A Gauss Markov approach to conditioning to block data

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Abstract

Integration of data defined at different scales is a recurring problem in Earth Science applications. Integrating seismic data with well data defined on a much smaller scale is one such application. Currently sequential and iterative simulation algorithms are not equipped to combine data which are non-linearly related with each other over different scales. This paper proposes a novel technique based on the theory of Gauss-Markov random functions (GMrf). The conditioning of point simulations to point data and non-linear block data is performed by using a Metropolis-Hastings sampler on a Markov-type random field. The efficiency of the algorithm is demonstrated with an example based on an exhaustive synthetic data set. Simulations honor a strongly anisotropic covariance, a bimodal histogram and are conditioned to point data and imprecise block data. A second example of conditioning to weighted harmonic averages along irregularly shaped blocks is discussed. The general applicability of the algorithm to problems of greater complexity is discussed.
**Notations**

\( f_Z(z) \): probability density distribution of a single random variable \( Z \) with outcome \( z \).

\( f_Z(z) \): probability density distribution of a random vector \( Z \) with outcome \( z \).

\( f_{Z|Y}(z|y) \): conditional probability density distribution of a single random variable \( Z \) given a random vector \( Y \).

\( Y(u) \): random function in the standard normal space.

\( Z(u) \): random function in the original data space.

\( u_{ijk} \): grid node at index \( i, j, k \) in a 3D grid.

\( z \): a realization in the original data space.

\( y \): a realization in the normal score space.

\( \tilde{y}(u) \): the set of all grid node values except the grid node value \( y(u) \).

\( \partial u \): the set of grid node values in the neighborhood of \( u \), excluding \( u \).

\( \partial Y \): the random vector of values in the neighborhood \( \partial u \).

\( d_V(u) \): a volume average datum centered at \( u \).

\( D: \{d_v(u_\beta), \beta = 1, \ldots, m\} \): the random vector of volume average data.

\( Z_V \): the random vector of grid node values in the volume \( V \).
1 Problem statement

In many applications, ancillary information is available through secondary data which are often expressed as non-linear volume or multiple-point averages of the primary variable. For example

- Surface seismic data can be calibrated to subsurface volume averages of rock properties such as porosity (Yao, 1998).

- Well-test data can be interpreted as an effective, non-linear averaged, permeability over a cylindric volume around the well (Srinivasan and Journel, 1998).

- History-matched realizations provide reservoir models defined over a large volume support (Tran et al., 1999). Production data integration is then performed on such large volume support at a coarse scale because of the CPU-limitation of current inverse methods. Hence after inversion, the reservoir models must be down-scaled in order to honor local well data and reflect small scale geological continuity. The original scale simulated values can be considered as linear averages of the small scale geological model.

Behrens et al. (1996) and Yao and Journel (2000) use a block kriging model to constrain a 3D reservoir model to 2D seismic data. The seismic data is integrated by a first 2D co-kriging of vertically averaged porosity using vertically averaged well data and seismic data as a soft variable. The resulting estimated horizontally-averaged porosity map is then used to condition the 3D stochastic simulation of a 3D porosity field. In this second step, the
authors use a block kriging approach to solve the difference of scale problem between well-data and the estimated vertically averaged porosity. Their methods operate in the Gaussian space, requiring a prior transformation of both well-data and the vertical averaged estimated values. A back-transformation into the original data space allows identifying the porosity histogram. However, this latter operation, which is a non-linear transformation, does not preserve the linear block-average type constraints. A similar approach is used to down-scale history matched production data (Tran et al., 1999).

Journel (1999) proposes to short-cut the problem caused by the back-transformation by performing direct sequential simulation instead of Gaussian simulation. Direct sequential simulation operates in the original data space, hence does not require a prior histogram transformation. Journel’s method relies on a property of co-kriging, stating that volume-average data values can be reproduced exactly as long as the averaging process is linear. However, the method still has various limitations:

- The volume averaging is restricted to a power average, or any other monotonous increasing or decreasing transformation of the original variable.

- Kriging is not a convex estimator, hence estimates can be negative, due to negative weights. Negative weights need to be corrected because the power-averaging requires strict positive values.

- The direct sequential simulation does not reproduce the histogram of the original primary variable. This calls for a posterior histogram identification of the resulting simulated realizations, which again affects the reproduction of the block-average data.
In this paper we propose to address any of the three above limitations using an iterative Markov chain Monte Carlo approach instead of a non-iterative sequential simulation approach. The main difference between iterative and non-iterative simulation methods is that the former operates on a "full grid", iteratively updating each node, while the latter simulates sequentially each node only once. In iterative simulation initial random values are assigned to all grid nodes, then an iterative process is started, randomly selecting nodes and assigning new values to each node according to some probabilistic rule. Since non-iterative sequential simulation fills the grid gradually, it becomes hard to condition realizations to block-average data, since the simulated block value is only known when all points within that block are visited. Therefore, convenient analytical theorems (Journel, 1999) are required to ensure the reproduction of the block averages in the resulting simulations. Such theorems only exist for linear or pseudo-linear averages such as the power-average.

In iterative simulation, one operates on a full grid, hence any kind of block or multi-point averages no matter how complex, can be calculated during the iteration and its reproduction can be checked. A common perturbation algorithm is simulated annealing simulation (eg. program sasim, Deutsch and Journel, 1998). Simulated annealing has been used for constraining realizations to block-average constraints (Deutsch et al., 1996) but has several drawbacks

- it requires extensive CPU-time on large 3D grids and skill in the delicate tuning of the cooling schedule.
- The resulting down-scaled models often exhibit higher nugget effect than desired (Tran
et al., 1999).

- Annealing aims at the exact reproduction of constraints within an objective function.

  Block averages are often of an imprecise nature (Behrens et al., 1996; Yao, 1998), i.e. the primary variable is averaged within the block up to a certain known precision.

  Annealing cannot deal with such imprecise constraints easily.

The Gauss-Markov iterative simulation method (Caers, 2000a) presently allows for the reproduction of a given variogram and histogram model through the specification of a Markov-Gaussian random function. We extend the algorithm to allow conditioning to any type of block average accounting for precision.

2 Gauss-Markov iterative simulation

The concept of the Gauss-Markov random functions is recalled briefly. Consider a stationary random function \( Y(u) \) defined on a three-dimensional regular grid of \( L \times M \times N \) nodes. The nodes in the grid are indexed as follows

\[
u_{ijk}, \ i = 1, \ldots L, \ j = 1, \ldots, M, \ k = 1, \ldots, N
\]

The set of all values of the grid is denoted by the vector \( y \)

\[
y = \{y(u_{ijk}), \ i = 1, \ldots L, \ j = 1, \ldots, M, \ k = 1, \ldots, N\}
\]

The random function \( Y(u) \) has stationary covariance model \( C(h) \) and is standard Gaussian.

Denote by \( C \) the full covariance matrix with entries \( C_{ij,lmn} \) representing the covariances
between \( Y(u_{ijk}) \) and any other \( Y(u_{lmn}) \). The diagonal entries \( C_{ii,i} \) are constant equal to unit variance.

The multivariate density distribution of \( Y(u) \) is denoted as

\[
f_Y(y) = f_Y(y(u_{ijk}), i = 1, \ldots, L, j = 1, \ldots, M, k = 1, \ldots, N) \tag{1}
\]

In the case of Gauss-Markov random functions the conditional distribution of \( Y(u_{ijk}) \), at a particular location \( u_{ijk} \) being the location on a regular 3D grid, depends only on its nearest neighbors, i.e.

\[
f_{Y|\tilde{Y}}(y(u_{ijk})|\tilde{Y} : \text{ all other } y(u_{lmn})) \approx f_{Y|\partial Y}(y(u_{ijk})|\partial Y : y(u_{lmn}), u_{lmn} \in \partial u_{ijk}) \tag{2}
\]

where \( \partial u_{ijk} \) is a set of neighboring sites to \( u_{ijk} \). \( \tilde{Y} \) is the vector of random variables at all grid locations except at \( u_{ijk} \). \( \partial Y \) is the vector of random variables in the neighborhood of \( u_{ijk} \) and excluding \( u_{ijk} \). The neighborhood \( \partial u_{ijk} \) is defined by the fixed geometry of a template. A large template size would provide a better Markov approximation in Eq. (2), lead to faster convergence in terms of iteration but would be more costly in terms of flops per iteration.

The local conditional distribution in (2) is Gaussian. The mean and variance of this Gaussian distribution are expressed through the following expressions

\[
E[Y(u_{ijk})|\partial y] = \sum_{(l,m,n): u_{lmn} \in \partial u_{ijk}} w_{ijk,lmn} y(u_{lmn}) \tag{3}
\]

\[
\text{Var}[Y(u_{ijk})|\partial y] = c^2 \tag{4}
\]
In Caers (2000a) an algorithm is presented for determine the non-kriging weights $w_{ijk,lmn}$ and the homoscedastic variance $c^2$ for any target covariance model $C(h)$. Caers shows that the Markov approximation in (2) is accurate enough to reproduce typical isotropic and anisotropic covariances, such as the spherical, exponential and Gaussian models.

A Metropolis-Hastings sampler, which is a particular type of Markov chain Monte Carlo simulation, uses the local conditional distribution (2) to simulate realizations that are standard normal and reproduce the covariance model $C(h)$. A conditional simulation using Markov chains commences by initializing each grid node with a random value drawn from the corresponding marginal distribution. Hard data values are frozen at the nearest grid node locations. The algorithm then visits each grid node along a purely random path. Each node is visited more than once as opposed to traditional sequential simulation. Hard data locations are never visited. At each visited node, the existing value at that node is updated using a the Metropolis-Hastings sampling criterion. Hastings (1970) shows that in order to reproduce the target multivariate distribution should proceed in two steps

- **A proposal step**: a new value $y^{\text{new}}(u_{ijk})$ for grid node $u_{ijk}$ is proposed as a possible replacement for the current value at that node, $y^{\text{current}}(u_{ijk})$. The value $y^{\text{new}}(u_{ijk})$ is obtained by a random draw from the marginal distribution. One then proposes to change the current grid values

  $$y^{\text{current}} = \{y(u_{111}), \ldots, y^{\text{current}}(u_{ijk}), \ldots, y(u_{LMN})\}$$

  to the new grid

  $$y^{\text{new}} = \{y(u_{111}), \ldots, y^{\text{new}}(u_{ijk}), \ldots, y(u_{LMN})\}$$
\( y^{\text{current}} \) and \( y^{\text{new}} \) thus differ at the grid node \( u_{ijk} \) only.

- **A transition step**: An acceptance probability is defined that determines the probability of accepting the new grid \( y^{\text{new}} \). According to Hastings (1970), in order to sample from the multivariate density \( f_Y(y) \) with marginal \( f_Y(y) \), this probability has to be equal to

\[
\alpha = \min \left\{ 1, \frac{f_Y(y^{\text{new}})}{f_Y(y^{\text{current}})} \frac{f_Y(y^{\text{current}})}{f_Y(y^{\text{new}})} \right\} \tag{5}
\]

The expression (5) is difficult to evaluate due to the presence of the ratio of multivariate densities. Therefore, the acceptance criterion is simplified by recognizing the following approximation

\[
\frac{f_Y(y^{\text{new}})}{f_Y(y^{\text{current}})} = \frac{f_{Y\mid \bar{Y}}(y^{\text{new}}(u_{ijk})\mid \bar{y}(u_{ijk}))}{f_{Y\mid \bar{Y}}(y^{\text{current}}(u_{ijk})\mid \bar{y}(u_{ijk}))} \approx \frac{f_{Y\mid \partial Y}(y^{\text{new}}(u_{ijk})\mid \partial y)}{f_{Y\mid \partial Y}(y^{\text{current}}(u_{ijk})\mid \partial y)} \tag{6}
\]

where \( \bar{Y} \) is the complement of \( Y(u_{ij}) \), i.e. the random vector containing all nodes except the variable at node \( u_{ijk} \):

\[
\bar{y}(u_{ijk}) = \{y(u_{inn}), \forall u_{inn} \neq u_{ijk}\}
\]

Given expression (6) the Metropolis acceptance criterion (5) is rewritten as function of only conditional densities:

\[
\alpha = \min \left\{ 1, \frac{f_{Y\mid \partial Y}(y^{\text{new}}(u_{ijk})\mid \partial y)}{f_{Y\mid \partial Y}(y^{\text{current}}(u_{ijk})\mid \partial y)} \frac{f_Y(y^{\text{current}}(u_{ijk}))}{f_Y(y^{\text{new}}(u_{ijk}))} \right\} \tag{7}
\]

\( f_Y(y) \) being the marginal distribution.
3 Conditioning to block average data

3.1 Model initialization and histogram identification

First, we need to deal with the issue of histogram reproduction. At the start of the simulation, the grid is initialized with standard normal random values. In order to simultaneously simulate in the original data space, a second grid consisting of the corresponding backtransforms (using the target histogram) is built. Hence one utilizes two grids: a standard normal space grid denoted as \( y \) and a data space grid denoted as \( z \). At each step of the iteration whenever a new value for location \( u \) is accepted in the normal space grid, its back-transform is calculated and stored in the data space grid. Hence when the Markov chain iteration is completed, one has two simulation results, one with simulated values in the normal space and one with simulated values in the original space. Since the method guarantees the reproduction of the standard normal distribution (see Caers, 2000a) in the normal space, it will also guarantee reproduction of the sample histogram (or any target histogram used to perform the back-transform) in the original space. Also because the method conditions without discontinuities properly to local hard data in the normal space, it also correctly conditions in the original space.

The existence of two grids is also important in the conditioning to block average data. A block average is function of the primary variable in the original data space, not in the normal space. Hence the block average reproduction at any iteration step is evaluated in the original data space.
A block average datum $d_V(u')$ is considered to be a general multi-point function of the primary variable $z$ over a volume $V(u')$, centered at $u'$

$$d_V(u') = \phi(z(u), u \in V(u'))$$  \hspace{1cm} (8)

Linear or power averages are but special types of this representation. We also consider the precision of the block average datum $d_V(u')$ to be known. This precision is provided either the a full distribution $f_D(d_V(u'))$, or, through its variance $\sigma^2_V(u')$.

### 3.2 Conditioning to fixed-volume, non-overlapping averages

Consider the case of non-overlapping, fixed-volume blocks. This case is relevant for the integration of seismic impedance data (Yao, 1998), production data (Tran et al., 1999) or single well-test data (Srinivasan and Journel, 1998). In order to constrain Gauss-Markov realizations to imprecise block data, we need to adapt the Metropolis sampling criterion (7).

Instead of sampling the joint distribution $f_Y(y)$ one needs to sample from the joint distribution $f_Y(y|d_V(u_\beta), \beta = 1, \ldots, m)$ conditioned to the block data $D : \{d_V(u_\beta), \beta = 1, \ldots, m \}$. The Metropolis-Hastings sampler relies on the ratio of two joint probabilities of type (5) which in this case can be decomposed, using Bayes’ rule, into the ratio of two
conditional distributions:

\[
\frac{f_Y(y_{\text{new}}|d_V(u_\beta), \beta = 1, \ldots, m) f_Y(y_{\text{cur}}(u_{ijk}))}{f_Y(y_{\text{cur}}|d_V(u_\beta), \beta = 1, \ldots, m) f_Y(y_{\text{new}}(u_{ijk}))} = \\
\frac{f_Y(y_{\text{new}}|y_{ijk}) f_Y(y_{\text{cur}}|y_{ijk})}{f_Y(y_{\text{cur}}|y_{ijk}) f_Y(y_{\text{new}}|y_{ijk})} \times \\
\frac{f_D|Y(d_V(u_\beta), \beta = 1, \ldots, m|y) f_Y(y_{ijk})}{f_D|Y(d_V(u_\beta), \beta = 1, \ldots, m|y) f_Y(y_{ijk})}
\]

The first term after the equality sign is related to the acceptance criterion (7) once the Markov assumption (2) is introduced; the second term is the ratio of likelihood distributions \( f_D|Y \) of all the block averages given the entire permeability field \( y = \{y(u_{ijk}), y(u_{ijk})\} \). In the Gauss-Markov method, each nodal value \( y(u) \) is assumed conditionally independent of values not inside a given neighborhood. We will also assume that, given nodal values \( y(u) \) within a given volume \( V(u') \), the block averages are conditionally independent one from each other if the blocks are disjoint, hence

\[
f_D|Y(d_V(u_\beta), \beta = 1, \ldots, m|y) \approx \prod_{\beta=1}^{m} f_D(d_V(u_\beta))
\]

\( f_D(d_V(u_\beta)) \) are the block distributions of the block average (8). A similar assumption is made in Yao (1998) and Tran et al. (1999). Given the approximation (10), the Metropolis criterion (9) can be rewritten

\[
\alpha = \min \left\{ 1, \frac{f_Y|\partial Y(y_{\text{new}}(u_{ijk})|\partial y) f_Y(y_{\text{cur}}(u_{ijk})) f_D^{\text{new}}(d_V(u'_{\beta_0}))}{f_Y|\partial Y(y_{\text{cur}}(u_{ijk})|\partial y) f_Y(y_{\text{new}}(u_{ijk})) f_D^{\text{cur}}(d_V(u'_{\beta_0}))} \right\}
\]

where \( u'_{\beta_0} \) is the location of the single volume \( V \) containing the nodal location \( u_{ijk} \) to be
updated. $f_D^{\text{new}}$ is the block distribution evaluated with the proposed value $y^{\text{new}}(u_{ijk})$, $f_D^{\text{cur}}$ for the current value $y^{\text{cur}}(u_{ijk})$.

Eq. (11) shows a clear decomposition of the Metropolis sampler into a prior conditional distribution ratio ensuring normal-score-variogram and histogram reproduction and a ratio of block conditional distributions, ensuring reproduction of the block averages up to a given precision.

4 Case Studies

4.1 2D synthetic case

The GMrf theory presented in this paper is applicable to the problems solved by Yao(1998) and Tran et. al. (1999), which consider single-point block averages, and also to the non-linear averages considered by Srinivasan(2000). Here, we present an example of conditioning to linear averages in a two-dimensional grid with the aim of demonstrating that the Gmrf methodology honors the covariance and histogram along with preserving the property of data exactitude without discontinuity and conditioning to the block data up to the specified precision.

We consider the case of conditioning point simulations to point data and non-overlapping linear block averages. The data are synthetic and taken from exhaustive data set defined on a $100 \times 100$ Cartesian grid shown in Figure 1a. The variogram map of the exhaustive data and the semi-variograms in the two principal directions of continuity are shown in Figure 1b.
and Figure 1c. The direction of maximum continuity is N45°E.

The simulations are conditioned to the 13 point data shown in Figure 2a. The bimodal target histogram of the reference data is shown in Figure 2b. The semi-variograms in the principal directions modeled from the reference data set directions are shown in 2c. These are spherical models with range values 31 and 8 grid nodes.

We then average linearly the reference data set over blocks of size 10 × 10. A heteroscedastic uncertainty measure proportional to the measurement is attached to each block average datum. We consider two different sets of precision measures, one with large uncertainty and the other with smaller uncertainty. The block means along with the larger uncertainty values are mapped in Figures 3a and b. The smaller uncertainty values are set to 0.2 times the values shown in Figure 3b. In practice, the point attribute could be porosity and the block averages could be obtained from a co-kriging of seismic data which correlate with the block-averaged porosity (Yao, 1998).

The block average data along with their precision could be interpreted as parameters of a Gaussian probability distribution defined with two parameters, for example, a normal distribution or a log-normal distribution. We condition our simulations to these probability distributions of the block data.

4.1.1 Determination of GMrf weights

The main idea of the GMrf approach lies in the approximation of the required Gaussian conditional distribution assuming them to be dependent only on neighboring values within a
fixed template. Since the semi-variogram model (Figure 2c) is anisotropic, the GMrf template shown in Figure 4 is also chosen anisotropic. The number of template nodes determines the accuracy of the GMrf covariance in reproducing the target covariance model. The weights of expression (3) of the conditional means are obtained by an optimization process which exploits the special block-circulant structure of the inverse covariance matrix for GMrf’s (Caers, 2000a). The resulting weights are shown color-coded in Figure 4. Since the weights are symmetric around the two directions of anisotropy, the actual number of weights (9) estimated by the optimization process is much less than the number of grid nodes (26) in the template.

The algorithm which computes the GMrf weights optimizes the reproduction of the covariance matrix at all lag distances including those beyond the template. The finally approximated covariance matrix contains the covariance at all lags and directions. Figure 5 shows the comparison between the covariance in the principal directions and the target covariance model. The fit is close at all lags.

### 4.1.2 Convergence Issues

Convergence of an MCMC algorithm is an important issue. Various asymptotic convergence proofs have been given in literature (Ripley, 1981). But these proofs provide only asymptotic results, they can be rarely used as a practical check for convergence. Our primary concern is however, data integration and not defining the resulting space of uncertainty. Therefore the stopping criterion, which inherently decides the amplitude of fluctuations between simulated realizations, is of little concern for now, as long as these realizations are conditioned to point
and block data and honor the histogram and the semi-variogram.

The choice of the initial seed image and the criteria for stopping the iterations are important issues which decide the time required for convergence.

- **The initial seed image** The choice of the initial seed realization has a strong effect upon the number of iterations required for convergence. Since the GMrf technique is very efficient in establishing the short range structures, any information provided by the initial seed image about the long range structures would be beneficial in helping the algorithm to converge quickly. Therefore, we begin the iterations with a realization in which all point values in each block datum are set equal to the corresponding block mean value.

- **The criterion for convergence** We use the global variance, i.e. the variance of the point values over the entire grid, as a measure of convergence. Figure 6 shows the convergence of the global variance as the iterations\(^1\) proceed, along with the semi-variogram in the principal directions. As the simulated semi-variogram converges to the model semi-variogram, the global variance also increases and stabilizes close to 1.0. Therefore for this example, the global variance can be considered as a measure of convergence of the variogram.

To decide a fixed number of iterations for stopping the MCMC, we run several simulations with different random seeds and observe the behavior of the global variance.

The number of iterations is then decided on an ad-hoc basis by visual inspection and

\(^1\)1 iteration = Visit to 10000 nodes in random order
is fixed for all realizations.

4.1.3 Comparison of different cases

Our primary motivation for the example considered here is to demonstrate point and block data conditioning along with honoring of the histogram and the variogram. We present several cases, each considering part or all of the conditioning data. For each case, the reproduction of statistics is verified over a set of 100 equiprobable realizations.

**Case A**: Conditioning is only to the 13 point data shown in Figure 2a. Although it is not advisable to use an iterative approach for the easy case of conditioning to point data only (sequential simulation will do the same work faster), this exercise demonstrates the appropriate conditioning of the GMrf approach to point data in addition to honoring the semi-variogram model. Figure 7a shows the global variance from 10 realizations with the number of iterations for case A. From these realizations, the number of iterations for convergence for case A is fixed at 1200.

Figures 8 and 9 show the results for case A. The semi-variograms in the major and minor directions for all 100 realizations and the variograms for the first 5 realizations are shown in Figures 8a and 8b respectively. Note that the semi-variogram is reproduced in both the major and minor directions for all ranges beyond the template dimensions.

The near-data semi-variogram is computed. To calculate this near-data semi-variogram, we consider only small lags distances and only those pairs which include a point conditioning datum. Figure 8c shows the average of the near-data semi-variogram over 100 realizations.
The average of the near-data semi-variograms shows that there is no discontinuity near the data locations which may be associated with some other iterative techniques such as spectral conditional simulation (Yao, 1998).

Figures 9a and 9b shows the histogram of the first realization and its quantile-quantile (q-q) plot versus the target histogram. The bimodal histogram is honored without having to perform a back transformation as is required in other techniques (Journel 1999; Behrens et al, 1996).

Figures 9c and 9d show the conditional mean and the conditional variance computed over the 100 realizations. The point conditioning data locations can be identified from the map of the conditional variance as the areas of low conditional variance.

Thus the GMrf approach is observed to be effective in reproduction of the semi-variogram at all scales, and also in the the honoring of the histogram.

**Case $B_1$** : Simulations are now conditioned to the 13 point data and the 100 block data shown in Figure 3. By "block datum", we understand a specific Gaussian block distribution with given mean and standard deviation. The purpose here is to observe the reproduction of the semi-variogram and of the bimodal target histogram and the conditioning to block data. Figure 7b shows the global variance from 10 realizations with the number of iterations for case $B_1$. From these realizations, the number of iterations for convergence for case $B_1$ is fixed at 1100.

Figures 10a and 10b show the semi-variograms for case $B_1$ over 100 realizations and the
semi-variogram for the first 5 realizations respectively. The semi-variogram is honored in this case also. Note that the ranges of the simulated semi-variograms are larger in Figures 10a-b case than in Figure 8a-b. This can be attributed to the fact that block conditioning performed in this case updates the range of the semi-variogram by imposing the long-range structures. The prior model of the semi-variogram is updated to the posterior model. Figures 10c and d show the histogram and the q-q plot for the first realization of case $B_1$ with the target histogram.

Figures 11a and b show the conditional mean and the conditional variance over 100 realizations for case $B_1$. Note that as compared with Figure 9d, the conditional variance in Figure 11b is lesser because exhaustive block conditioning reduces considerably the space of uncertainty.

Figure 11c shows the scatter plot of the simulated block averages for the first realization versus the conditioning block data. The simulated block averages are scattered tightly around the corresponding block data mean with correlation 0.92. A more complete measure of the effectiveness of conditioning to block data distribution is to compare the distribution of the block average for a single block with the distribution of the block datum for that location. Figure 11d shows the q-q plot of the block averages over 100 realizations for the block located at index location (8,1) versus the block data distribution for that block which has mean 15.11 and standard deviation 1.51. The match is very close.

**Case $B_2$ :**
Conditioning is only to the sole point datum located at index location (45,56) with a value of 23.03 and the sole block datum centered at index location (5,6) which includes this point datum. This block datum has a mean of 16.31 and standard deviation of 1.63. Simulations were generated across the entire grid conditioned with, then without the point datum, but always with the block datum. The correct conditioning to the point datum can be evaluated by calculating the conditional variance for every node belonging to this block over the 100 realizations. The conditional variance should gradually increase from zero at the point data locations. The conditioning to the block datum is evaluated by comparing the empirically observed distribution of the 100 simulated block averages with the target block distribution. Figure 7c and d show the global variance from 10 realizations with the number of iterations for case $B_2$ with and without the point datum. From these realizations, the number of iterations for convergence for case $B_2$ is fixed at 1100 and 1200 respectively for the conditions of with and without the point datum at index location (45, 56).

The purpose behind this case is to demonstrate the conditioning to point datum in presence of a block datum enclosing that point datum. Figures 12 and 13 show the results with and without conditioning to the point datum. The maps of the conditional mean and the conditional variance are presented in Figures 12a and b for the simulations performed with the point datum and in Figures 13a and b for the simulations performed without the point datum. A comparison between Figures 12b and 13b shows that there is a gradual increase of the conditional variance from the center of block outwards along the direction of continuity. This trend in the conditional variance is more pronounced in Figure 12b than in 13b. Even
without the presence of any local datum, conditioning to a block datum causes a decrease in
the conditional variance, towards the center of that block. The reason for this effect is that,
when simulating the central nodes of the block, most conditioning point data are constrained
by the block datum. Nodes which lie along the edges of the block are also constrained by
the nodes outside the block which are not constrained by the block value and have an overall
mean of 12.45, lower than 16.31. Addition of the point conditioning datum increases this
effect as can be observed from Figures 12b and 13b.

Figures 12c and 13c show the reproduction of the block averages distribution. Note that
the reproduction of the block averages are not very good in Figure 12c. This is due to the
contrast in the point conditioning datum with value 23.03 and the mean of the block datum,
16.31. The point datum updates the ”prior” block distribution.

**Case C₁:** All 100 block data shown in Figure 3 are used for conditioning, but the error
standard deviation is taken much smaller (0.1 times the standard deviations shown in Figure
3). The simulations are also conditioned to all 13 point data. Figure 7e shows the global vari-
ance from 10 realizations with the number of iterations for case $C_1$. From these realizations,
the number of iterations for convergence for case $C_1$ is fixed at 400.

The results for case $C_1$ are presented in Figure 14 which shows the semi-variograms over
100 realizations, the semi-variogram for the first 5 realizations, the histogram of the first
realization and the q-q plot of the first realization versus the target histogram. It is observed
that the semi-variogram and the histogram are reproduced in this case also.
The conditional mean and the conditional variance over the 100 realizations for case $C_1$ are shown in Figure 15a-b. Note that the conditional variance in this case is much less than that shown in Figures 9d and 11b, because block data distributions have a smaller variance than in Case $B_1$. Figure 15c shows the scatter plot of the simulated block averages for the first realization with the means of the block data. Figure 15d shows the block datum reproduction for the block located at index location (8,1). Note the quasi-reproduction of the block datum.

**Case $C_2$:** The same single conditioning block datum located at index location (5,6) used in case $B_2$ is considered. No conditioning point data is used. The block data mean is the same, 16.31. But the block datum is assumed to be log-normally distributed with mean 16.29 and standard deviation 1.63, as shown in Figure 16. Figure 7f shows the global variance from 10 realizations with the number of iterations for case $C_2$. From these realizations, the number of iterations for convergence for case A is fixed at 1200.

The normal-quantile plot of simulated block average over 100 realizations is shown in Figure 16b. The simulated block averages are also skewed as the target block cdf. But the simulated mean of the block average (15.53) is slightly smaller than the mean of the target distribution (16.31).

### 4.2 Blocks of irregular geometry

Now, we consider another example of conditioning to block data. Here, we consider the conditioning of geostatistical realizations to weighted harmonic block averages. The block
averages defined in (8) can be written as:

\[ d_V(u') = \frac{\sum w(u)}{\sum \frac{w(u)}{z(u)}}, \quad u' \in V(u') \]  

(12)

where \( w(u) \) are the weights associated with the harmonic averages.

Examples of conditioning to such types of weighted harmonic averages are encountered in applications involving flow modeling in porous media where the effective permeability along a streamline can be represented, under restrictive assumptions, as a harmonic average weighted by the time of flights along the streamline (Wang and Kovscek, 2000). Therefore as opposed to the earlier case, where we had rectangularly shaped blocks, here we have blocks which are irregularly shaped.

The reference field along with the histogram and variogram are shown in Figure 17. The variogram is isotropic with range of 10 grid nodes. The GMrf weights corresponding to this variogram model are shown in Figure 18. The block data are shown in Figures 19 and 20. There are 53 block data. The vertical direction denotes the magnitude of the block harmonic average data. The weights associated with the weighted harmonic averages are shown in Figure 21. The linearly color coded streamlines in each figure denote the relative weights along the streamline. For the probability distributions of the block averages, we use symmetric triangular distributions with a width of 20.0 i.e. for a block average of 450.0, the distribution would range from 440.0 to 460.0.

We perform GMrf iterations conditional to these 53 harmonic averages and obtain the
realization shown in Figure 22. The variogram and histogram are honored. The conditioning to the harmonic averages is exact upto the defined accuracy as shown in Figure 23.

5 Conclusions

The theory of GMrf's allows an iterative algorithm reproducing covariance and histogram. This algorithm when combined with efficient sampling techniques leads to a powerful tool to integrate block data in point simulations. The theory presented in this paper is applicable only to non-overlapping block averages.

Demonstration of the theory with an example shows that there is exact conditioning to the point conditioning datum without any discontinuity. Block data are honored along with their precision measures. In the presence of both point and block datum, there is an updating of the prior block data distribution. Thus, the example demonstrates the applicability of the theory in integrating point and block data.

The application to the example involving harmonic block averages along irregularly shaped blocks demonstrates two things: first that of conditioning to a non-linear average and second, that of conditioning to irregularly shaped blocks.

The general applicability of the proposed algorithm extends beyond the examples shown here. In the context of a grid of nodes, the blocks are any set of nodes (possibly connected). The relation between point and block data could be any many-to-one non-linear function. Further research would focus on such applications which exploit the flexibility provided by
the theory.

6 References


Mathematical Geology, 931–954.


Yao, T., 1998. Porosity modeling in a W. Texas carbonate reservoir conditioned to seismic data: solving the difference of scale and precision problem. SCRF report 11.
Figure 1: **Reference Data Set**: A) Exhaustive Data Set over 100 × 100 grid., B) Variogram Map of Reference Data, C) Semi-variograms in principal directions of continuity.
Figure 2: **Data and Models**: A) Conditioning point data, B) Target histogram, C) Standardized model semi-variogram, D) Point and block datum for case $B_2$
Figure 3: **Block Data Distributions**: (a) The map of the means of the block data distributions, b) The map of the standard deviations of the block data distributions.
Figure 4: **The GMrf Neighborhood and weights**: Relative coordinates are referred to here. In case of edges, the grid is wrapped around.

Figure 5: **Comparison between the true covariance model and the GMrf Fit**: The model covariance (thick line) and covariance obtained using the GMrf weights (dots). *(The covariances are standardized)*
Figure 6: **Convergence of the global variance and the semi-variogram**: The global variance in the normal space is marked by the red line. The thick black line is the model semi-variogram. The semi-variogram in the normal space have been plotted for the 50th, 100th, 200th, 300th and the 500th iteration.
Figure 7: **Variation of global variance in the normal space for the first ten realizations:** a) case A, b) case $B_1$, c) case $B_2$ with the point datum at (45, 56), d) case $B_2$ without any point datum, e) case $C_1$, f) case $C_2$. 
Figure 8: **Results from case A**: a) Semi-variogram for all 100 realizations over the entire grid, b) Semi-variograms from the first 5 realizations over the entire grid, c) The average of the near data semi-variograms over the 100 realizations.
Figure 9: Results from case A: a) Histogram of the first realization, b) q-q plot of the first realization with the target histogram, c) Conditional mean over the 100 realizations, d) Conditional variance over the 100 realizations.
Figure 10: **Results from case $B_1$**: a) Semi-variogram over the entire grid for all 100 realizations, b) Semi-variogram over the entire grid for the first 5 realizations, c) Histogram of the first realization, d) q-q plot of the first realization with the target histogram.
Figure 11: **Results from case $B_1$**: Conditional mean over the 100 realizations, b) The conditional variance over the 100 realizations, c) The scatter plot of the simulated block averages with the means of the block data, d) The q-q plot of the simulated block distribution in block (8,1) with the block data distribution for block (8,1).
Figure 12: **Results from case** $B_2$ with conditioning to the point datum: a) Conditional mean over 100 realizations of the point values in block(5,6), b) Conditional variance of the point values over 100 realizations in block (5,6), c) q-q plot of the simulated block average in block(5,6) with the block datum for block(5,6). (*Relative coordinate of the point datum inside the block is (5,6)*)
Figure 13: **Results from case $B_2$**: with no conditioning to the point datum: a) Conditional mean over 100 realizations of the point values in block(5,6), b) Conditional variance of the point values over 100 realizations in block (5,6), c) q-q plot of the simulated block average in block(5,6) with the block datum for block(5,6).
Figure 14: Results from case $C_1$: a) Semi-variogram over the entire grid for all 100 realizations, b) Semi-variogram over the entire grid for the first 5 realizations, c) Histogram of the first realization, d) q-q plot of the first realization with the target histogram.
Figure 15: **Results from case $C_1$:** Conditional mean over the 100 realizations, b) The conditional variance over the 100 realizations, c) The scatter plot of the simulated block averages with the means of the block data, d) The q-q plot of the simulated block distribution in block (8,1) with the block data distribution for block (8,1).
Figure 16: **Results from case $C_2$** : a) The log-normal probability plot of the block datum for the central block (5,6), b) The log-normal probability of the simulated block averages over the 100 realizations for block (5,6), c) The q-q plot of the simulated block averages with the block data shown in Figure a)
The Reference Permeability Field

Histogram of Reference Field

Variogram in North and South directions

Figure 17: Reference: A) Reference field, B) Target histogram, C) Standardized model semi-variogram
Figure 18: The GMrf Neighborhood and weights: Relative coordinates are referred to here. In case of edges, the grid is wrapped around.
Figure 19: **The Geometry of blocks numbered 1 through 26**: Vertical direction refers to block datum value

Figure 20: **The Geometry of blocks numbered 27 through 53**: Vertical direction refers to block datum value
Figure 21: The harmonic average weights: Weights associated with nodes for each of the 53 streamlines, color is on a linear scale.
Figure 22: Reference: A) Resultant field, B) Reproduced histogram, C) Standardized semi-variogram
Figure 23: Reproduction of 53 block averages