-1,\omega,p) = \gamma_1 \frac{\partial \overline{\overline{\Phi}}_{2}}{\partial \xi}(-1,\omega,p) \tag{23}$$

$$\overline{\overline{\Phi}}_{2}(0,\omega,p) = \overline{\overline{\Phi}}_{3}(0,\omega,p) \tag{24}$$

$$\frac{\partial \overline{\overline{\Phi}}_{2}}{\partial \xi}(0,\omega,p) = \gamma_2 \frac{\partial \overline{\overline{\Phi}}_{3}}{\partial \xi}(0,\omega,p) \tag{25}$$

Evaluation of the constants is a straightforward but tedious algebraic exercise. Once the constants are found, they are substituted back into equations (14)-(16) and (18) to obtain the following expressions:

$$\overline{\overline{\Phi}}_1(\xi,\omega,p)=(T1)[e^{2\lambda_1 X_{LB}+\lambda_1\xi}+e^{-\lambda_1\xi}] \tag{26}$$

$$\overline{\overline{\Phi}}_2(\xi,\omega,p)=(T1)[(A1)e^{\lambda_2\xi}+(B1)e^{-\lambda_2\xi}] \tag{27}$$

$$\overline{\overline{\Phi}}_3(\xi,\omega,p)=(T1)[(D1)e^{\lambda_3\xi}+(E1)e^{-\lambda_3\xi}],0\le\xi\le\alpha \tag{28}$$

$$\overline{\overline{\Phi}}_3(\xi,\omega,p)=(\frac{(T1)(G1)}{(H1)})[e^{\lambda_3\xi}+e^{2\lambda_3 X_{RB}-\lambda_3\xi}],\alpha<\xi\le X_{RB} \tag{29}$$

where

$$\text{A1}=\frac{1}{2}(e^{2\lambda_1 X_{LB}-\lambda_1+\lambda_2}+e^{\lambda_1+\lambda_2})+\frac{\lambda_1}{2\gamma_1\lambda_2}(e^{2\lambda_1 X_{LB}-\lambda_1+\lambda_2}-e^{\lambda_1+\lambda_2})$$

$$\text{B1}=\frac{1}{2}(e^{2\lambda_1 X_{LB}-\lambda_1-\lambda_2}+e^{\lambda_1-\lambda_2})-\frac{\lambda_1}{2\gamma_1\lambda_2}(e^{2\lambda_1 X_{LB}-\lambda_1-\lambda_2}-e^{-\lambda_1-\lambda_2})$$

$$\text{D1}=\frac{1}{2}[(A1)+(B1)]+\frac{\lambda_2}{2\gamma_2\lambda_3}[(A1)-(B1)]$$

$$\text{E1}=\frac{1}{2}[(A1)+(B1)]-\frac{\lambda_2}{2\gamma_2\lambda_3}[(A1)-(B1)]$$

$\text{F1}=-e^{i\omega\beta}/(\lambda_3 p\sqrt{2\pi})$
$\text{G1}=(D1)e^{\lambda_3\alpha}+(E1)e^{-\lambda_3\alpha}$
$\text{H1}=e^{\lambda_3\alpha}+e^{2\lambda_3 X_{RB}-\lambda_3\alpha}$
$\text{J1}=(D1)e^{\lambda_3\alpha}-(E1)e^{-\lambda_3\alpha}$
$\text{K1}=e^{\lambda_3\alpha}-e^{2\lambda_3 X_{RB}-\lambda_3\alpha}$
T1=(F1)(H1)/[(G1)(K1)-(J1)(H1)]

Equations (26)-(29) form the Fourier-Laplace space solutions to (1)-(10). Substitution of (26)-(29) into the transform-space analogues of (1)-(10) will demonstrate the viability of the proposed solutions.

The Fourier-Laplace space solution must be transformed back to real space for practical applications. Butler et al. (in review) discuss the numerical inversion schemes used in this work and compare the numerically inverted solution, which is computed using Butler and Tsou (1999), with existing analytical and numerical models.

III. Stream Depletion Solution

The solution for stream depletion was obtained following the approach outlined by Hunt (1999). Butler et al. (in review) define stream depletion in equation (11), the dimensionless form of which is:

$$\Delta Q(\tau) = \frac{B}{\gamma_2} \int_{-\infty}^{\infty}\int_{1}^{0} \Phi_2 \, d\xi \, d\eta, \quad \tau > 0 \tag{30}$$

Application of the Laplace transform to equation (30) and switching the ξ and η integrals results in:

$$\Delta \overline{Q}(p) = \frac{B}{\gamma_2} \int_{-1}^{0} \left(\int_{-\infty}^{\infty} \overline{\Phi}_2 \, d\eta \right) d\xi \tag{31}$$

where

$\Delta \overline{Q}(p)$ =Laplace transform of ΔQ.

The term in parentheses is simply the Fourier-Laplace transform of Φ_2 for ω=0:

$$\Delta \overline{Q}(p) = \frac{B}{\gamma_2} \int_{-1}^{0} \overline{\overline{\Phi}}_2(\xi, 0, p) \, d\xi \tag{32}$$

Substitution of equation (27) into (32) and performing the integration results in:

$$\Delta \overline{Q}(p) = \frac{\mathrm{B}}{\gamma_2 \lambda_2^*} (\mathrm{T1})[(\mathrm{A1})(1 - \mathrm{e}^{-\lambda_2^*}) - (\mathrm{B1})(1 - \mathrm{e}^{\lambda_2^*})] \tag{33}$$

where

λ_2^* =$(\mathrm{B}+\mathrm{P}_2\mathrm{p})^{0.5}$.

Equation (33) is the Laplace-space solution for stream depletion. Butler et al. (in review) describe the numerical scheme used to invert equation (33) to real space, and compare the resulting solution to existing analytical and numerical models. As with the drawdown solution, the numerical inversion scheme is implemented in Butler and Tsou (1999).

IV. References

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